The Move–to–Partner Rule for Self–Organizing Task Allocation on a Linear Array

Walter J. Gutjahr
Department of Statistics and Decision Support Systems
University of Vienna

Abstract: The well–known Move–to–Front rule for self–organizing lists is modified to a so–called Move–to–Partner rule for dynamic task allocation on a chain of processors. We enable an analysis of the sequence of expected communication costs by reducing their computation to that of three–dimensional arrays of certain probabilities, for which a recursion formula and an asymptotic expansion can be given. The costs depend on the underlying communication profile; two types of such profiles are investigated in more detail, namely a two–dimensional version of Zipf's law, and profiles based on a class structure of tasks. In the considered cases, the Move–to–Partner rule effects fast convergence to a stationary state, but comparably high expected stationary costs.

Keywords: Move–to–front, self–organizing systems, task allocation, Markov chains.

1 Introduction and Definitions

Let us consider the following simple allocation problem: Suppose that an array of size N or (in graph-theoretical terms) a chain 1—2— ... —N of N nodes, is given. The nodes represent *processors* of a parallel or distributed system. To these N processors, N *tasks* are to be assigned by a permutation π, where $\pi(i) \in \{1, \ldots, N\}$ denotes the index of the processor to which the task with index i is assigned ($i = 1, \ldots, N$). Furthermore, let an $[N \times N]$–matrix $(p(i, j))$ of *communication probabilities* be given, $p(i, j)$ denoting the probability that in a fixed intercommunication step of the system, task i exchanges a message with task j. In each intercommunication step, exactly one pair of tasks communicates, which implies $\sum_{i=1}^{N} \sum_{j=1}^{N} p(i, j) = 1$. Usually, $p(i, i) = 0$, but it is convenient not to exclude the case $p(i, i) > 0$. The *distance* of two processors with indices k and l is defined as $d(k, l) = |k - l|$, i.e., as the (graph–theoretical) distance between the two nodes in the chain. Now, the aim is to minimize the *weighted communication cost*,

$$\bar{C}_\pi = \sum_{i=1}^{N} \sum_{j=1}^{N} p(i, j)\, d(\pi(i), \pi(j)), \tag{1}$$

over all possible permutations π of $1, \ldots, N$. Formula (1) expresses the assumption that the cost of a communication between two processors k and l is proportional to their distance, such that $d(\pi(i), \pi(j))$ is a measure for the cost caused by communication between task i and task j.

Formally, the problem of minimizing (1) is a *Quadratic Assignment Problem* (QAP) (see Burkard [2]) with a special distance matrix $(d(k, l))$ determined by the structure of the linear array. Solving analogous QAPs for different types of interconnection network topologies has found considerable attention in the literature on task allocation in parallel and distributed systems (see, e.g., Sinclair [20], Ho and Johnsson [13], Dahl [6], Chakrapani and Skorin–Kapov [3], Gutjahr et al. [11]). Nevertheless, most articles dealing with the communication cost aspect only treat the *static* version of the problem: The communication profile $(p(i,j))$ is assumed to be *known*, so that tasks may be allocated in advance, and the allocation needs not to be modified later. In many applications, this assumption is unrealistic. In these cases, it is better to do the task allocation in a *dynamic* way using task migration, i.e., a procedure by which certain tasks change the processors they have been assigned to before. (Let us remark that there are many publications on dynamic task allocation with the aim of achieving criteria like load balance, see, e.g., Xu and Lau [22], Xu et al. [23], Gosh and Muthukrishnan [10]. These approaches, however, usually pay little attention to the intertask communication aspect.)

In the present article, we want to investigate how dynamic task allocation on a linear array can be managed by a self–organizing system similar to a *self–organizing linear list*. Self–organizing data structures, especially self–organizing lists, have been studied in a vast number of articles (see, e.g., Hendricks [12], the two Lecture Notes volumes [16] and [7], Courcoubetis and Weber [5], Phatarfod [18], Hofri and Shachnai [14], Lam and Cowan [15], McGeoch [17], Flajolet et al. [9], Valiveti and Oomen [21], Rodrigues [19], Fill [8], Chong and Lam [4]). The efficiency of some basic strategies of maintaining a self–organizing list, such as *Move–to–Front* (MTF) or the *Transpose Rule* (TR), has been analyzed in much detail. Here, we borrow the MTF rule from self–organizing lists and adapt it to our problem context in the form of a "Move–to–Partner" (MTP) rule.

MTF prescribes that each time a data element in a list is addressed by a user request, it is moved to the front of the current list. E.g., if the list currently has the form 4—2—5—3—1 and element 3 is required, then the list is re–organized to 3—4—2—5—1. In this way, the expected sequential search cost for an access to a data element is gradually decreased, since frequently required elements tend to get located near the front of the list.

We use an analogous principle in the following definition:

Move–to–Partner Rule:

Let π be the current task allocation, let i and j be the tasks communicating in the current intercommunication step, and let $k = \pi(i)$ and $l = \pi(j)$ be the current positions of the tasks i and j. Then π is re–organized to the following allocation π' (note that $\pi^{-1}(s)$ and $(\pi')^{-1}(s)$ refer to the task at position s before resp. after the re–organization):

If $k < l-1$, set $(\pi')^{-1}(l-1) = i$, $(\pi')^{-1}(s) = \pi^{-1}(s+1)$ for $s = k, \ldots, l-2$, and $(\pi')^{-1}(s) = \pi^{-1}(s)$ for $s < k$ or $s \geq l$.

If $k > l+1$, set $(\pi')^{-1}(l+1) = i$, $(\pi')^{-1}(s) = \pi^{-1}(s-1)$ for $s = l+2, \ldots, k$, and $(\pi')^{-1}(s) = \pi^{-1}(s)$ for $s \leq l$ or $s > k$.

If $k \in \{l-1, l, l+1\}$, set $\pi' = \pi$.

Informally speaking, MTP shifts task i immediately next to task j, and it shifts all tasks that were located between i and j one position away from j. E.g., if the current allocation is 4—2—5—3—1, and task $i = 2$ communicates with task $j = 1$, then the new allocation is 4—5—3—2—1; if, instead, task $i = 1$ communicates with task $j = 2$ in this example, then the new allocation is 4—2—1—5—3. As it can be seen, we admit a role distinction between the two communicating tasks i and j. This distinction may already be present in the application, e.g., if in a communication a *sender* may be distinguished from a *receiver*. Where this is not the case, one may simply "flip a coin" in order to decide which of the two communicating tasks i and j is moved to the other one. Formally, this situation can be represented by splitting the overall communication probability between i and j into two equal parts $p(i, j) = p(j, i)$.

Now let us assume that initially, an allocation $\pi^{(0)}$ is chosen according to some probability distribution on the set of permutations of $1, \ldots, N$. (This distribution may also degenerate to the point mass in a certain initial permutation; then, the start allocation is deterministic.) By repeated application of the MTP rule according to the pairs $(i^{(n)}, j^{(n)})$ of tasks communicating in intercommunication step n ($n = 1, 2, \ldots$), we get a sequence of permutations $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots)$, where $\pi^{(n)}$ is the result of MTP applied to $\pi^{(n-1)}$ in case of communication of $i^{(n)}$ with $j^{(n)}$. In our theoretical model, each allocation $\pi^{(n)}$ may be evaluated by computing its weighted communication cost $\bar{C}_{\pi^{(n)}}$ according to (1). Note that this implies a kind of meta– perspective: The dynamic allocation procedure itself is based on the assumption that the communication profile $(p(i, j))$ is *not* known, but an external evaluator knowing this profile might nevertheless judge each current $\pi^{(n)}$ according to formula (1).

Let us abbreviate $\bar{C}_{\pi^{(n)}}$ by $\bar{C}^{(n)}$. Then, to the (random) sequence $(\pi^{(n)})$ of allocations, there corresponds a (random) sequence $(\bar{C}^{(n)})$ of weighted communication costs. Fixing an initial distribution of $\pi^{(0)}$, we can further define the deterministic sequence $(E(\bar{C}^{(n)}))$ of *expected* weighted communication costs after intercommunication step n.

We would like to have answers to the following questions:

1. If a communication profile is given and the initial distribution is the uniform distribution on the set of all permutations of $1, \ldots, N$, can it be verified that the values $E(\bar{C}^{(n)})$ decrease?

2. Do the values $E(\bar{C}^{(n)})$ *converge* to a stationary expected cost value, and if yes, how near is the value $\lim_{n\to\infty} E(\bar{C}^{(n)})$ to the weighted communication costs of the *optimal* allocation?

3. In case of convergence — what can be said about the *speed* of convergence?

For the counterpart of self–organizing lists, many interesting results answering analogous questions are known. We cannot expect to obtain a comparable amount of analytical results for our problem: In the self–organizing–list case, the determination of the optimal configuration for *known* frequencies is trivial and requires nothing but *sorting*, whereas in our case, the corresponding problem for known communication profile is the QAP, which is notorious as one of the hardest problems of combinatorial optimization. So also the structure of self–organizing task allocation, as described above, can be expected to be distinctly more involved than that of self–organizing lists. As a consequence, the aim of our investigation must necessarily be more modest. Nevertheless, some analytical insights can also be obtained in our problem context. As it will be shown, they enable at least numerical evaluations of the performance of the system for arbitrary given communication profiles in polynomial time, as well as certain general asymptotic assertions for particular profiles.

It is interesting to note that MTF for the self–organizing linear list is a special case of MTP for the self–organizing allocation considered here: Let $f(i)$ denote the probability of an access to element i a linear list ($i = 1, \ldots, N$). Construct a linear array with elements (tasks) $1, \ldots, N$

on places $1, \dots, n$ in any arrangement, and an additional (fictitious) element 0 positioned to the l.h.s. of element 1, i.e., on place 0. Now, by setting $p(i, 0) = f(i)$ $(i = 1, \dots, N)$ and $p(i, j) = 0$ for all other pairs (i, j), MTP on this array reduces to MTF on the corresponding linear list. In this article, however, we shall mainly concentrate on symmetric communication profiles, where this construction is not possible anymore.

In order to avoid misunderstandings, let us emphasize that the MTP rule as introduced above is *not* intended as a suggestion for the practical implementation of self–organizing task allocation on a linear array, but rather as a paradigm allowing the mathematical investigation of certain important aspects of such systems. In order to become practically applicable, the use of the MTP rule has to undergo modifications. E.g., note that the cost of task migration is usually higher than that of intercommunication between already allocated tasks, so that it is inefficient to apply MTP after *each* intercommunication step. The most obvious modification for these cases would be to apply MTP only after each Rth intercommunication step (or: at the end of time periods of constant length), which leads to the same stationary expected cost, but to a convergence time multiplied by R.

The plan for the rest of the paper is the following: In section 2, a basic recursion formula enabling us to compute the values $E(\bar{C}^{(n)})$ will be derived. Section 3 provides the tools for an asymptotic expansion of these values, and in section 4, the derived results are applied to two specific types of communication profiles: a two–dimensional version of Zipf's law, and a type of profiles based on a decomposition of the tasks into classes with homogenous communication behavior. Section 5 contains concluding remarks.

2 A Basic Recursion

Throughout the article, it will be assumed that the pairs $(i^{(n)}, j^{(n)})$ of communicating tasks are selected *independently* from each other in each intercommunication step according to the distribution $(p(i, j))$. Then the random sequence $(\pi^{(0)}, \pi^{(1)}, \dots)$ of permutations created by application of the MTP rule is a *Markov chain*, because the distribution of $\pi^{(n)}$ only depends on $\pi^{(n-1)}$. At least in the case where $p(i, j) > 0$ for all $i \neq j$, this Markov chain is guaranteed to be ergodic, hence it has a unique stationary distribution, and the values $E(\bar{C}^{(n)})$ converge to the expected cost value $E(\bar{C}^{(\infty)})$ under this distribution. (We shall not restrict ourselves exclusively to situations where $p(i, j) > 0$ for $i \neq j$; in subsection 4.2, also cases where certain pairs of tasks never communicate will occur.) Before addressing the determination of $E(\bar{C}^{(\infty)})$, we shall deal with the case of finite n, deriving a basic recursion formula for quantities that admit the computation of $E(\bar{C}^{(n)})$.

We set

$$b^{(n)}(i, j) = P\{\pi^{(n)}(i) < \pi^{(n)}(j)\} \tag{2}$$

and

$$b^{(n)}(i, j, k) = P\{\pi^{(n)}(i) < \pi^{(n)}(j) < \pi^{(n)}(k)\}, \tag{3}$$

i.e., $b^{(n)}(i, j)$ and $b^{(n)}(i, j, k)$ are the probabilities that after the nth communication step, task i is allocated (somewhere) to the left of task j, resp. that task i is allocated to the left of task j *and* task j is allocated to the left of task k.

Lemma 2.1. The expected communication cost after intercommunication step n is given by

$$E(\bar{C}^{(n)}) = 1 - \sum_{i=1}^{N} p(i,i) + \sum_{i=1}^{N}\sum_{j=1}^{N} p(i,j) \sum_{k=1}^{N} (b^{(n)}(i,k,j) + b^{(n)}(j,k,i)). \tag{4}$$

Proof. Since $d(\pi(i),\pi(i)) = 0$, it suffices to consider the terms in (1) with $i \neq j$. For $i \neq j$, the distance $d(\pi^{(n)}(i),\pi^{(n)}(j)) = |\pi^{(n)}(i) - \pi^{(n)}(j)|$ is one plus the number of other tasks allocated between task i and task j, i.e., (with $1_{\{\text{assertion}\}} = 1$ if assertion $=$ true and 0 else)

$$d(\pi^{(n)}(i),\pi^{(n)}(j)) = 1 + \sum_{k=1}^{N} (1_{\{\pi^{(n)}(i)<\pi^{(n)}(k)<\pi^{(n)}(j)\}} + 1_{\{\pi^{(n)}(j)<\pi^{(n)}(k)<\pi^{(n)}(i)\}}).$$

Taking expectation with respect to the distribution of $\pi^{(n)}$, we obtain

$$E(d(\pi^{(n)}(i),\pi^{(n)}(j))) = 1 + \sum_{k=1}^{N} (b^{(n)}(i,k,j) + b^{(n)}(j,k,i)).$$

Multiplication by $p(i,j)$ and summation over all (i,j) with $i \neq j$ yields (4), since $b^{(n)}(i,k,i) = 0$. $\square$

The reader should notice that $1 - \sum_i p(i,i)$ is independent of the allocation π, so instead of $E(\bar{C}^{(n)})$, we may as well consider the cost function

$$K^{(n)} = E(\bar{C}^{(n)}) - [1 - \sum_{i=1}^{N} p(i,i)] = \sum_{i=1}^{N}\sum_{j=1}^{N} p(i,j) \sum_{k=1}^{N} (b^{(n)}(i,k,j) + b^{(n)}(j,k,i)). \tag{5}$$

Remark 2.1. If $p(i,j) = p(j,i)$ for all (i,j), then (5) simplifies to

$$K^{(n)} = 2 \sum_{i=1}^{N}\sum_{j=1}^{N} p(i,j) \sum_{k=1}^{N} b^{(n)}(i,k,j). \tag{6}$$

The same simplified equation holds in the case where the initial distribution of $\pi^{(0)}$ is *left–right–symmetric*, i.e., where

$$P\{\pi^{(0)} = (k_1, \ldots, k_N)\} = P\{\pi^{(0)} = (k_N, \ldots, k_1)\}, \tag{7}$$

since (7) entails

$$b^{(n)}(i,j,k) = b^{(n)}(k,j,i) \quad \text{for all } i,j,k \text{ and all } n \geq 0. \tag{8}$$

(Observe that also the MTP rule itself is symmetric with respect to a left–right–interchange, such that the distribution of $\pi^{(n)}$ "inherits" left–right–symmetry from that of $\pi^{(n-1)}$.) In particular, the *uniform* distribution on the set of permutations of $1, \ldots, N$ is left–right–symmetric.

Now we are able to derive the basic recursion formula. Therein, we use the abbreviation

$$p(i,.) = \sum_{j=1}^{N} p(i,j).$$

Proposition 2.1. For $n \geq 0$ and i, j, k pairwise different,

$$b^{(n+1)}(i,j,k) = [1 - p(i,.) - p(j,.) - p(k,.) + p(i,j) + p(j,i) + p(k,j) + p(j,k)]\; b^{(n)}(i,j,k)$$
$$+\, p(j,i)\, b^{(n)}(i,k,j) + p(j,k)\, b^{(n)}(j,i,k)$$
$$+ \sum_{m=1}^{N} [p(i,m)\, b^{(n)}(m,j,k) + p(j,m)\, b^{(n)}(i,m,k) + p(k,m)\, b^{(n)}(i,j,m)]. \tag{9}$$

Proof. Consider the nth intercommunication step. For abbreviation, we write b, b', π and π' instead of $b^{(n)}$, $b^{(n+1)}$, $\pi^{(n)}$ and $\pi^{(n+1)}$, respectively. In the considered step, a random pair of tasks communicates. Let us denote the indices of these tasks by X and Y (capitalization is used to indicate that X and Y are random variables). X is the moved task and Y is the task to which task X will be neighbor. Furthermore, let us fix three pairwise different indices i, j, k from the range $\{1, \ldots, N\}$. We distinguish four mutually exclusive cases and ask in each case under which circumstances the situation $\pi'(i) < \pi'(j) < \pi'(k)$ can occur:

Case (a): $X = i$. Since task $X = i$ is the moved task, $\pi(j) < \pi(k)$ must already hold if $\pi'(j) < \pi'(k)$ is to be reached. According to MTP, there are exactly two situations that lead to $\pi'(X) = \pi'(i) < \pi'(j) < \pi'(k)$:

- $\pi(Y) < \pi(j) < \pi(k)$, and
- $\pi(Y) = \pi(j) < \pi(k)$ and $\pi(i) < \pi(j)$.

In both situations, task $X = i$ is drawn to a position to the left of task j. By considering all possible indices m for Y, we get as the probability for the first situation the expression

$$\sum_{m=1}^{N} p(i,m)\, b(m,j,k),$$

whereas the probability for the second situation is $p(i,j)\, b(i,j,k)$. In total, we obtain

$$P\{\pi'(i) < \pi'(j) < \pi'(k) \mid \text{case (a)}\} = \sum_{m=1}^{N} p(i,m)\, b(m,j,k) + p(i,j)\, b(i,j,k). \tag{10}$$

Case (b): $X = j$. For $\pi'(i) < \pi'(k)$, already $\pi(i) < \pi(k)$ must hold. Now, there are exactly three situations that lead to $\pi'(i) < \pi'(j) < \pi'(k)$:

- $\pi(i) < \pi(Y) < \pi(k)$,
- $\pi(Y) = \pi(i) < \pi(k)$ and $\pi(j) > \pi(i)$, and
- $\pi(Y) = \pi(k) > \pi(i)$ and $\pi(j) < \pi(k)$.

Using

$$P\{\pi(i) < \pi(j) \text{ and } \pi(i) < \pi(k)\} = b(i,j,k) + b(i,k,j),$$

and adding the respective probabilities for the three situations, we obtain in a similar way as in case (a):

$$P\{\pi'(i) < \pi'(j) < \pi'(k) \mid \text{case (b)}\}$$
$$= \sum_{m=1}^{N} p(j,m)\, b(i,m,k) + p(j,i)\, [b(i,j,k) + b(i,k,j)] + p(j,k)\, [b(i,j,k) + b(j,i,k)]. \tag{11}$$

Case (c): $X = k$. This case is treated quite analogously as case (a). One obtains

$$P\{\pi'(i) < \pi'(j) < \pi'(k) \mid \text{case (c)}\} = \sum_{m=1}^{N} p(k,m)\, b(i,j,m) + p(k,j)\, b(i,j,k). \tag{12}$$

Case (d): $X \neq i,j,k$. In this case, the relative order of the tasks i, j and k is not changed, such that $\pi'(i) < \pi'(j) < \pi'(k)$ results exactly if $\pi(i) < \pi(j) < \pi(k)$. The probability of the last event is $b(i,j,k)$, such that

$$P\{\pi'(i) < \pi'(j) < \pi'(k) \mid \text{case (d)}\} = (1 - p(i,.) - p(j,.) - p(k,.))\, b(i,j,k). \tag{13}$$

The assertion of the proposition follows by adding equations (10) – (13). □

Let us shortly outline the significance of Proposition 2.1: The Markov chain $\pi^{(0)}, \pi^{(1)}, \ldots$ has $N!$ possible states, so the distribution of $\pi^{(n)}$ is represented by an $(N!)$–dimensional probability vector. Computing the distributions of $\pi^{(0)}, \pi^{(1)}, \ldots$ recursively is therefore already infeasible for relatively small N. Proposition 2.1, together with Lemma 2.1, allows it to compute $E(\bar{C}^{(n)})$ based on the values $b^{(n)}(i,j,k)$, which reduces the complexity from the computation on $N!$ real numbers for each n to that of only N^3 real numbers.

Remark 2.2. Consider any *fixed* $m \in \{1, \ldots, N\} \setminus \{i, j\}$. Then $\pi^{(n)}(i) < \pi^{(n)}(j)$ is equivalent to the event that either $\pi^{(n)}(i) < \pi^{(n)}(j) < \pi^{(n)}(m)$, $\pi^{(n)}(i) < \pi^{(n)}(m) < \pi^{(n)}(j)$, or $\pi^{(n)}(m) < \pi^{(n)}(i) < \pi^{(n)}(j)$, and these three particular events are mutually exclusive, such that their probabilities can be added. Hence,

$$b^{(n)}(i,j) = b^{(n)}(i,j,m) + b^{(n)}(i,m,j) + b^{(n)}(m,i,j). \tag{14}$$

Using this relation, (9) can also be written in the following alternative form:

$$b^{(n+1)}(i,j,k) = [p(j,i) + p(j,k)]\; b^{(n)}(i,k)$$
$$+[1 - p(i,.) - p(j,.) - p(k,.) + p(i,j) + p(k,j)]\; b^{(n)}(i,j,k)$$
$$- p(j,k)\; b^{(n)}(i,k,j) - p(j,i)\; b^{(n)}(j,i,k)$$
$$+ \sum_{m=1}^{N} [p(i,m)\; b^{(n)}(m,j,k) + p(j,m)\; b^{(n)}(i,m,k) + p(k,m)\; b^{(n)}(i,j,m)]. \tag{15}$$

3 Asymptotic Expansion

By the basic recursion formula (9), the probabilities $b^{(n)}(i,j,k)$ and therefore also the costs $K^{(n)}$ can be computed up to an arbitrary value of n. However, for obtaining $K^{(n)}$ to some given large n (say $n = 10^6$) by means of the recursion, the required computational effort gets prohibitive. Besides the values $K^{(n)}$, we may also be interested in the *stationary* cost $K^{(\infty)} = \lim_{n\to\infty} K^{(n)}$, and in the *time–to–$\epsilon$–quasi-stationarity*, defined for a given $\epsilon > 0$ as the least n_0 such that $|K^{(n)} - K^{(\infty)}| < \epsilon$ for all $n \geq n_0$, a value which cannot be obtained by (9). Finally, for the analysis of particular communication profiles (cf. section 4), it is desirable to have a more explicit representation of the numbers $K^{(n)}$ than that provided by the results of section 2. For these reasons, we shall derive a complete *asymptotic expansion* of the probabilities $b^{(n)}(i,j,k)$ (and hence also of the costs $K^{(n)}$) in this section.

From now on, we restrict ourselves to a *symmetric communication profile*, i.e., to the condition $p(i,j) = p(j,i)$ for all (i,j), and to a *left–right–symmetric initial distribution*, as defined by (7), such that (8) holds. For numerical purposes, it is very convenient to make explicit use of this equation which reduces the number of variables to the half (and therefore certain computations described later to a fourth). Hence we define an equivalence relation on the set of triples of pairwise different indices by $(i,j,k) \sim (k,j,i)$. The equivalence classes with respect to this equivalence relation shall be denoted by $< i,j,k >$. Each equivalence class $< i,j,k > = < k,j,i >$ consists of exactly two triples, namely (i,j,k) and (k,j,i). As the representative of the class $< i,j,k >$, we may take that triple for which the first component is smaller than the third component.

Under the assumptions above,

$$b^{(n)}(< i,j,k >) = b^{(n)}(i,j,k) + b^{(n)}(k,j,i) = 2\, b^{(n)}(i,j,k)$$

defines values $b^{(n)}(< i,j,k >)$ in an unambigous sense. Eq. (9) may then be rewritten by replacing each $b^{(n)}(i,j,k)$ with $b^{(n)}(< i,j,k >)/2$, etc. Furthermore, under the assumptions above, $b^{(n)}(i,k) = 1/2$ holds. Extracting the terms for $m = i, j$ and k from the sum, we obtain in total:

$$\begin{aligned} b^{(n+1)}(< i,j,k >) \;&=\; p(i,j) + p(j,k) \\ &+ \alpha_{<i,j,k>}\, b^{(n)}(< i,j,k >) - p(j,k)\, b^{(n)}(< i,k,j >) - p(i,j)\, b^{(n)}(< j,i,k >) \\ &+ \sum_{m\neq i,j,k} [p(i,m)\, b^{(n)}(< m,j,k >) + p(j,m)\, b^{(n)}(< i,m,k >) + p(k,m)\, b^{(n)}(< i,j,m >)], \end{aligned} \quad (16)$$

where

$$\alpha_{<i,j,k>} = 1 - p(i,.) - p(j,.) - p(k,.) + p(i,j) + p(j,k) + p(i,i) + p(j,j) + p(k,k). \quad (17)$$

Note that $\alpha_{<i,j,k>}$ is well–defined since interchanging i and k does not change the values on the r.h.s. of (17).

Next, we introduce *generating functions* by setting

$$f_{<i,j,k>}(z) = \sum_{n=0}^{\infty} b^{(n)}(< i,j,k >)\, z^n.$$

Multiplication of (16) by z^{n+1}, summation over $n \geq 0$, and index transformation on the l.h.s. yields after some short manipulations:

$$(z\,\alpha_{<i,j,k>} - 1)\, f_{<i,j,k>}(z) + z \cdot \{-p(j,k)\, f_{<i,k,j>}(z) - p(i,j)\, f_{<j,i,k>}(z)$$

$$+ \sum_{m \neq i,j,k} [p(i,m)\, f_{<m,j,k>}(z) + p(j,m) f_{<i,m,k>}(z) + p(k,m) f_{<i,j,m>}(z)]\}$$

$$= -b^{(0)}(< i,j,k >) - [p(i,j) + p(j,k)] \cdot z(1-z)^{-1}. \tag{18}$$

To rewrite (18) in a more compact form, we use matrix notation. Since there are exactly $N(N-1)(N-2)$ triples (i,j,k) of pairwise different indices, there exist

$$M := (1/2)\; N(N-1)(N-2)$$

equivalence classes $< i,j,k >$. We construct a $[M \times M]$–matrix A. Both its lines and its columns are labeled with the equivalence classes $< i,j,k >$, arranged in some fixed order which is identical for lines and columns. Now, the elements of the line of A labeled with $< i,j,k >$ are the following:

- $\alpha_{<i,j,k>}$ in column $< i,j,k >$,
- $-p(j,k)$ in column $< i,k,j >$,
- $-p(i,j)$ in column $< j,i,k >$,
- $p(i,m)$ in column $< m,j,k >$ for each $m \neq i,j,k$,
- $p(j,m)$ in column $< i,m,k >$ for each $m \neq i,j,k$,
- $p(k,m)$ in column $< i,j,m >$ for each $m \neq i,j,k$.

All other columns of line $< i,j,k >$ contain the element 0.

It is easy to verify that the columns corresponding to the $3 + 3(N-3)$ nontrivial entries of line $< i,j,k >$ listed above are pairwise different. Moreover, observe that it does not matter whether the label $< i,j,k >$ of the considered line of A is written in this form or in the form $< k,j,i >$: In the latter case, $-p(j,i)$ is inserted in column $< k,i,j >$, and $-p(k,j)$ is inserted in column $< j,k,i >$, etc., with the same final effect. So the definition of A is unambigous.

Lemma 3.1. Under the assumptions at the beginning of this section, the matrix A is symmetric.

Proof. Fix the line with label $< i,j,k >$. Column $< i,j,k >$ of this line contains the element of the main diagonal. In column $< i,k,j >$ of this line, the entry is $-p(j,k)$. On the other hand, in column $< i,j,k >$ of line $< i,k,j >$, the entry is the same, namely $-p(k,j) = -p(j,k)$. In an analogous way, the symmetry is verified for the remaining non–trivial entries in line $< i,j,k >$ as well. □

While the transition matrix of the Markov chain $(\pi^{(0)}, \pi^{(1)}, \ldots)$ has $O(N!)$ lines and columns, A has only $O(N^3)$ lines and columns and is sparse, containing not more than $O(N^2)$ nonzero elements in total.

Let us introduce some further abbreviations. We set

$$\bar{p}_{<i,j,k>} = p(i,j) + p(j,k)$$

and

$$r_{<i,j,k>}(z) = -b^{(0)}(< i,j,k >) - \bar{p}_{<i,j,k>} \cdot z(1-z)^{-1},$$

and define the M-dimensional column vectors

$$\begin{aligned} \bar{p} &= (\bar{p}_{<i,j,k>}), \\ b^{(n)} &= (b^{(n)}(< i,j,k >)) \quad (n = 0, 1, \ldots), \\ r(z) &= (r_{<i,j,k>}(z)) = -b^{(0)} - z(1-z)^{-1}\bar{p}, \end{aligned}$$

and

$$f(z) = (f_{<i,j,k>}(z)).$$

By arranging the M equivalence classes $< i,j,k >$ in a fixed order, we obtain an index

$$ind(< i,j,k >) \in \{1, \ldots, M\}$$

for each $< i,j,k >$. We may identify $< i,j,k >$ with $ind(< i,j,k >)$. Then, the line of A labeled by $< i,j,k >$ may also be indicated by the line number $u = ind(< i,j,k >)$, and the column of A labeled by $< i',j',k' >$ may also be indicated by the column number $v = ind(< i',j',k' >)$.

Finally, let $(A; v; b)$ be the result of replacing in A the column with index v by the column vector b, and let

$$[A; b] = \begin{pmatrix} \det(A; 1; b) \\ \det(A; 2; b) \\ \cdots \\ \det(A; M; b) \end{pmatrix}.$$

Using the introduced notation, we can now write (18) as

$$(zA - I)\, f(z) = r(z). \tag{19}$$

As A is symmetric by Lemma 3.1, all eigenvalues of A are real numbers. Let $\lambda_1, \ldots, \lambda_k$ be these eigenvalues, ordered in such a way that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k|$, and let m_s be the (algebraic) multiplicity of eigenvalue λ_s $(s = 1, \ldots, k)$. We have $m_1 + \ldots + m_k = M$. The characteristic polynomial $\varphi(\lambda)$ of A is then

$$\varphi(\lambda) = \det(A - \lambda I) = (-1)^M\, (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_k)^{m_k}. \tag{20}$$

For $z \neq 0$,

$$(A - z^{-1}I)\, f(z) = r(z)/z, \tag{21}$$

and if z^{-1} is not an eigenvalue of A, i.e., if $z \notin \{\lambda_1^{-1}, \ldots, \lambda_k^{-1}\}$, the matrix $A - z^{-1}I$ is regular. In this case,

$$f(z) = (A - z^{-1}I)^{-1} r(z)/z. \tag{22}$$

Alternatively, the solution of (21) can be represented by Cramer's rule: The uth component $f_u(z)$ of the vector $f(z)$ is given by

$$f_u(z) = \frac{\det(A - z^{-1}I;\ u;\ r(z)/z)}{\det(A - z^{-1}I)} = \frac{\det(A - z^{-1}I;\ u;\ r(z))}{z\ \det(A - z^{-1}I)}. \tag{23}$$

Now let us set $\lambda = z^{-1}$ in (23). This yields:

$$f_u(\lambda^{-1}) = \frac{\lambda\ \det(A - \lambda I;\ u;\ -b^{(0)} - \bar{p}/(\lambda - 1))}{\det(A - \lambda I)} = \frac{(-1)^M\ \lambda\ \det(A - \lambda I;\ u;\ -(\lambda - 1)b^{(0)} - \bar{p})}{(\lambda - 1)\,(\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_k)^{m_k}}.$$

Here, the numerator is a polynomial of degree less or equal $M + 1$ in λ, and the denominator is a polynomial of degree $M + 1$ in λ. Hence $f_u(\lambda^{-1})$ has a partial fraction decomposition with a (possibly vanishing) constant as remainder:

$$f_u(\lambda^{-1}) = \sum_{s=0}^{k} \sum_{t=1}^{m_s} \frac{c_{st}^{(u)}}{(\lambda - \lambda_s)^t} + \text{ const},$$

where $\lambda_0 = 1$ and $m_0 = 1$. Expressed in z,

$$f_u(z) = \sum_{s=0}^{k} \sum_{t=1}^{m_s} c_{st}^{(u)}\ (-1)^t (\lambda_s - z^{-1})^{-t} + \text{ const}. \tag{24}$$

As usual, we denote the coefficient of z^n in the power series $g(z)$ by $[z^n]g(z)$. Taylor expansion of $(\lambda z - 1)^{-t}$ at $z = \lambda^{-1}$ yields

$$[z^n]\ (\lambda z - 1)^{-t} = \lambda^n\ (-1)^{-t} \binom{n + t - 1}{n},$$

so

$$[z^n]\ (\lambda - z^{-1})^{-t} = [z^{n-t}]\ (\lambda z - 1)^{-t} = \lambda^{n-t}\ (-1)^t \binom{n - 1}{n - t}.$$

Inserting this into (24), we finally obtain the following result:

Theorem 3.1. With A, λ_s, $\bar{p}$ and $(A; v; b)$ as defined at the beginning of this section, the vector $b^{(n)}$ of the probabilities $b^{(n)}(< i, j, k >)$ $(n \geq 1)$ is given by

$$b^{(n)} = \sum_{s=0}^{k} \sum_{t=1}^{m_s} c_{st}\ \lambda_s^{n-t} \binom{n - 1}{t - 1}, \tag{25}$$

where, with $\lambda_0 = 1$ and $m_0 = 1$,

$$\sum_{s=0}^{k} \sum_{t=1}^{m_s} \frac{c_{st}^{(u)}}{(\lambda - \lambda_s)^t} + \text{ const}$$

is the partial fraction decomposition of the rational function

$$\Phi_u(\lambda) = \frac{(-1)^M\ \lambda\ \det(A - \lambda I;\ u;\ -(\lambda - 1)b^{(0)} - \bar{p})}{(\lambda - 1)\,(\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_k)^{m_k}}, \tag{26}$$

and c_{st} is the column vector $(c_{st}^{(u)})_{u=1,\ldots,M}$.

In the following Corollary 1, the two leading terms in the asymptotic expansion (25) are computed. We restrict ourselves to the case where $|\lambda_1| < 1$. This is no essential restriction: Let $b_u^{(n)}$ be the uth coordinate of the vector $b^{(n)}$ $(u = 1, \ldots, M)$. If there was an eigenvalue λ_s

of A with $|\lambda_s| > 1$, eq. (25) would yield unbounded values of $b_u^{(n)}$, which is impossible since the numbers $b_u^{(n)}$ are probabilities. Also $\lambda_s = 1$ ($s \geq 1$) cannot occur, since in this case the root $\lambda = 1$ of the denominator of (26) would have a multiplicity larger than one, which would again imply unboundedness of the values $b_u^{(n)}$. An eigenvalue $\lambda_1 = -1$ with $m_1 = 1$ is excluded at least in the case of an ergodic Markov chain (as it is guaranteed, e.g., by the condition that all $p(i,j) > 0$ ($i \neq j$)), since $\lambda_1 = -1$ would lead to oscillating behavior instead of convergence to a stationary state.

Corollary 1. If $1 > |\lambda_1| > |\lambda_2|$, then

$$b^{(n)} = -(A-I)^{-1}\,\bar{p}$$
$$+(-1)^{m_1}\frac{[A-\lambda_1 I;\; r(\lambda_1^{-1})]}{(\lambda_2-\lambda_1)^{m_1}\ldots(\lambda_k-\lambda_1)^{m_k}}\,\lambda_1^{n-m_1+1}\binom{n-1}{m_1-1}\,(1+O(1/n)) \quad (n\to\infty).$$

If $1 > |\lambda_1| = |\lambda_2|$ (i.e., $\lambda_2 = -\lambda_1$), then a further term is added to the second term, wherein λ_1 is replaced by $\lambda_2 = -\lambda_1$.

Proof. The highest–order term in (25) is the constant term $c_{01}^{(u)}$. In view of (26) and (20),

$$c_{01}^{(u)} = \lim_{\lambda\to 1}\,(\lambda-1)\,\Phi_u(\lambda) = \frac{(-1)^M\;\det(A-I;\;u;\;-\bar{p})}{(1-\lambda_1)^{m_1}\ldots(1-\lambda_k)^{m_k}} = \frac{\det(A-I;\;u;\;-\bar{p})}{\det(A-I)}.$$

By Cramer's rule, the last expression is the solution of $(A-I)x = -\bar{p}$, that is, the vector $-(A-I)^{-1}\bar{p}$. (Note that $A-I$ is regular, since $\lambda = 1$ is not an eigenvalue of A by assumption.)

The term of second–highest order in (25) is yielded by $s = 1$ and $t = m_1$, since as $n \to \infty$,

$$\binom{n-1}{0} << \binom{n-1}{1} << \ldots << \binom{n-1}{m_1-1}.$$

So we have to compute $c_{1m_1}^{(u)}$. From

$$\Phi_u(\lambda) = \sum_{0\leq s\leq k, s\neq 1}\sum_{t=1}^{m_s}\frac{c_{st}^{(u)}}{(\lambda-\lambda_s)^t} + \sum_{t=1}^{m_1-1}\frac{c_{1t}^{(u)}}{(\lambda-\lambda_1)^t} + \frac{c_{1m_1}^{(u)}}{(\lambda-\lambda_1)^{m_1}} + \text{const}$$

it follows that

$$(\lambda-\lambda_1)^{m_1}\,\Phi_u(\lambda) = O(\lambda-\lambda_1) + c_{1m_1}^{(u)} \quad (\lambda\to\lambda_1),$$

and hence

$$c_{1m_1}^{(u)} = \lim_{\lambda\to\lambda_1}(\lambda-\lambda_1)^{m_1}\,\Phi_u(\lambda) = \frac{(-1)^M\,\lambda_1\;\det(A-\lambda_1 I;\;u;\;-b^{(0)}-\bar{p}/(\lambda_1-1))}{(\lambda_1-\lambda_2)^{m_2}\ldots(\lambda_1-\lambda_k)^{m_k}}$$
$$= \frac{(-1)^{m_1}\,\lambda_1\;\det(A-\lambda_1 I;\;u;\;r(\lambda_1^{-1}))}{(\lambda_2-\lambda_1)^{m_2}\ldots(\lambda_k-\lambda_1)^{m_k}}.$$

The second–order term is obtained by multiplying this expression by

$$\lambda_1^{n-m_1}\binom{n-1}{m_1-1},$$

which yields the assertion after comprehension of the scalars to vectors. The case $|\lambda_1| = |\lambda_2|$ is treated analogously. $\square$

For concrete communication profiles, it could often be observed that the eigenvalues of A turn out as pairwise different. In this case, Theorem 3.1 simplifies considerably. Furthermore, we can make the computation more efficient by using *diagonalization* of the matrix A:

Corollary 2. If all eigenvalues λ_s of A have multiplicity $m_s = 1$, and $\{w_1, \ldots, w_M\}$ is an orthonormal base of eigenvectors (written as column vectors) corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_M$, then for $n \geq 1$,

$$b^{(n)} = -(A-I)^{-1}\,\bar{p} + \sum_{j=1}^{M} w_j w_j^t\,(b^{(0)} - \bar{p}/(1-\lambda_j))\,\lambda_j^n.$$

Proof. We compute the terms of the asymptotic expansion similarly as in the proof of Corollary 1. Note, however, that in the case $m_s = 1$, the term $c_{s1}^{(u)}\,\lambda_s^{n-1}$ with

$$c_{s1}^{(u)} = \lim_{\lambda \to \lambda_s} (\lambda - \lambda_s)\,\Phi_u(\lambda)$$

is already the *exact* (instead of asymptotically approximated) $(s+1)$th term in the expansion. This produces

$$b^{(n)} = -(A-I)^{-1}\bar{p} - \sum_{j=1}^{M} \frac{[A - \lambda_j I;\; r(\lambda_j^{-1})]}{\prod_{1 \leq s \leq M, s \neq j} (\lambda_s - \lambda_j)}\,\lambda_j^n. \tag{27}$$

Let now $W = (w_1, \ldots, w_M)$ be the orthonormal matrix composed of the column vectors w_j. We have

$$W^t A W = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_M)$$

and hence $A = WDW^t$ and $A - \lambda I = W(D - \lambda I)W^t$. Observe that $-r(\lambda_j^{-1}) = b^{(0)} - \bar{p}/(1-\lambda_j)$. For deriving the result from (27), it remains to show that for each M-dimensional vector x,

$$\frac{[A - \lambda_j I;\; x]}{\prod_{s \neq j} (\lambda_s - \lambda_j)} = w_j w_j^t x.$$

It is sufficient to derive this equation for $x = e_i$, the ith unit vector ($i = 1, \ldots, M$). It can easily be verified that

$$[A - \lambda_j I;\; e_i] = W[D - \lambda_j I;\; W^t e_i].$$

With $W = (w_{ij})$,

$$[D - \lambda_j I; W^t e_i]_k = \det(D - \lambda_j I;\; k;\; W^t e_i) = w_{ik} \prod_{s \neq k} (\lambda_s - \lambda_j)$$

$$= w_{ij} \prod_{s \neq j} (\lambda_s - \lambda_j) \text{ for } k = j \text{ and } 0 \text{ else.}$$

Therefore,

$$W[D - \lambda_j I;\; W^t e_i] = w_j \cdot (w_{1j}, \ldots, w_{Mj})\,e_i \prod_{s \neq j} (\lambda_s - \lambda_j) = w_j w_j^t e_i \cdot \prod_{s \neq j} (\lambda_s - \lambda_j).$$

$\square$

From the vector $b^{(n)}$, the cost $K^{(n)}$ can easily be computed as an inner product of vectors: By (5), we get

$$K^{(n)} = 2 \sum_{i<j} p(i,j) \sum_{k=1}^{N} b^{(n)}(< i,k,j >) \tag{28}$$

under the assumptions of this section. Defining the M-dimensional column vector μ by

$$\mu(< i,k,j >) = p(i,j),$$

we obtain

$$K^{(n)} = 2\, \mu^t\, b^{(n)}. \tag{29}$$

In particular, the stationary cost $K^{(\infty)}$ is given by

$$K^{(\infty)} = 2\, \mu^t\, b^{(\infty)} = -2\, \mu^t\, (A - I)^{-1}\, \bar{p} \tag{30}$$

according to Corollary 1, provided that $1 > |\lambda_1| > |\lambda_2|$.

4 Special Communication Profiles

In this section, the developed technique is applied to the analysis of the MTP rule for two different types of communication profiles.

4.1 Zipf's Law

In theoretical investigations of self–organizing *lists*, a frequently used model distribution is *Zipf's law*, named after the linguist G.K. Zipf. Without loss of generality, the elements of the list can be considered as indexed in such a way that the element with index $1, 2, \ldots$ is the most frequently requested, the second–frequently requested, etc. Then Zipf's law assumes that the probability of a request concerning element i is proportional to $1/i$ (see, e.g., [17], [9]).

We may investigate a two–dimensional analogue of this model distribution as the communication profile of the task allocation problem: For $i, j \in \{1, \ldots, N\}$, set

$$p(i,j) = \begin{cases} \text{const} \cdot \frac{1}{i} \cdot \frac{1}{j}, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases} \tag{31}$$

Obviously, the normalization constant in (31) has to be chosen as $(H_N^2 - \sum_{i=1}^{N} i^{-2})^{-1}$, where $H_N = \sum_{i=1}^{N} i^{-1}$ is the Nth harmonic number.

As an illustration, let us start by considering the special case $N = 3$: The communication profile is given by the matrix

$$(p(i,j)) = \frac{1}{2} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

The matrix A defined in section 3 is then the following:

$$A = \frac{1}{12} \begin{pmatrix} 4 & -1 & -3 \\ -1 & 3 & -2 \\ -3 & -2 & 5 \end{pmatrix}.$$

Therein, the lines and columns correspond to the equivalence classes of index triples arranged in the order $< 1, 2, 3 >$, $< 1, 3, 2 >$, $< 2, 1, 3 >$.

Short calculation yields the eigenvalues of A:

$$\lambda_1 = (6 + \sqrt{3})/12 = 0.6443376\ldots, \quad m_1 = 1,$$

$$\lambda_2 = (6 - \sqrt{3})/12 = 0.3556624\ldots, \quad m_2 = 1,$$

$$\lambda_3 = 0, \quad m_3 = 1.$$

Corollary 2 of the last section can immediately be applied. For the stationary cost, e.g., we obtain the value $K^{(\infty)} = 0.2727\ldots$.

In a similar way, we can analyze the cases $N = 4, 5, \ldots$. Numerical calculations with the help of MATLAB produced the results in Table 4.1.

For all investigated choices of N, we obtained only eigenvalues with multiplicities 1, such that Corollary 2 of section 3 was always applicable. This allows (for given N and given initial distribution of $\pi^{(0)}$) a plot of the exact numbers $K^{(n)}$ for arbitrary domains of n, as well as of the time–to–ϵ–quasi-stationarity (as defined at the beginning of section 3) for arbitrary $\epsilon > 0$. For example, Fig. 4.1 shows a plot of $K^{(n)}$ in the case of $N = 10$ tasks for uniform initial distribution on the set of permutations of $1, \ldots, N$. For this distribution, $b^{(0)}(< i, j, k >) = 1/3$ for each $< i, j, k >$, such that (28) implies

$$K^{(0)} = 2 \sum_{i<j} p(i,j) \cdot \frac{N-2}{3} = \frac{N-2}{3}. \tag{32}$$

As it can be seen from Fig. 4.1, the convergence to the stationary cost is rather fast.

N	$K^{(\infty)}$	λ_1
3	0.2727	0.6443
4	0.5645	0.7503
5	0.8615	0.8187
6	1.1610	0.8661
7	1.4622	0.8983
8	1.7648	0.9192
9	2.0684	0.9361
10	2.3730	0.9489

Table 4.1. Stationary costs and largest eigenvalues of A for Zipf–type profiles.

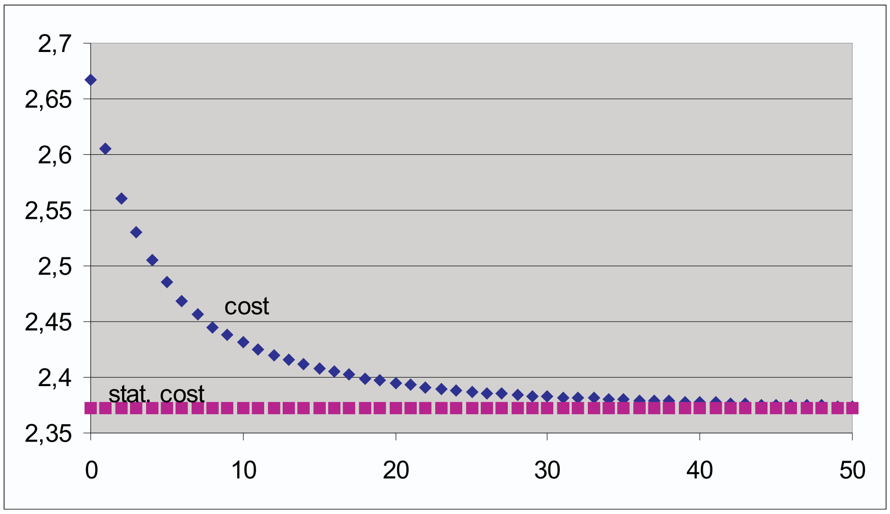

Fig. 4.1. Cost values $K^{(n)}$ (as a function of n) and stationary cost $K^{(\infty)}$ for Zipf's law on $N = 10$ tasks.

Fig. 4.2 shows the growth of the stationary costs $K^{(\infty)}$ for $N = 3, \ldots, 15$, which turns out to be almost linear. Compared to the (exactly linear) growth of $K^{(0)}$ in N, we observe a nearly constant saving rate of about 10 percent.

Of special interest is a comparison of $K^{(\infty)}$ with K^{opt}, the cost value (reduced according to eq. (5)) of an *optimal* allocation. For $N \leq 9$ we were able to verify by complete enumeration that the following allocation $\pi_N^\star$ is optimal under Zipf's law (we conjecture that this also holds for $N > 9$): Arrange, from left to right, first the tasks with odd indices in decreasing order, and then the tasks with even indices in increasing order. (E.g.: $\pi_8^\star = (7, 5, 3, 1, 2, 4, 6, 8)$.) In Fig. 4.2, the cost values K^{opt} corresponding to these optimal allocations are plotted in comparison to the values of $K^{(0)}$ and $K^{(\infty)}$. As it can be seen, the difference between K^{opt} and $K^{(\infty)}$ increases relatively fast for growing N. So we can state the same trend that has been found for the MTF rule for self–organizing lists (cf. [17]): The speed of convergence to stationarity is favorable, but the stationary state itself is not particularly good, compared to the optimum. We have done some simulations for task allocation using a dynamic rule inspired by the *Transpose* rule for self–organizing lists, and found a worse convergence behavior, but better stationary cost values, compared to MTP.

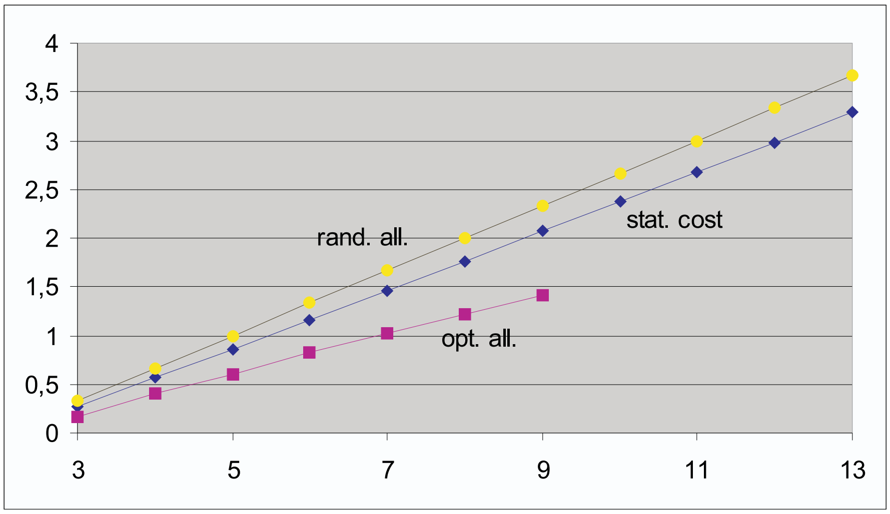

Fig. 4.2. Stationary costs $K^{(\infty)}$, costs $K^{(0)}$ in case of uniform random allocation, and costs K^{opt} of the optimal allocations as a function of N under Zipf's law.

4.2 Classes of Tasks

It would be interesting to study the behavior of MTP for a large number of tasks. While our approach allows a complete asymptotic analysis as $n \to \infty$, asymptotic results as $N \to \infty$ are harder to obtain. For the following type of communication profiles, however, results of this kind can be derived. We assume that there are r different classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ of tasks. Each task belongs to one of these classes. Tasks of the same class are equivalent with respect to their communication profile, i.e., the communication probability between two tasks only depends on the classes to which they belong. We refer to classes by their indices $1, \ldots, r$. Class index variables will be denoted by $\alpha, \beta, \gamma, \ldots$. Let $\phi(i)$ be the index of the class to which task i belongs.

Now let a matrix $(\tilde{p}(\alpha, \beta))_{\alpha=1,\ldots,r;\,\beta=1,\ldots,r}$ with

$$\sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \tilde{p}(\alpha, \beta) = 1$$

be given. For $\alpha \neq \beta$, the value $\tilde{p}(\alpha, \beta)$ represents the overall "inter–class" communication probability between the classes $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$. For $\alpha = \beta$, the value $\tilde{p}(\alpha, \alpha)$ represents the overall "intra–class" communication probability of the class $\mathcal{C}_\alpha$. As in the previous section, we assume symmetry of the communication profile, i.e., $\tilde{p}(\alpha, \beta) = \tilde{p}(\beta, \alpha)$. Let $\mu_\alpha = |\mathcal{C}_\alpha|$ denote the number of tasks in class $\mathcal{C}_\alpha$. We set

$$p(i, j) = \frac{\tilde{p}(\phi(i), \phi(j))}{\mu_{\phi(i)}\, \mu_{\phi(j)}}.$$

(Note that now $p(i, i) > 0$ is possible!) Obviously, the sum of the probabilities $p(i, j)$ over all pairs (i, j) of task indices is unity.

As the initial distribution of $\pi^{(0)}$, we choose the uniform distribution on the set of all permutations $1, \ldots, N$, such that $b^{(0)}(i, j, k) = 1/6$ for all pairwise different indices i, j, k. In particular, $b^{(0)}(i, j, k) = b^{(0)}(i', j', k')$ for $\phi(i) = \phi(i')$, $\phi(j) = \phi(j')$ and $\phi(k) = \phi(k')$, and in view of the equivalent communication profiles of tasks of the same class, this entails for such indices $b^{(n)}(i,j,k) = b^{(n)}(i',j',k')$ for all n. Hence the definition

$$\tilde{b}^{(n)}(\alpha,\beta,\gamma) = b^{(n)}(i,j,k) \quad (i \in \mathcal{C}_\alpha,\ j \in \mathcal{C}_\beta,\ k \in \mathcal{C}_\gamma) \tag{33}$$

for pairwise different α,β,γ is unambigous. If α,β,γ are *not* pairwise different, (33) must be slightly modified: We set

$$\tilde{b}^{(n)}(\alpha,\alpha,\alpha) = b^{(n)}(i,j,k) \quad (i,j,k \in \mathcal{C}_\alpha,\ i,j,k \text{ pairwise different}),$$

$$\tilde{b}^{(n)}(\alpha,\alpha,\beta) = b^{(n)}(i,j,k) \quad (i,j \in \mathcal{C}_\alpha,\ i \neq j,\ k \in \mathcal{C}_\beta),$$

and define $\tilde{b}^{(n)}(\alpha,\beta,\alpha)$ and $\tilde{b}^{(n)}(\beta,\alpha,\alpha)$ in an analogous way.

Let us set $\tilde{p}(\alpha,.) = \sum_{\beta=1}^{r} \tilde{p}(\alpha,\beta)$, such that

$$p(i,.) = \sum_{\rho=1}^{r} \sum_{j \in \mathcal{C}_\rho} p(i,j) = \sum_{\rho=1}^{r} \mu_\rho \, \frac{1}{\mu_{\phi(i)}\,\mu_\rho}\, \tilde{p}(\phi(i),\rho) = \frac{1}{\mu_{\phi(i)}}\, \tilde{p}(\phi(i),.).$$

Furthermore, observe that on the present conditions, $b^{(n)}(i,j) = 1/2$ for all (i,j) with $i \neq j$. Using this, we can now re–formulate (15) for the considered particular situation:

Proposition 4.1. Let $\tilde{b}'(\alpha,\beta,\gamma) = \tilde{b}^{(n+1)}(\alpha,\beta,\gamma)$ and $\tilde{b}(\alpha,\beta,\gamma) = \tilde{b}^{(n)}(\alpha,\beta,\gamma)$. For (not necessarily pairwise different) class indices α,β,γ,

$$\begin{aligned}
\tilde{b}'(\alpha,\beta,\gamma) &= \frac{1}{2}\left[\frac{\tilde{p}(\beta,\alpha)}{\mu_\beta\,\mu_\alpha} + \frac{\tilde{p}(\beta,\gamma)}{\mu_\beta\,\mu_\gamma}\right] \\
&+ \left[1 - \frac{\tilde{p}(\alpha,.)}{\mu_\alpha} - \frac{\tilde{p}(\beta,.)}{\mu_\beta} - \frac{\tilde{p}(\gamma,.)}{\mu_\gamma} + \frac{\tilde{p}(\alpha,\beta)}{\mu_\alpha\,\mu_\beta} + \frac{\tilde{p}(\gamma,\beta)}{\mu_\gamma\,\mu_\beta}\right] \cdot \tilde{b}(\alpha,\beta,\gamma) \\
&- \frac{\tilde{p}(\beta,\gamma)}{\mu_\beta\,\mu_\gamma}\,\tilde{b}(\alpha,\gamma,\beta) - \frac{\tilde{p}(\beta,\alpha)}{\mu_\beta\,\mu_\alpha}\,\tilde{b}(\beta,\alpha,\gamma) \\
&+ \sum_{\rho=1}^{r}\left[\frac{\tilde{p}(\alpha,\rho)}{\mu_\alpha}\,\tilde{b}(\rho,\beta,\gamma) + \frac{\tilde{p}(\beta,\rho)}{\mu_\beta}\,\tilde{b}(\alpha,\rho,\gamma) + \frac{\tilde{p}(\gamma,\rho)}{\mu_\gamma}\,\tilde{b}(\alpha,\beta,\rho)\right] \\
&- \left[\frac{\tilde{p}(\alpha,\beta)}{\mu_\alpha\,\mu_\beta}\,\tilde{b}(\beta,\beta,\gamma) + \frac{\tilde{p}(\alpha,\gamma)}{\mu_\alpha\,\mu_\gamma}\,\tilde{b}(\gamma,\beta,\gamma) + \frac{\tilde{p}(\beta,\alpha)}{\mu_\beta\,\mu_\alpha}\,\tilde{b}(\alpha,\alpha,\gamma)\right. \\
&\left. + \frac{\tilde{p}(\beta,\gamma)}{\mu_\beta\,\mu_\gamma}\,\tilde{b}(\alpha,\gamma,\gamma) + \frac{\tilde{p}(\gamma,\alpha)}{\mu_\gamma\,\mu_\alpha}\,\tilde{b}(\alpha,\beta,\alpha) + \frac{\tilde{p}(\gamma,\beta)}{\mu_\gamma\,\mu_\beta}\,\tilde{b}(\alpha,\beta,\beta)\right]
\end{aligned} \tag{34}$$

Proof. The formula is obtained without difficulty from (15). The terms in the last square brackets arise as follows: With

$$b_0(m,j,k) = \begin{cases} b(m,j,k), & \text{if } m \neq j,k, \\ \tilde{b}(\phi(m),\phi(j),\phi(k)), & \text{else,} \end{cases}$$

the sum

$$\sum_m p(i,m)\, b(m,j,k) = \sum_{m \neq j,k} p(i,m)\, b(m,j,k)$$

in (15) is represented as

$$\sum_m p(i,m)\, b_0(m,j,k) - p(i,j)\, b_0(j,j,k) - p(i,k)\, b_0(k,j,k)$$

$$= \sum_\rho \mu_\rho \frac{\tilde{p}(\alpha,\rho)}{\mu_\alpha\, \mu_\rho}\, \tilde{b}(\rho,\beta,\gamma) - \frac{\tilde{p}(\alpha,\beta)}{\mu_\alpha\, \mu_\beta}\, \tilde{b}(\beta,\beta,\gamma) - \frac{\tilde{p}(\alpha,\gamma)}{\mu_\alpha\, \mu_\gamma}\, \tilde{b}(\gamma,\beta,\gamma),$$

where $\phi(i) = \alpha$, $\phi(j) = \beta$, $\phi(k) = \gamma$ and $\phi(m) = \rho$, and analogously for $\sum_m p(j,m)\, b(i,m,k)$ and $\sum_m p(k,m)\, b(i,j,m)$. This leads to (34). $\square$

For the cost function $K^{(n)}$, we obtain from (6):

$$\begin{aligned} K^{(n)} &= 2 \sum_{i,k} p(i,k) \sum_j b^{(n)}(i,j,k) \\ &= 2 \sum_{\alpha,\beta,\gamma} \frac{\mu_{\alpha\beta\gamma}}{\mu_\alpha \mu_\gamma}\, \tilde{p}(\alpha,\gamma)\, \tilde{b}^{(n)}(\alpha,\beta,\gamma), \end{aligned} \tag{35}$$

where

$$\mu_{\alpha\beta\gamma} = \begin{cases} \mu_\alpha \mu_\beta \mu_\gamma, & \text{if } \alpha,\beta,\gamma \text{ pairwise different,} \\ \mu_\alpha(\mu_\alpha - 1)\mu_\gamma, & \text{if } \alpha = \beta \neq \gamma, \\ \mu_\alpha(\mu_\alpha - 1)\mu_\beta, & \text{if } \alpha = \gamma \neq \beta, \\ \mu_\beta(\mu_\beta - 1)\mu_\alpha, & \text{if } \beta = \gamma \neq \alpha, \\ \mu_\alpha(\mu_\alpha - 1)(\mu_\alpha - 2), & \text{if } \alpha = \beta = \gamma. \end{cases}$$

Remark 4.1. Quite analogously as in section 3, we can define equivalence classes $< \alpha,\beta,\gamma >$ of triples of indices, but now without restriction to pairwise different indices. Let us distinguish three cases:

(i) All indices are equal: One equivalence class $< \alpha,\alpha,\alpha >$ to the given index α is obtained.

(ii) Two of the three indices are equal: Let $\alpha < \beta$ be the different indices. We obtain four equivalence classes, $< \alpha,\alpha,\beta >$, $< \alpha,\beta,\beta >$, $< \alpha,\beta,\alpha >$ and $< \beta,\alpha,\beta >$.

(iii) The indices α,β,γ are pairwise different. Then we obtain three equivalence classes, $< \alpha,\beta,\gamma >$, $< \alpha,\gamma,\beta >$, and $< \beta,\alpha,\gamma >$.

In case (i), we can set $\tilde{b}(< \alpha,\alpha,\alpha >) = 1/6$. In case (ii), by means of the relations

$$\tilde{b}(< \alpha,\alpha,\beta >) + \tilde{b}(< \alpha,\beta,\alpha >) = 1/2, \quad \tilde{b}(< \alpha,\beta,\beta >) + \tilde{b}(< \beta,\alpha,\beta >) = 1/2$$

following from (14) and $b^{(n)}(i,k) = 1/2$, two of the four variables can be expressed by the other two, say $\tilde{b}(< \alpha,\beta,\alpha >)$ and $\tilde{b}(< \beta,\alpha,\beta >)$ by $\tilde{b}(< \alpha,\alpha,\beta >)$ and $\tilde{b}(< \alpha,\beta,\beta >)$, respectively. In case (iii), by means of the equation

$$\tilde{b}(< \alpha,\beta,\gamma >) + \tilde{b}(< \alpha,\gamma,\beta >) + \tilde{b}(< \beta,\alpha,\gamma >) = 1,$$

one of the three variables can be expressed by the other two, say $\tilde{b}(< \beta,\alpha,\gamma >)$ by $\tilde{b}(< \alpha,\beta,\gamma >)$ and $\tilde{b}(< \alpha,\gamma,\beta >)$. In this way, the r^3 variables $\tilde{b}(\alpha,\beta,\gamma)$ are reduced to

$$\tilde{M} = \binom{r}{2} \cdot 2 + \binom{r}{3} \cdot 2$$

variables $\tilde{b}(<\alpha,\beta,\gamma>)$. Furthermore, the homogenous linear system (34) of r^3 equations in the r^3 pairs of variables $(\tilde{b}'(\alpha,\beta,\gamma),\tilde{b}(\alpha,\beta,\gamma))$ is reduced to an inhomogenous linear system of $\tilde{M}$ equations in $\tilde{M}$ pairs of variables. Setting $\tilde{b}'(\alpha,\beta,\gamma)=\tilde{b}(\alpha,\beta,\gamma)$, which yields the stationary probability values, produces a system of $\tilde{M}$ linear equations in $\tilde{M}$ variables.

Now let us admit that both the class sizes μ_ρ and the inter– and intra–class communication probabilities $\tilde{p}(\alpha,\beta)$ depend on N, i.e., $\mu_\rho=\mu_\rho(N)$, and $\tilde{p}(\alpha,\beta)=\tilde{p}_{\alpha\beta}(N)$, whereas r is kept fixed. (When it is clear from the context, the argument N in this notation will be omitted.) In this way, a sequence of communication profiles for growing N is obtained, and we may investigate the efficiency of the MTP rule for a "large" number N of tasks by examining the asymptotic behavior of $K^{(n)}=K^{(n)}(N)$ as $N\to\infty$.

It will turn out that frequently, the stationary cost $K^{(\infty)}(N)$ is not essentially smaller than the cost $K^{(0)}(N)$ of a (uniform) random allocation, provided that N is large. Let us capture this situation by a formal definition: We say that MTP is *asymptotically inefficient* for a special sequence of communication profiles, if

$$K^{(\infty)}(N)\sim K^{(0)}(N)\quad (N\to\infty),$$

i.e., if the relative gain $(K^{(0)}(N)-K^{(\infty)}(N))/K^{(0)}(N)$ tends to zero as $N\to\infty$. Therein, $K^{(0)}(N)$ is based on the uniform distribution on the set of permutations of $1,\dots,N$.

Let us now deal with a situation where the sizes $\mu_\alpha(N)$ of all equivalence classes grow in such a way that each of these sizes is at least of order N^q ($q>0$) in N, i.e., $\mu_\alpha(N)\geq \text{const}\cdot N^q$ for each α. We compute the stationary distribution for each N by setting $\tilde{b}'(\alpha,\beta,\gamma)=\tilde{b}(\alpha,\beta,\gamma)$ in (34). Similarly as before, we set $\tilde{p}_{\alpha.}=\sum_{\beta=1}^{r}\tilde{p}_{\alpha\beta}$. Observe that all probabilities $\tilde{p}_{\xi.}$ and $\tilde{p}_{\xi\eta}$ are bounded by unity, such that, e.g., $\tilde{p}_{\alpha.}/\mu_\alpha=O(N^{-q})$ and $\tilde{p}_{\alpha\beta}/(\mu_\alpha\,\mu_\beta)=O(N^{-2q})$. This leads to

$$\begin{aligned}&-\left[\frac{\tilde{p}_{\alpha.}}{\mu_\alpha}+\frac{\tilde{p}_{\beta.}}{\mu_\beta}+\frac{\tilde{p}_{\gamma.}}{\mu_\gamma}\right]\tilde{b}(\alpha,\beta,\gamma)\\ &+\sum_{\rho=1}^{r}\left[\frac{\tilde{p}_{\alpha\rho}}{\mu_\alpha}\,\tilde{b}(\rho,\beta,\gamma)+\frac{\tilde{p}_{\beta\rho}}{\mu_\beta}\,\tilde{b}(\alpha,\rho,\gamma)+\frac{\tilde{p}_{\gamma,\rho}}{\mu_\gamma}\,\tilde{b}(\alpha,\beta,\rho)\right]+O(N^{-2q})=0.\end{aligned}\tag{36}$$

Note that after the substitution process described in Remark 4.1, the system of these equations gets *inhomogenous*. Multiplying (36) by N^q, we obtain a system of equations where all coefficients are (at most) of order $O(1)$ in N. It turns out that "usually", MTP is asymptotically inefficient in this situation:

Proposition 4.2. Suppose that $\mu_\alpha(N)\geq\text{const}\cdot N^q$ for each α with some $q>0$. Let $A^{(N)}\,x^{(N)}=r^{(N)}$ be the system of $\tilde{M}$ linear equations in $\tilde{M}$ variables defined in Remark 4.1. If

$$A^\star=\lim_{N\to\infty}N^q\,A^{(N)}\quad\text{and}\quad r^\star=\lim_{N\to\infty}N^q\,r^{(N)}$$

exist, and $\det A^\star\neq 0$, then MTP is asymptotically inefficient.

Proof. Multiplying $A^{(N)}\,x^{(N)}=r^{(N)}$ by N^q, we get

$$N^q\,A^{(N)}\,x^{(N)}=N^q\,r^{(N)}.\tag{37}$$

By standard continuity arguments, it can easily be verified that for a sequence of systems of linear equations, $B^{(N)}\,x^{(N)}=b^{(N)}$, the following assertion holds: If $B^{(N)}\to B$, $b^{(N)}\to b$, and

$\det B \neq 0$, then there is an N_0 such that the systems $B^{(N)}\, x^{(N)} = b^{(N)}$ possess for $N \geq N_0$ unique solutions $x^{(N)}$ which converge to the unique solution x of $Bx = b$. Applying this to $B^{(N)} = N^q\, A^{(N)}$ and $b^{(N)} = N^q\, r^{(N)}$, we conclude that there is an N_0 such that (37) has unique solutions $x^{(N)}$ for all $N \geq N_0$, and that the solutions $x^{(N)}$ converge to the unique solution of $A^\star x = b^\star$ (which must exist in view of $\det A^\star \neq 0$). What remains to do is to determine this solution x. We assert that x is given by setting $\tilde{b}(\alpha,\beta,\gamma) = 1/6$ for all α,β,γ. The systems of equations $A^{(N)}\, x^{(N)} = r^{(N)}$ are of the form of the equation (36), considering that some of the variables therein have still to be substituted by others according to Remark 4.1. Multiplying (36) by N^q and setting $\tilde{b}(\alpha,\beta,\gamma) = 1/6$ for all α,β and γ produces $O(N^{-q}) = 0$, so taking the limits for $N \to \infty$ on both sides yields $0 = 0$ indeed. Therefore, the current solutions converge to the solution defined by $\tilde{b}(\alpha,\beta,\gamma) = 1/6$, which is that corresponding to (uniform) random allocation. □

Remark 4.2. For "smooth" functions $\mu_\rho(N)$ and $\tilde{p}_{\alpha\beta}(N)$, the limits $A^\star$ and $r^\star$ in Proposition 4.1 exist (cf. (36) and the comments to that equation). The condition $\det A^\star \neq 0$, however, needs not to be satisfied in *all* cases, as it can be seen in the following example:

Example. With $[x]$ denoting the largest integer smaller or equal x, let $r = 2$ classes of tasks be given, and assume $\mu_1(N) = [cN]$ $(0 < c < 1)$, $\mu_2(N) = N - \mu_1(N)$, and $\tilde{p}_{\alpha\beta}$ independent of N $(\alpha,\beta \in \{0,1\})$. For simplicity, we assume that c is a *rational* number n_1/n_2 $(n_1, n_2$ positive integers), and concentrate on the values $N = n_3 \cdot n_2$, $n_3 = 1, 2, \ldots$. The condition $\mu_\rho(N) \geq \text{const} \cdot N^q$ is then satisfied with $q = 1$. Short calculation yields

$$\det A^\star = 2\, \tilde{p}_{12}\, (1/c + 1/(1-c))^2,$$

which is $\neq 0$ except if $\tilde{p}_{12} = 0$. So in all cases where there is at least *some* intercommunication between the two classes of tasks, MTP is asymptotically inefficient.

The case $\det A^\star \neq 0$ is more likely to occur if for several (α,β), the values $\tilde{p}_{\alpha\beta}(N)$ tend to zero, i.e., if intercommunication between certain classes gets negligible for large N. Such situations occur in applications; we should therefore not rush to the conclusion that MTP is inefficient for all large systems of practical interest with class–structured profile.

To complement our investigation, we consider a particular case where the condition $\mu_\alpha(N) \geq \text{const} \cdot N^q$ of Proposition 4.2 is not satisfied: As in the example above, let $r = 2$ and $\tilde{p}_{\alpha\beta}$ independent of N, but assume now that $\mu_1(N) = \mu_1$ is a fixed constant, and $\mu_2(N) = N - \mu_1$. By a (somewhat tiresome) computation along the lines of section 3, we are able not only to obtain the stationary costs $K^{(\infty)}(N)$ for large N, but even the transient behavior $K^{(n)}(N)$. Without proof we state the following result:

Proposition 4.3. For $r = 2$ classes with $\mu_1 = \text{const}$ and $\mu_2 = N - \mu_1$,

$$K^{(n)}(N) \sim (c_1 + c_2 \lambda_1^n) \cdot N \quad (N \to \infty),$$

where

$$c_1 = \frac{1}{3} - \frac{\tilde{p}_{11}\, (\tilde{p}_{11} + \tilde{p}_{12})}{3\, (\tilde{p}_{11} + \mu_1 \tilde{p}_{12})},$$

$$c_2 = \frac{(\mu_1 - 1)\, \tilde{p}_{11}^2}{3\, \mu_1 (\tilde{p}_{11} + \mu_1 \tilde{p}_{12})},$$

and

$$\lambda_1 = 1 - 2\,\tilde{p}_{11}/\mu_1^2 - 2\,\tilde{p}_{12}/\mu_1.$$

Only in the marginal case where $\tilde{p}_{11} = 0$, MTP is asymptotically inefficient in this situation. The best case for MTP is again given by $\tilde{p}_{12} = 0$.

5 Conclusion

We have defined a rule for self–organizing task allocation on a linear array, the Move–to–Partner rule, which is an analogue to the well–known Move–to–Front rule for self–organizing linear lists. When this rule is applied, the random intertask communication induces a Markov chain, the states of which are allocations, i.e., permutations of the N tasks. We have reduced the analysis of the expected cost connected with intercommunication step n from the investigation of the $N!$ possible permutations to that of $O(N^3)$ probabilities $b^{(n)}(i,j,k)$ connected with the current step. A basic recursion formula for the numbers $b^{(n)}(i,j,k)$ has been derived, and it has been shown that a full asymptotic expansion of these numbers can be given, enabling a numerical analysis of the expected stationary cost as well as that of the convergence behavior of the expected cost values for any given communication profile.

Two particular types of profiles have been investigated in more detail: A two–dimensional generalization of Zipf's law, and a profile based on the assumption that the set of tasks can be decomposed into classes of tasks which are homogenous in their expected communication behavior. For the last type of profiles, asymptotic results for the case where the task number N tends to infinity have been derived. It has been shown that in some of these situations, the Move–to–Partner rule is asymptotically inefficient, which means that for growing N, its relative gain over uniform random allocation tends to zero. Also for Zipf's law, it has turned out that Move–to-Partner has the advantage of fast convergence, but the disadvantage of rather poor efficiency in the stationary state: savings of only about 10%, compared to random allocation, can be observed. This result resembles the general trend of theoretical and experimental findings for the Move–to–Front rule. Nevertheless, since in practical applications, a re–organization of the task allocation will usually be performed only occasionally, fast convergence may be more important in this area than excellent stationary behavior. An attractive alternative could consist in a two–phase procedure with MTP in the first phase and a slower, but asymptotically more efficient reorganization rule in the final phase.

Our investigation should be understood as a first step to a stochastic analysis of dynamic task allocation rules for interconnection topologies. Let us mention some topics for future research: From a theoretical point of view, it would be interesting to know whether our stochastic model fits into the framework of random walks on chambers of hyperplane arrangements, as studied by Brown and Diaconis [1]. (Since the MTF rule is covered by this framework and can be considered as a special case of the MTP rule considered here, there is some hope that also MTP could be interpreted in terms of hyperplane arrangements.) Brown and Diaconis show very strong results for the eigenstructure of transition matrices fitting into the hyperplane arrangement framework, and it would be worthwile to investigate whether results of this type can also be derived for the MTP process.

From a practical point of view, obvious research topics are: (a) other types of topologies, (b) other communication profiles, (c) dependent subsequent communication requests (e.g., described by a Markov chain), and (d) alternative reorganization rules (e.g., an analogue to the well–known Transpose rule). It is likely that not all of these topics are accessible to an analytical treatment; for some of them, simulation experiments will be required.

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