1 Motivating examples:

Ex. 1:

An oil company has to decide on a new location of a petrol station. There are two alternatives: the suburbs (a_1) or the city center (a_2). The attainable profits and turnovers are known and given in the following table:

	profit	turnover
a_1 (suburbs)	150.000	1 800.000
a_2 (center)	125.000	2 000.000

Assume, that the only goal of the company is to maximize the profit $\Rightarrow$ choose a_1

If additionally the turnover should be maximized $\Rightarrow$ "conflict of goals" $\Rightarrow$ multi objective optimization

Relationships of goals:

- congruent: higher realization of goal 1 implies higher realization of goal 2
- conflict: higher realization of goal 1 implies less realization of goal 2
- neutral: degree of realization of goal 1 has no influence on goal 2

Ex.2:

Consider the oil company from ex. 1 and assume that the firm wants to maximize profits. The government has plans to build a by-pass and the expected profits depend on whether this by-pass will be built or not; i.e. the possible "states of the world" are "$z_1 \cdots$ no by-pass" and "$z_2 \cdots$ by-pass".

	z_1 (no by-pass)	z_2 (by-pass)
a_1 (suburbs)	$e_{11} = 150.000$	$e_{12} = 80.000$
a_2 (center)	$e_{21} = 125.000$	$e_{12} = 125.000$

Decision under ambiguity, if the decision maker has no information on the probabilities of the different states of the world. **Decision under risk,** if these probabilities are known.

2 Basic model

- The decision maker has to choose one action out of a set of alternatives, the **"Alternative space"** $A = \{a_1, a_2, \cdots a_m\}$.

  Assumptions:

  – The decision maker **has** to choose one action out of the set of alternatives under consideration (**completeness**)

– at the same time **only one** alternative can be realized, i.e. the alternatives are mutually exclusive (**exclusivity**).

- The consequences of the chosen alternative depend on the "state of nature" $z_1, z_2, \cdots z_n$. The set of all relevant states of nature are summarized in the state space $Z = \{z_1, z_2, \cdots z_n\}$.

- The **Decision matrix** (a $[m \times n]$-matrix) describes the outcome for each of the alternatives and each of the states of nature. e_{ij} is the outcome if alternative a_i is chosen and state z_j occurs.

	z_1	z_2	$\cdots$	z_n
	p_1	p_2	$\cdots$	p_n
a_1	e_{11}	e_{12}	$\cdots$	e_{1n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	e_{m1}	e_{m2}	$\cdots$	e_{mn}

Principles of dominance:

1. **absolute dominance**

   An alternative a_i **dominates** an alternative a_j **absolutely**, iff the worst outcome of a_i is not worse than the best outcome of a_j.

$$a_i \text{ dominates } a_j \text{ absolutely } \Leftrightarrow \min_k e_{ik} \geq \max_l e_{jl}$$

(under the assumption that higher values of e_{ij} are better.)

Ex.

	z_1	z_2	z_3	min	max
a_1	70	80	10	10	80
a_2	50	90	20	20	90
a_3	20	10	20	10	20

a_2 dominates a_3 absolutely.

2. **statewise dominance**

An alternative a_i **dominates** an alternative a_j statewise, iff a_i is not worse than a_j for each state $z_j, j = 1, \cdots m$ and better for at least one state, i.e.

$$a_i \text{ dominates } a_j \text{ statewise } \Leftrightarrow \begin{cases} e_{ik} \geq e_{jk} & \text{for all } k \\ e_{il} > e_{jl} & \text{for at least one } l \end{cases}$$

Ex.

	z_1	z_2	z_3	min	max
a_1	70	80	10	10	80
a_2	50	90	20	20	90
a_3	20	10	20	10	20
a_4	60	95	30	30	95

a_4 dominates a_2 statewise.

3. **stochastic dominance:**

(only for decisions under risk!)

For each alternative a_i the outcome is a random variable. Let us denote with $f_i(.)$ the corresponding density function and $F_i(.)$ the cumulative distributon function of these random variables.

- a_i dominates a_j according to 1^{st} order stochastic dominance, iff

  $F_i(x) \leq F_j(x), \forall x, \quad$ and $F_i(x^*) < F_j(x^*)$ for at least one x^*.

  $$F_i(x) \leq F_j(x) \Leftrightarrow 1 - F_i(x) \geq 1 - F_j(x)$$

  i.e. the probability to achieve an outcome better than x is higher when choosing alternative a_i than when choosing a_j.

- a_i dominates a_j according to 2^{st} order stochastic dominance, iff

$$\int_{-\infty}^{x} F_i(\xi)d\xi \leq \int_{-\infty}^{x} F_j(\xi)d\xi, \forall x$$

where strict inequality should hold for at least one value of x.

Ex. 1:

	z_1	z_2	z_3	z_4
	$p_1 = 0.3$	$p_2 = 0.2$	$p_3 = 0.4$	$p_4 = 0.1$
a_1	20	40	10	50
a_2	60	30	50	20

$$F_1(x) = \begin{cases} 0 & x < 10 \\ 0.4 & 10 \leq x < 20 \\ 0.7 & 20 \leq x < 40 \\ 0.9 & 40 \leq x < 50 \\ 1 & 50 \leq x \end{cases} \qquad F_2(x) = \begin{cases} 0 & x < 20 \\ 0.1 & 20 \leq x < 30 \\ 0.3 & 30 \leq x < 50 \\ 0.7 & 50 \leq x < 60 \\ 1 & 60 \leq x \end{cases}$$

$\Rightarrow$ a_2 dominates a_1 according to 1^{st} order stochastic dominance.

Ex. 2:

	z_1	z_2	z_3
	$p_1 = 0.4$	$p_2 = 0.2$	$p_3 = 0.4$
a_1	120	100	30
a_2	60	0	110

$$F_1(x) = \begin{cases} 0 & x < 30 \\ 0.4 & 30 \leq x < 100 \\ 0.6 & 100 \leq x < 120 \\ 1 & 120 \leq x \end{cases} \qquad F_2(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \leq x < 60 \\ 0.6 & 60 \leq x < 110 \\ 1 & 110 \leq x \end{cases}$$

$$F_2(x) - F_1(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \leq x < 30 \\ -0.2 & 30 \leq x < 60 \\ 0.2 & 60 \leq x < 100 \\ 0 & 100 \leq x < 110 \\ 0.4 & 110 \leq x < 120 \\ 0. & 120 \leq x \end{cases}$$

a_1 dominates a_2 according to 2^{st} order dominance, because $\int_{-\infty}^{x}(F_2(\xi) - F_1(\xi))d\xi \geq 0$

3 Decisions under Ambiguity

3.1 Maximin-criterion

(Mini-max criterion, Wald rule)

Choose the alternative which maximizes the worst outcome

Ex.

	z_1	z_2	z_3	z_4	min
a_1	60	30	50	60	30
a_2	50	90	20	20	20
a_3	60	95	30	-30	-30

$\Rightarrow$ choose a_1.

Properties:

extremely risk avers, values the alternatives only by their worst outcome.

	z_1	z_2	z_3	z_4	min
a_1	1000	1000	1	1000	1
a_2	1.1	1.1	1.1	1.1	1.1

$\Rightarrow$ chooses a_2.

Maximax-criterion

Choose the alternative which maximizes the best outcome.

Ex.

	z_1	z_2	z_3	z_4	min	max
a_1	60	30	50	60	30	60
a_2	50	90	20	20	20	90
a_3	60	95	30	-30	-30	95

$\Rightarrow$ Choose a_3.

Properties:

extremly risk seeking, values each alternative by its best outcome.

Hurwicz-Rule

Combines Maximin-criterion and Maximax-criterion.

$\lambda \in [0, 1] \cdots$ parameter of optimism.

Choose the alternative which maximizes the preference function $\Phi(a_i) = \lambda \max_j(e_{ij}) + (1 - \lambda) \min_j(e_{ij})$.

	z_1	z_2	z_3	z_4	min	max	$\Phi(\lambda = 0.4)$
a_1	60	30	50	60	30	60	42
a_2	50	90	20	20	20	90	48
a_3	60	95	30	-30	-30	95	20

$\Rightarrow$ choose a_2.

Savage-Niehans-Rule

Minimax Regret criterion

- Find for each column of the decision matrix, i.e. for each state of nature the best outcome.

- Compute the regret matrix $R = (r_{ij})$. Fo each state of nature substract the outcomes of the alternatives from the best outcome possible for this state. i.e. $r_{ij} = max_k(e_{kj}) - e_{ij}$
- find for each row of the regret matrix (i.e. for each alternative) the largest regret value.
- choose the alternative which minimizes the maximum regret.

	z_1	z_2	z_3	z_4
a_1	**60**	30	**50**	**60**
a_2	50	90	20	20
a_3	**60**	**95**	30	-30

Regret-Matrix:

	z_1	z_2	z_3	z_4	max
a_1	0	65	0	0	65
a_2	10	5	30	40	40
a_3	0	0	20	90	90

$\Rightarrow$ choose a_2.

Laplace-criterion:

Assume that each of the n states of nature have the same probability $p = 1/n$. Compute for each alternative the expected outcome and choose the alternative with the highest expected outcome.

	z_1	z_2	z_3	z_4	Φ
a_1	60	30	50	60	50
a_2	50	90	20	20	45
a_3	60	95	30	-30	38.75

$\Rightarrow$ choose a_1.

In a $[m \times n]$ **Decision matrix**

	z_1	z_2	$\cdots$	z_n
	p_1	p_2	$\cdots$	p_n
a_1	e_{11}	e_{12}	$\cdots$	e_{1n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	e_{m1}	e_{m2}	$\cdots$	e_{mn}

e_{ij} is the outcome if alternative a_i is chosen and state z_j occurs.

Which properties should hold for a "reasonable" decision rule?

A decision rule assigns a relation " $\geq$ " between the alternatives $\{a_i, i = 1, \cdots m\}$.

3.2 10 Axioms of Milnor:

(John Milnor, 1954, "Games Against Nature", in R.M. Thrall, C.H. Combes, R.L. Davis (eds.), Decision Processes, J. Wiley, New York.)

For this relation the following 10 axioms should hold:

1 **Ordering:** The relation $" \geq "$ is a complete ordering of the alternatives $\{a_i, i = 1, \cdots m\}$, i.e. a transitive relation such that any pair of alternatives can be compared.

2 **Symmetry:** The ordering does not depend on the numbering of alternatives or numbering of states of the world.

3 **strong statewise domination** If $e_{ij} > e_{kj} \quad \forall j = 1, \ldots n$, i.e. if alterantive a_i is strictly better than a_k for all states z_j, then a_i should be strictly preferred to a_k.

4 **Continuity**

If the matrices (e^k_{ij}) converge to (e_{ij}) and if alternative a_i is strictly prefered to a_l with respect to the outcomes (e^k_{ij}), then a_i is prefered to a_j with respect to outcome (e_{ij}).

5 **Linearity:**

The ordering relation between the alternatives a_i does not change if all outcomes $\{e_{ij}\}$ are transformed according to the positive linear transformation

$$e'_{ij} = \alpha + \beta e_{ij}, \quad \beta > 0$$

6 **Row adjunction:**

The ordering between two alternatives depends only on the direct comparison of these two alternatives and does not change when further alternatives are added.

7 **Column linearity:**

The ordering is not changed if a constant is added to a column, i.e. if for state j the outcomes are change according to $e'_{ij} = e_{ij} + \alpha_j, i = 1, \cdots m$.

8 **Column duplication:**

The ordering is not changed if a new column, identical with an existing column, is added to the matrix.

9 **Convexity:**

If the outcome of an alternative a_i is the average of the outcome of two equivalent alternatives a_l and a_k, i.e. $e_{it} = (e_{lt} + e_{kt})/2, t =, \cdots n$, then a_i is at least as good as a_l.

10 **Special row adjunction:**

The ordering between the old alternatives is not changed by the adjunction of a new alternative, provided that no outcome of this new alternative is greater than the corresponding outcomes of all old alternatives.

criterion	not compatible with
Savage-Niehans	row adjunction
Hurwicz	column linearity, convexity
Laplace	column duplication
maximin	column linearity

4 Decisions under risk

4.1 Expected value

Decision matrix

	z_1	z_2	$\cdots$	z_n	$\phi(.)$
	p_1	p_2	$\cdots$	p_n	
a_1	e_{11}	e_{12}	$\cdots$	e_{1n}	$\Sigma_{j=1}^{n} p_j e_{1j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	e_{m1}	e_{m2}	$\cdots$	e_{mn}	$\Sigma_{j=1}^{n} p_j e_{mj}$

Choose the alternative with the highest expected outcome. (suggested by Fermat, Pascal, etc.)

- Many repetitions: average outcome converges to expected value.
- No risk preferences

Ex.

	z_1	z_2	$\phi(.)$
p_i	0.5	0.5	
a_1	0	0	0
a_2	10	-10	0
a_3	100000	-100000	0

A (risk neutral) decision maker is indifferent between these 3 alternatives.

Ex.: A decision maker is confronted with the following "catastrophe" $\tilde{x}$.

x	p(x)
0	0.9
-1000	0.1

$$\mathrm{E}(\tilde{x}) = -100, \sigma^2(\tilde{x}) = 90000$$

Assume that 10 000 persons are in an identical situation, where the catastrophes are mutually independent and the outcomes are shared between these people.

$$\tilde{y} = \frac{1}{10000} \sum_{i=1}^{10000} \tilde{x}_i$$

$$Prob\{\tilde{y} = \frac{-1000k}{10000}\} = \begin{pmatrix} 10000 \\ k \end{pmatrix} 0.1^k 0.9^{10000-k}, k = 0, \cdots, 10000$$

$$\mathrm{E}(\tilde{y}) = -100, \sigma^2 = \frac{90000}{10000} = 9$$

4.2 expected value/variance criterion

$\mu - \sigma -$ criterion

$$\Phi(.) = h[\mathrm{E}(\tilde{x}), \sigma^2(\tilde{x})]$$

Properties of h :

- Monotonicity

$$\frac{\partial h(\mathrm{E}, \sigma^2)}{\partial \mathrm{E}} > 0$$

- Risk attitudes

1. $\partial h(\mathrm{E}, \sigma^2)/\partial \sigma^2 = 0$: the decision maker is **risk-neutral.**
2. $\partial h(\mathrm{E}, \sigma^2)/\partial \sigma^2 < 0$: the decision maker is **risk-avers.**
3. $\partial h(\mathrm{E}, \sigma^2)/\partial \sigma^2 > 0$: the decision maker is **risk-seeking.**

$\Phi(\tilde{x}) = \mathrm{E}(\tilde{x}) - k\sigma^2(\tilde{x})$

4.3 ”safety-first” criterion

Consider the following two lotteries:

x	$p(x)$
-8	$\frac{1}{11}$
3	$\frac{10}{11}$

y	$p(y)$
1	$\frac{10}{11}$
12	$\frac{1}{11}$

Computing expected value and variance leads to

$$E(x) = -\frac{8}{11} + \frac{30}{11} = 2, \qquad E(x^2) = \frac{64}{11} + \frac{90}{11} = \frac{154}{11}$$

$$E(y) = \frac{12}{11} + \frac{10}{11} = 2, \qquad E(y^2) = \frac{144}{11} + \frac{10}{11} = \frac{154}{11}$$

Using the $\mu/\sigma-$ criterion a decision maker would be indifferent.

Risk $\cong$ variability of $\tilde{x}$ below the threshold t.

Semi-Variance:

$$\sigma^{2-}(t) = \int_{-\infty}^{t} (x-t)^2 f(x) dx$$

Variance vs. Semi-Variance:

Variance: mean squared distance to the expected value

Semi-Variance: mean squared distance to the threshold, given that the values are below the threshold.

The decision is made according to the preference function

$$\Phi[\tilde{x}] = \mathrm{E}(\tilde{x}) - k\sigma^{2-}(t)$$

Ex.

Consider the following 2 alternatives:

A		B	
x	p	x	p
0	0.2	-20	0.01
+5	0.3	+7	0.49
+11	0.5	+8	0.50

μ/σ **criterion:**

- B has a higher gain than A:

$$\mathrm{E}(\tilde{x}|A) = 7 < \mathrm{E}(\tilde{x}|B) = 7.23$$

- A has higher risk than B

$$\sigma^2(\tilde{x}|A) = 19 > \sigma^2(\tilde{x}|B) = 7.737$$

- $\Rightarrow$ a risk avers ($k = 1 > 0$) decision maker chooses B, as $\Phi(\tilde{x}|A) < \Phi(\tilde{x}|B)$.

"safety first"-criterion

Assume that the threshhold is $t = 0$.

- A has no risk, as $\tilde{x}$ can never be below the threshold.

- Semi-Variance for B:

$$\sigma^{2-}(t = 0) = .01(-20 - 0)^2 = 4$$

- using the preference function $\Phi(\tilde{x}) = \mathrm{E}(\tilde{x}) - \sigma^{2-}(t = 0)$ one obtains

$$\Phi(\tilde{x}|A) = 7 - 1(0) = 7 \text{ and } \Phi(\tilde{x}|B) = 7.23 - 1(4) = 3.23$$

4.4 Bernoulli-Principle (expected utility principle)

Daniell Bernoulli (1738), John von Neumann & Oskar Morgenstern (1944).

Petersburg paradoxon

How much money would you bet on the following game?

A fair coin is tossed until "tail" appears for the first time. If "tail" appears at the n-th time you gain 2^n euros.

	$i = 1$	$i = 2$	$i = 3$	$\cdots$	n
p_i	2^{-1}	2^{-2}	2^{-3}	$\cdots$	2^{-n}
gain(i)	2	2^2	2^3	$\cdots$	2^n
p_i*gain(i)	1	1	1	$\cdots$	1

The expected value of this game is $+\infty$.

Decisions are based on the expected value of the utility of outcomes rather than on the expected value of outcomes themselves.

- assign to the outcomes e_{ij} of each alternative a utility $u(e_{ij})$ by means of a utility function u. $\Rightarrow$ Utility matrix

$$\begin{array}{c|cccc} & z_1 & z_2 & \cdots & z_n \\ & p_1 & p_2 & \cdots & p_n \\ \hline a_1 & u(e_{11}) & u(e_{12}) & \cdots & u(e_{1n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_m & u(e_{m1}) & u(e_{m2}) & \cdots & u(e_{mn}) \end{array}$$

- The expected value of the utility is then used as preference function $\Phi(a_i)$.

$$\Phi(a_i) = E(u(a_i)) = \sum_{k=1}^{n} p_k u(e_{ik})$$

Ex.

	z_1 $p_1 = 0.4$	z_2 $p_2 = 0.1$	z_3 $p_3 = 0.2$	z_4 $p_4 = 0.3$
a_1	80	70	100	90
a_2	60	90	150	80

Utility function $u(x) = \log(x)$

	z_1 $p_1 = 0.4$	z_2 $p_2 = 0.1$	z_3 $p_3 = 0.2$	z_4 $p_4 = 0.3$	$\Phi(a_i)$
a_1	4.38	4.25	4.61	4.50	4.449
a_2	4.09	4.50	5.01	4.38	4.402

Remark: unbounded utility functions do not solve the Petersberg Paradoxon.

	$i = 1$	$i = 2$	$i = 3$	$\cdots$	n	$\cdots$
p_i	2^{-1}	2^{-2}	2^{-3}	$\ldots$	2^{-n}	$\cdots$
gain (i)	x_1	x_2	x_3	$\cdots$	x_n	$\cdots$
utility(i)	$u(x_1) = 2$	$u(x_2) = 2^2$	$u(x_3) = 2^3$	$\cdots$	$u(x_n) = 2^n$	$\cdots$
p_i*utility(i)	1	1	1	$\cdots$	1	$\cdots$

the expected utility is " $+\infty$"

4.5 Axioms of Decision Making under Risk:

- **Ordinality:**
  - **Completeness**, i.e. for all outcomes (alternatives) a and b it has to hold that: $a \succ b$ or $a \sim b$ or $a \prec b$.
  - **Transitivity,** i.e. $a \succ b$ and $b \succ c$ implies $a \succ c$.
- **Dominance principle** (Monotony)

  If there are two alternatives $a = (\bar{e}, p_1; \underline{e}, 1 - p_1)$ and $b = (\bar{e}, p_2; \underline{e}, 1 - p_2)$ where $(\bar{e} > \underline{e})$ then: $p_1 > p_2 \Rightarrow a \succ b$.
- **Continuity:**

  For all outcomes a, b, c where $a \succ b \succ c$ there is a probability p, such that $b \sim (a, p; c, 1 - p)$

- **Substitution**

  for any outcomes (alternatives) a, b, where $a \succ b$, and an arbitrary third outcome c and an arbitrary probability p, the condition $(a, p; c, 1-p) \succ (b, p; c, 1-p)$ holds.

  Ex:

  Consider the following lotteries: $a = (2000, 1), b = (10000, 0.25; 0, 0.75), c = (10000, 1/3; 0, 2/3)$.

  According to the axiom of substitution: $a \sim b \Rightarrow (a, 0.25; c, 0.75) \sim (b, 0.25; c, 0.75)$

4.6 Elicitation of Utility functions:

Theorem:

Two utility functions, where one function is a positive linear transformation of the other one, lead to the same preference order.

Proof: 2 utility functions: $u_1(x), u_2(x)$

positive linear transformation: $u_2(x) = \alpha + \beta u_1(x), \beta > 0$,

Preference functions: $\Phi_1(a_i) = E(u_1(a_i)), \quad \Phi_2(a_i) = E(u_2(a_i))$,

$\Phi_2(a_i) = E(u_2(a_i)) = E(\alpha + \beta u_1(a_i)) = \alpha + \beta E(u_1(a_i)) = \alpha + \beta \Phi_1(a_i)$

$\Phi_2(a_i) > \Phi_2(a_j) \Leftrightarrow \Phi_1(a_i) > \Phi_1(a_j)$

Elicitation of utility functions:

Choose the best possible outcome $\bar{e}$ and the worst outcome $\underline{e}$ from the set of all outcomes.

Define: $u(\bar{e}) = 1, \quad u(\underline{e}) = 0.$

Probability Equivalent Method

To estimate the utilities $u(e_{ij})$ of all other outcomes e_{ij} the decision maker has to choose between

- the certain outcome e_{ij}

and • the lottery $(\bar{e}, p; \underline{e}, 1-p)$ $\quad\Rightarrow u(e_{ij}) = p^*.$

Certainty Equivalent Method

The lottery $(\bar{e}, p; \underline{e}, 1-p)$ is given and the decision maker has to decide, which certain outcome e^* is considered to be equivalent to this lottery.

$\Rightarrow u(e^*) = p.$

4.7 Value of Information:

$$
\begin{aligned}
z_k, k = 1, \cdots n \quad & \cdots \quad \text{States of the world} \\
p(z_k) \quad & \cdots \quad \text{probability of state } k \\
b_r, r = 1, \cdots m, \quad & \cdots \quad \text{additional information} \\
q(b_r) \quad & \cdots \quad \text{probability of information } b_r \\
p(z_k|b_r) \quad & \cdots \quad \text{conditional probability, that state } k \text{ occurs,} \\
& \text{when information } b_r \text{ is observed}
\end{aligned}
$$

$$p(z_k) = \sum_{r=1}^{m} p(z_k|b_r)q(b_r)$$

the maximized expected utility is

without information:

$$\mathrm{E}U = \max_i \sum_{k=1}^{n} p(z_k)u_{ik} =$$

$$= \max_i \sum_{k=1}^{n} \sum_{r=1}^{m} p(z_k|b_r)q(b_r)u_{ik} =$$

$$= \max_i \sum_{r=1}^{m} \sum_{k=1}^{n} p(z_k|b_r)q(b_r)u_{ik}$$

$$\mathrm{E}U = \max_i \sum_{r=1}^{m} q(b_r) \sum_{k=1}^{n} p(z_k|b_r)u_{ik}$$

with complete information

$$\mathrm{E}U = \sum_{r=1}^{m} q(b_r) \sum_{k=1}^{n} p(z_k|b_r) \max_i u_{ik} = \sum_{k=1}^{n} p(z_k) \max_i u_{ik}$$

with partial information

$$\mathrm{E}U = \sum_{r=1}^{m} q(b_r) \max_i \sum_{k=1}^{n} p(z_k|b_r)u_{ik}$$

Movie co-branding example

A company has to decide, whether to develop and launch a new product co-branded to a movie, without knowing whether the movie will be a success or will flop.

Profits & Costs:

Costs of developing the product	1 000 000 $
Profits, in case movie is a hit	2 500 000 $
Profits, in case movie is a flop	100 000 $
costs of focus group	4 000 $

From other comparable movies, one knows, that

$Prob$(movie is a hit) $= 0.8$ and $Prob$(movie is a flop) $= 0.2$.

Company may ask focus group, whether the movie will be a success. From the past, one knows:

		A_j Movie was	
	$P(B_i\|A_j)$	a success	a flop
B_i Focus	likes movie	0.95625	0.675
group	dislikes movie	0.04375	0.325

The following events may occur:

$A_1 \cdots$ movie is a success

$A_2 \cdots$ movie is a flop

$B_1 \cdots$ focus group likes movie

$B_2 \cdots$ focus group dislikes movie

According to *Total Probability Theorem* one can compute

$$P(\text{ Focus group likes movie}) \;=\; 0.95625 \times 0.8 + 0.675 \times 0.2 \;=\; 0.9$$

$$P(\text{ Focus group dislikes movie}) \;=\; 0.04375 \times 0.8 + 0.325 \times 0.2 \;=\; 0.1$$

and by *Bayes Theorem:*

$$P(\text{ success} \mid \text{group likes movie}) = \frac{0.95625 \times 0.8}{0.9} = 0.85$$

$$P(\text{ flop} \mid \text{group likes movie}) = 0.15$$

$$P(\text{ success} \mid \text{group dislikes movie}) = \frac{0.04375 \times 0.8}{0.1} = 0.35$$

$$P(\text{ flop} \mid \text{group dislikes movie}) = 0.65$$

4.8 Initial Wealth, Certainty Equivalent, Risk Premum

Decision does not only depend on risk preferences, but also on initial wealth.

The initial wealth can be taken into account:

$w_0 \cdots$ initial wealth, (not random)

$\tilde{w}_f \cdots$ final wealth, random variable.

$\tilde{w}_f = w_0+\tilde{x}$, or $\tilde{w}_f = w_0(1+\tilde{x})$, or $\tilde{w}_f = (w_0-a)(1+i)+a(1+\tilde{x})$

Decisions are based on the expected utility of final wealth $\mathrm{E}(U(\tilde{w}_f))$. The final wealth depends on initial wealth, the states of nature(i.e. the outcome of the lottery $\tilde{x}$), and the chosen alternative a.

Def: The **certainty equivalent** w^* of a lottery $\tilde{x}$ is the final wealth, which gives the same utility to the decision maker as the initial wealth plus the uncertain lottery.

i.e. $U(w^*) = E(U(w_0 + \tilde{x}))$.

Def: The **asking price** p_a of a lottery $\tilde{x}$ is the price at which the decision maker is indifferent between keeping the lottery and selling it; i.e.

$U(w_0 + p_a) = E(U(w_0 + \tilde{x})) \Rightarrow p_a = w^* - w_0$.

- $p < p_a$ $\Rightarrow$ the decision maker has a higher utility, if he keeps the lottery

- $p > p_a \quad \Rightarrow$ the decision maker has a higher utility if he sells the lottery.

Ex.

$w_0 = 20$

	z_1	z_2	z_3	z_4	
	$p_1 = 0.4$	$p_2 = 0.1$	$p_3 = 0.2$	$p_4 = 0.3$	
x	60	50	80	70	
w_f	80	70	100	90	
$u(w_f) = \log(w_f)$	4.38	4.25	4.61	4.50	4.449

$\log(w^*) = 4.449 \Rightarrow w^* = \exp(4.449) = 85.5414 \Rightarrow p_a = 65.5414$.

The asking price may also be negative:

$U(w_f) = \sqrt{w_f}$, initial wealth $w_0 = 100$

x	p(x)
-50	0.5
50	0.5

the expected utility is

$$\mathrm{E}(U(\tilde{w}_f)) = 0.5\sqrt{50} + 0.5\sqrt{150} = 9.659 = U(w^*) = \sqrt{w^*}$$

$\Rightarrow w^* = 93.296 \quad \Rightarrow p_a = -6.699$

$\Rightarrow$ the decision maker is willing to pay 6.699, go get rid of the lottery!!

On the other hand, a decision maker may prefer to buy a risky asset.

Def: The **bid price** p_b of a lottery $\tilde{x}$ is the price at which the decision maker is indifferent between buying the lottery and not buying it;

p_b is implicitly given by the equation

$$U(w_0) = \mathrm{E}[U(w_0 - p_b + x)] = \int_{-\infty}^{\infty} U(w_0 + x - p_b) f(x) dx$$

Theorem:

In case of a linear utility function, the asking price for a additive lottery is equal to the expected value of the lottery.

Proof:

U linear $\Rightarrow U(w_f) = g + dw_f, \quad$ with $d > 0$

This implies:

$$U(w^*) = \mathrm{E}[U(\tilde{w}_f)] = \mathrm{E}[g + d(w_0 + \tilde{x})]$$

$$g + dw^* = g + dw_0 + d\mathrm{E}(\tilde{x})$$

$$w^* = w_0 + \mathrm{E}(\tilde{x})$$

$$\Rightarrow p_a = w^* - w_0 = E(\tilde{x})$$

$\Rightarrow$ decision maker with a linear utility function values the lottery only by its expected value

$\Rightarrow$ utility function is linear $\Rightarrow$ decision maker is risk neutral

$\Rightarrow$ if the utility function is linear and the decision maker has two lotteries, which have the same expected value, the decision maker will ask the same price, even if the lotteries differ considerably with respect to other moments, e.g. the variance.

The **Risk-Premium** π is defined as the difference between expected value of the lotterie and the asking price i.e. $\pi = \mathrm{E}(\tilde{x}) - p_a$

$\pi = 0$ the decision maker is **risk neutral.** The asking price is equal to the expected value of the lottery.

$\pi > 0$ The decision maker is **risk avers.** $\pi > 0 \Rightarrow p_a < \mathrm{E}(\tilde{x})$, i.e. the asking price der is less than the asking price in case of risk neutrality.

$\pi < 0$ the decision maker is **risk seeking.** $\pi < 0 \Rightarrow p_a > \mathrm{E}(\tilde{x})$

Ex.

$w_0 = 20$

	z_1	z_2	z_3	z_4	
	$p_1 = 0.4$	$p_2 = 0.1$	$p_3 = 0.2$	$p_4 = 0.3$	
x	60	50	80	70	
w_f	80	70	100	90	
$u(w_f) = \log(w_f)$	4.38	4.25	4.61	4.50	4.449

$\log(w^*) = 4.449 \Rightarrow w^* = \exp(4.449) = 85.5414 \Rightarrow p_a = 65.5414.$

$E(\tilde{x}) = 66, \Rightarrow \pi = 0.4586.$

Now assume a different utility function: $u(w) = w^2$.

	z_1	z_2	z_3	z_4	
	$p_1 = 0.4$	$p_2 = 0.1$	$p_3 = 0.2$	$p_4 = 0.3$	
x	60	50	80	70	
w_f	80	70	100	90	
$u(w_f) = w_f^2$	6400	4900	10 000	8100	7480

$(w^*)^2 = 7480 \Rightarrow w^* = \sqrt{7480} = 86.487 \Rightarrow p_a = 66.487.$ $\pi = -0.487.$

5 Critiques and Alternativ Decision Rules

Ellsberg paradox: people are ambiguity-averse and prefer decisions under risk (i.e. with given or estimated probabilities) to decisions under abiguity.

Kahneman and Tversky, "Prospect Theory: An Analysis of Decision under Risk", Econometrica 47, 1979.

Tversky and Kahneman, "Advances in Prospect Theory: Cumulative Representation of Uncertainty", J. of Risk and Uncertainty 5, 1992.

Certainty effect:

expected utility theory: utilities are weighted by their probabilities

Problem: outcomes being considered as certain are overweigthed relative to outcomes which are merely probable. (Alais Paradox)

probability/possibility vs. gain

Compare lotteries:

$A : (6000, 0.45; 0., 0.55)$ with $B : (3000, 0.90; 0, 0.10)$

as well as

$C : (6000, 0.001; 0, 0.999)$ with $D : (3000, 0.002; 0, 0.998)$

prospects with high probability $\Rightarrow$ choose prospect, where probability is higher prospects possible, but probability low $\Rightarrow$ choose higher gain

Isolation effect

Consider the following two stage game:

Stage 1: games end with probability 0.75 (without winning anything), or proceed to stage 2 with probability 0.25.

Stage 2: either lottery $A : (4000, 0.80, 0, 0.20)$ or $B : (3000, 1.; 0, 0.)$.

The choice between A and B has to be made before the game starts.

This two stage game is equivalent to lotteries

$C : (4000, 0.20; 0, 0.80)$ and $D : (3000, 0.25; 0, 0.75)$.

Usually $C \succ D$ but $B \succ A$. People consider the stage 2 decision independently from stage 1.

Reflection effect:

You are given 1000 \$ and then you have to choose between

$A : (1000, 0.50; 0, 0.50)$ and $B : (500, 1.00)$

Now consider the situation, that you get 2000 \$ and then you have to choose between

$C : (-1000, 0.50; 0, 0.50)$ and $D : (-500, 1.00)$.

Most people prefer B in the first case but C in the second case.

$\Rightarrow$risk aversion for positive prospects, risk seeking for negative prospects.

5.1 Prospect Theory:

5.1.1 Phase 1: editing phase

Coding: people perceive outcomes as gaines or losses (relative to a reference point), rather than final state of wealth. Choice of the reference point (i.e. formulate the prospect as gain or loss) may have an influence on the decision.

Combination: Simplify prospects by combining probabilities associated with identical outcomes.

Segregation: separate certain component from uncertain components, i.e.

$(300, 0.80; 200, 0.20) \Rightarrow 200 + (100, 0.80; 0, 0.20)$

Cancellation:

Discard components that are shared by the prospects under consideration, i.e.

The choice between $(200, 0.20; 100, 0.50; -50, 0.30)$ and

$(200, 0.20; 150, 0.50; -100, 0.30)$ is reduced to a choice between $(100, 0.50; -50, 0$

and

$(150, 0.50; -100, 0.30)$.

Simpification and detection of dominance

Outcomes and probabilities are rounded, extremely unprobable events are discarded, dominated prospects are eliminated.

5.1.2 Phase 2: evaluation phase

Consider in the following only simple prospects of the form $(x, p; y, q)$, with at most two non-zero outcomes; i.e. receive x with probability p, receive y with probability q or nothing with probability $1-p-q$.

A prospect is called **strictly positive**, iff outcomes are strictly positive, (i.e. $x, y > 0, p + q = 1$), **strictly negative**, iff all outcomes are strictly negative (i.e. $x, y < 0, p+q = 1$), or **regular** otherwise.

$$
\begin{aligned}
V \quad & \cdots \quad \text{overall value of an edited prospect} \\
\pi = \pi(p) \quad & \cdots \quad \text{decision weight associated with each probability} \\
v = v(x) \quad & \cdots \quad \text{subjective value of outcome } x
\end{aligned}
$$

with properties: $v(0) = 0, \pi(0) = 0, \pi(1) = 1$.

Value of **regular** prospects:

$V(x, p; y, q) = \pi(p)v(x) + \pi(q)v(y)$

Vaue of **strictly positive** prospects:

in the editimg phase, such prospects are devided into two components: a riskless component and a risky component, i.e. the additional gain or loss.

Consider $(x, p; y, q)$ with $p + q = 1$ and $x > y > 0$.

$V(x, p; y, q) = v(y) + \pi(p)[v(x) - v(y)]$.

Example: $V(400, 0.25; 100, 0.75) = v(100) + \pi(0.25)[v(400) -$

$v(100)]$.

5.1.3 Properties of the Value Function

According to Kahneman & Tversky:

$v(200) - v(100) > v(1200) - v(1100)$, i.e. the difference in value between a gain of 200 and a gain of 100 is greater than the difference between a gain of 1200 and 1100, $\Rightarrow$ the value function $v(.)$ is concave for gains.

Similarly, $v(-100) - v(-200) > v(-1100) - v(1200)$, i.e. the difference between an loss of 100 and 200 is greater than the difference between a loss of 1100 and 1200, $\Rightarrow$ the value function is convex for losses.

Most people consider the disutility resulting from losses being greater than the utility from winning the same amount of money; symmetric bets of the form $(x, 0.50; -x, 0.50)$ are unattractive. Moreover,

$$(y, 0.50; -y, 0.50) \succ (x, 0.50; -x, 0.50), \qquad \text{for } 0 \leq y < x$$

$$\Rightarrow v(y)+v(-y) > v(x)+v(-x) \quad \text{and } v(-y)-v(-x) > v(x)-v(y)$$

$$\Rightarrow v(x) < -v(-x) \quad \text{and } \frac{v(-y) - v(-x)}{-y - (-x)} > \frac{v(x) - v(y)}{x - y} \Rightarrow v'(-x) > v'(x)$$

i.e. the value function is steeper for losses than for gains.

5.1.4 Properties of the Weighting Function

1. If the expectation principle holds, then $\pi(p) = p$ (expected utility theory).

2. In general: $\pi(p)$ is monotonically increasing, $\pi(0) = 0$, $\pi(1) = 1$.

3. For small probabilities, $\pi(p)$ is sub-additive, i.e. $\pi(rp) > r\pi(p)$ for $0 < r < 1$, e.g.

$$(6000, 0.001; 0, 0.999) \succ (3000, 0.002; 0, 0.998)$$

$$\Rightarrow \pi(0.001)v(6000) > \pi(0.002)v(3000)$$

$$\Rightarrow \frac{\pi(0.001)}{\pi(0002} > \frac{v(3000)}{v(6000)} > \frac{1}{2}$$

   where the last inequality follows from concavity of $v(.)$.

4. Rare events are overweighted, i.e. $\pi(p) > p$ for small p; e.g.

   $(5000, .001; 0, 0.999) \succ (5, 1.)$ and $(-5, 1.) \succ (-5000, .001; 0, .999)$.

   This implies $\pi(0.001)v(5000) > v(5)$ hence $\pi(0.001) > v(5)/v(5000) > 0.001$. (The last inequality follows from concavity.)

5. Allais Paradoxon: $(2400, 1.) \succ (2400, 0.66; 2500, 0.33)$ implies

   $v(2400) > \pi(0.66)v(2400) + \pi(0.33)v(2500)$

and $(2500, 0.33; 0, 0.67) \succ (2400, 0.34; 0, 0.66)$ implies $\pi(0.33)v(2500) > \pi(0.34)v(2400)$.

This leads to

$$(1 - \pi(0.66))v(2400) > \pi(0.33)v(2500) > \pi(0.34)v(2400)$$

$$\Rightarrow 1 - \pi(0.66) > \pi(0.34)$$

i.e. Allais paradoxon can be explained by prospect theory, if one assumes **subcertainty**, i.e. $\pi(p) + \pi(1-p) < 1$, for $0 < p < 1$.

6. **sub-proportionality**, i.e.

$$\frac{\pi(pq)}{\pi(p)} \leq \frac{\pi(pqr)}{\pi(pr)}, \quad 0 < p, q, r < 1$$

which means that for a fixed ratio of probabilities the ratio of corresponding decision weights is closer to unity when the probabilities are low, than when they are high.

Multiobjective Decisions

1. one discrete choice variable, n "Consequences" or "Goals".

   Ex.: buy a car

	Price	fuel consumption	horse power
VW	16 200	7.2 l/100 km	66
Opel	14 900	<u>7.0 l/100 km</u>	62
Ford	<u>14 000</u>	7.2 l /100 km	55
Toyota	15 200	8.2 l/100 km	<u>71</u>
	↓	↓	↓
	min	min	max

2. several decision variables (discrete and continuous), including constraints

   Ex: reservoir for hydro power plant

   decision variables: number of workers and time schedule for construction, size of the reservoir, Design of the dam

   criteria: capacity (max.), costs of construction (min.), evaporation (min.)

   constraints: minimal thickness of the dam

3. **Ex:** 1 continuous decision variable, 2 functions

   $\min_{x\geq 0}(f_1(x), f_2(x)), \quad$ where $f_1(x) = \sqrt{x+1}$, $f_2(x) = (x-2)^2+1$

The general model

$A = \{a_1, a_2, \cdots\} \quad \cdots \quad$ Set of alternatives

n consequences $C_1, \cdots C_n$ ("goals", "criteria")

$a \mapsto (C_1(a), C_2(a), \cdots, C_n(a))$

$$R = \{(C_1(a), C_2(a), \cdots, C_n(a)) | a \in A\} \cdots \text{"Range-set"}$$

How can the "best" alternative be chosen?

1. ordinal decision rules: lexicographic order, Borda's method,
2. by preference function (value function, utility function)

   $v : C_1 \times \cdots \times C_n \mapsto \mathbb{R}, \quad (C_1(a), \cdots, C_n(a)) \to v(C_1(a), \cdots, C_n(a))$

   which induces a preference order of the set of consequences and thus also a preference order on the set of alternatives A.

Def: The relation between two goals may be

- **indifferent** (or neutral) , if the realization of one goal has no influence on the degree of realization of the other goal
- **complementary** if a higher realization of the first goal also leads to a higher realization of the other goal
- **competitive**, if a higher realization of one goal reduces the degree of realization of the other goal

Def.:

- An alternative $a^* \in A$, which is optimal with respect to any of the criteria $C_i, i = 1, \cdots n$ is called a **perfect solution.**

  (but in most cases one is confronted with a conflict of goals.

- a solution $\hat{a} \in A$ dominates a solution $\tilde{a} \in A$, if $\hat{a}$ is at least as good as $\tilde{a}$ regarding any goal $C_i(a)$, and better with respect to at least one goal.

  e.g. for maximization: $C_k(\hat{a}) \geq C_k(\tilde{a}) \quad \forall k = 1, \cdots n$ and $C_i(\hat{a}) > C_i(\tilde{a})$ for at least one i.

- each admissable solution which is not dominated by another admissable solution is called **efficient solution** (or **pareto-optimal solution**).

- The **complete solution** is the set of all efficient solutions.

Example

A company has short-listed 9 applicants for a position. The personnel manager decides to consider 5 criteria:

C1: time spent in higher studies (in years)

C2: Professional experience (in years)

C3: Age (in years)

C4: evaluation from the interview (on a scale 0 - 10)

C5: psychometric test (on a scale 0 - 10)

Alternatives	C1 Max	C2 Max	C3 Min	C4 Max	C5 Max
1. Albert	6	5	28	5	5
2. Blanche	4	2	25	10	9
3. Charles	7	7	38	5	10
4. Donald	5	7	35	9	6
5. Emily	6	1	27	6	7
6. Frank	5	7	31	7	8
7. Georgia	6	8	30	7	9
8. Helen	5	6	26	4	8
9. Irving	3	8	34	8	7

There is **No** perfect solution as no applicant dominates all others.

Frank is dominated by Georgia, therefore Frank would certainly not be chosen.

The complete solution consists of all applicants except Frank.

pre-analysis of satisfaction

Define a level of satisfaction for each criterion (minimum requirement) eliminate alternatives, for which one (or more) of these minimum requirements do not hold

In above example the applicant must have at least 4 years of higher education and has to be younger than 35 years.

This requirements would exclude:

Charles, as he is 38 years old, and

Irving, as he only has 3 years of higher education.

Special decision rules:

ordinal multicriterion methods

Lexicografic Order:

- order the goals with respect to their importance
- optimize with respect to the most important goal
- in case alternatives are equally good with respect to the most important goal, they are ordered with respect to the second goal
- ...

Ex:

	fuel consumption	Price	horse power
Opel	7.0 l/100 km	14 900	62
Ford	7.2 l/100 km	14 000	55
VW	7.2 l/100 km	16 200	66
Toyota	8.2 l/100 km	15 200	71

Interchange "Price" and "power" $\Rightarrow$ order of Ford and VW changes.

Drawback: an alternative A is prefered to B if A is marginally better than B with respect to a more important goal, even if A is much worse than B in a less important goal.

Borda's method

Chevalier Jean-Charles de Borda (1733-1799), physicist, mathematician, sailor, well known for discoveries in ballistics, fluid mechanics; member of Academie des Sciences.

Suppose we have m alternatives and n criteria.

- choose m integers $k_1 > k_2 > \cdots > k_m \geq 0$, the Borda coefficients.
- rank the alternatives for each criterion j, $j = 1, \cdots, n$.
- assign the k_1 to the best alternative (with respect to the individual criterion)

- in case of ties (indifferences) take the arithmetic mean.
- for each alternative sum its rankings over all criteria
- rank the alternative according to this sum

Alternatives	C1 Max		C2 Max		C3 Min		C4 Max		C5 Max		sum
1. Albert	6	5	5	3	28	3	5	2	5	1	14
2. Blanche	4	1	2	2	25	6	10	6	9	5.5	20.5
4. Donald	5	2.5	7	5	35	1	9	5	6	2	15.5
5. Emily	6	5	1	1	27	4	6	3	7	3	16
7. Georgia	6	5	8	6	30	2	7	4	9	5.5	22.5
8. Helen	5	2.5	6	4	26	5	4	1	8	4	16.5

Condorcet method

Marquis Caritat de Condorcet (1743-1794)

	C_1	C_2	C_3
A	15	16	03
B	11	13	17
C	08	04	12
D	02	10	09

Condorcet aggregation procedure (or simple majority vote):

compare, e.g., A and B: how many criteria "vote" for A compared to B?

2 criteria prefer A over B, and only 1 prefers B over A: $\Rightarrow A \succ B$.

Condorcet's paradox:

This method can lead to a non-transitiv relation between the alternatives.

main goals and subordinate goals:

- Define **one** goal as main goal, which should be maximized/minimized ert/minimiert. The remaining goals are the subordinate goals.
- define bounds for the subordinate goals.

  in case the subordinate goal should be minimized $\Rightarrow$ upper bound

  in case the subordinate goal should be maximized $\Rightarrow$ lower bound

  Ex: Reservoir for hydro power plant:

  Main goal: minimize cost of construction

  subordinate goals: evaporation $\leq V_{max}$, capacity $\geq K_{min}$

Outranking Methods

Basic Idea:

an alternativ a_i outranks an alternativ a_k (a_i S a_k) if

- a_i is at least as good as alternative a_k for the majority of criteria
- there is no criterion, such that a_i is substantially less good as a_k

Ex. 4 alternatives, 3 criteria, which are equally important:

	C_1	C_2	C_3
a	90	10	100
b	100	0	100
c	90	100	90
d	50	50	100

- a S b, b S a, c S a,
- b and d may be incomparable
- relation is not transitive, c S a, a S b but c does not outrank b

Basic Concepts:

For each criterion $C_j, j = 1, \cdots, n$ define an indifference threshold q_j and a preference threshold p_j, such that $p_j > q_j$.

Define preference relation $\succ_j$ with respect to criterion C_j

$$a \succ_j b \leftrightarrow U_j(a) > U_j(b) + p_j$$

and the indifference relation

$$a \sim_j b \leftrightarrow |U_j(a) - U_j(b)| \leq q_j$$

Define the following sets:

Concordance set: set of criteria, such that a is not strictly worse than b

$$C(a,b) = \{C_j | U_j(a) \geq U_j(b) - q_j\}$$

Discordance set: set of criteria, such that b is strictly preferred to a

$$D(b,a) = \{C_j | U_j(a) \leq U_j(b) - p_j\}$$

Obviously, there is a third set of criteria, where b is weakly preferred to a

$$C_w(b,a) = \{C_j | U_j(b) - p_j \leq U_j(a) < U_j(b) - q_j\}$$

The three sets $C(a,b), C_w(b,a), D(b,a)$ form a partition of the set of criteria

$f_C(a,b), f_{C_w}(b,a), f_D(a,b) \cdots$ "importance" associated with the above sets.

Then define the

concordance coefficient $c_{a,b} = f_C(a,b) + f_{C_w}(b,a)$

discordance coefficient $d_{a,b} = f_D(b,a)$

and form the Concordance matrix $C = (c(a_i, a_k))$ and Discordance matrix $D = (d(a_i, a_k))$. Outranking methods are based on these matrices

ELECTRE

ELimination **E**t **C**hoix **T**raduisant la **RE**alité

ELECTRE 1:

the thresholds $p_j = q_j = 0$.

$$C(a,b) = \{C_j | a \succ_j b \text{ or } a =_j b\}$$

and

$$D(b,a) = \{C_j | b \succ_j a\}$$

Assign normalized weights w_j to the criteria ($0 \le w_j \le 1, \Sigma_j w_j = 1$).

Then

$$c(a,b) = \sum_{j \in C(a,b)} w_j$$

$$d(a,b) = \frac{\max_{j \in D(b,a)}(U_j(b) - U_j(a))}{\delta} \text{ with } \delta = \max_j \max_{e,f}(U_j(e) - U_j(f))$$

Now define two thresholds t_c and t_d, $0 < t_c, t_d < 1$

alternative a outranks b iff $c(a,b) \geq t_c$ and $d(a,b) \leq t_d$.

Ex.

Alternatives	C1 Max	C2 Max	C3 Min	C4 Max	C5 Max
weights	0.25	0.25	0.10	0.20	0.20
1. Albert	6	5	28	5	5
2. Blanche	4	2	25	10	9
3. Charles	7	7	38	5	10
4. Donald	5	7	35	9	6
5. Emily	6	1	27	6	7
6. Frank	5	7	31	7	8
7. Georgia	6	8	30	7	9
8. Helen	5	6	26	4	8
9. Irving	3	8	34	8	7

Eliminate Frank (is dominated by Georgia) and Charles and Irving, as they are non satisficing (in the example C_1 should be at least 4 and C_3 should be below 35).

Now normalize the criteria by dividing by the sum, i.e. $a_i / \sum_j a_j$. By criterion C_3 we take first the reciprocals, and then normilize. This yields

Alternatives	C1 Max	C2 Max	C3 Max	C4 Max	C5 Max
weights	0.25	0.25	0.10	0.20	0.20
1. Albert	0.188	0.172	0.168	0.122	0.114
2. Blanche	0.125	0.069	0.188	0.244	0.205
4. Donald	0.156	0.241	0.134	0.220	0.136
5. Emily	0.188	0.034	0.174	0.146	0.159
7. Georgia	0.188	0.276	0.156	0.171	0.205
8. Helen	0.156	0.207	0.180	0.098	0.182

Compute the concordance matrix

C =

	Al	Bl	Do	Em	Ge	He
1. Albert		0.50	0.35	0.50	0.35	0.45
2. Blanche	0.50		0.50	0.75	0.50	0.50
4. Donald	0.65	0.50		0.45	0.20	0.70
5. Emily	0.75	0.25	0.55		0.35	0.45
7. Georgia	0.90	0.70	0.80	0.90		0.90
8. Helen	0.55	0.50	0.55	0.55	0.10	

and the discordance matrix

D =

	Al	Bl	Do	Em	Ge	He
1. Albert		0.51	0.40	0.19	0.43	0.28
2. Blanche	0.43		0.71	0.26	0.86	0.57
4. Donald	0.14	0.28		0.16	0.28	0.19
5. Emily	0.57	0.40	0.86		1.00	0.71
7. Georgia	0.05	0.30	0.20	0.07		0.10
8. Helen	0.13	0.61	0.51	0.20	0.30	

Set the thresholds e.g. $t_c = 0.6, t_d = 0.4$ then we get the matrix

S=

	Al	Bl	Do	Em	Ge	He
1. Albert		0	0	0	0	0
2. Blanche	0		0	1	0	0
4. Donald	1	0		0	0	1
5. Emily	0	0	0		0	0
7. Georgia	1	1	1	1		1
8. Helen	0	0	0	0	0	

e.g., Ge S Do, as $c_{Ge,Do} = 0.80 > t_c, d_{Ge,Do} = 0.20 < t_d$.