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Stable games and their dynamics

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Abstract

We study a class of population games called *stable games*. These games are characterized by *self-defeating externalities*: when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. We prove that the set of Nash equilibria of a stable game is globally asymptotically stable under a wide range of evolutionary dynamics. Convergence results for stable games are not as general as those for potential games: in addition to monotonicity of the dynamics, integrability of the agents' revision protocols plays a key role.

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1. Introduction

This paper studies a class of population games that we call *stable games*. These games are characterized by a condition we call *self-defeating externalities*, which requires that when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. Our main results show that the set of Nash equilibria of a stable game is globally asymptotically stable under a wide range of evolutionary dynamics, including the BNN dynamic, the best response dynamic, and the Smith dynamic. Related global stability results hold for the logit, replicator, and projection dynamics. But we argue that convergence results for stable games are not as general as those for potential games: in addition to monotonicity of the dynamics, integrability of the agents' revision protocols plays a key role.

Our treatment of stable games builds on ideas from a variety of fields. From the point of view of mathematical biology, one can view stable games as a generalization of the class of symmetric normal form games with an interior ESS (Maynard Smith and Price [37]) to settings with multiple populations and nonlinear payoffs. Indeed, for games with an interior Nash equilibrium, the ESS condition reduces to the negative definiteness of the payoff matrix, and this latter property characterizes the "strictly stable" games in this symmetric normal form setting.[1] Bishop and Cannings [5] shows that the war of attrition satisfies the weaker semidefiniteness condition that characterizes stable games. Stable single population games appear in the work of Akin [1], and stable multipopulation games with linear payoff functions are studied in Cressman et al. [10]. Stable games can be found in the transportation science literature in the work of Smith [54,55] and Dafermos [11], where they are used to extend the network congestion model of Beckmann et al. [3] to allow for asymmetric externalities between drivers on different routes. Alternatively, stable games can be understood as a class of games that preserves many attractive properties of concave potential games: in a sense to be made explicit soon, stable games preserve the concavity of these games without requiring the existence of a potential function at all. Stable games can also be viewed as examples of objects called monotone operators from the theory of variational inequalities.[2] Finally, as we explain in Section 2.4, stable games are related to the diagonally concave games introduced by Rosen [43].

To analyze the behavior of deterministic evolutionary dynamics in stable games, we first derive these dynamics from an explicit model of individual choice. This model is specified in terms of *revision protocols*, which determine the rates at which an agent who is considering a change in strategies opts to switch to his various alternatives. Most of our presentation focuses on a general class of dynamics called *target dynamics*, under which the rate at which an agent switches to any given strategy is independent of his current strategy choice. The name of this class of dynamics comes from their simple geometric description: for each current population state, the dynamics specify a *target state* toward which the current state moves, as well as a rate at which the target state is approached. To this we add the additional restriction that the rates at which agents switch to alternative strategies can be expressed as a function of the *excess payoff vector*: a vector whose ith component is the difference the ith strategy's payoff and the population's average payoff. Dynamics with these properties are prominent in the literature: three fundamental

[1] Conversely, symmetric normal form games with such a negative definite payoff matrix have a unique ESS which is also the unique Nash equilibrium; see [29].

[2] See [20,31,34,38,40].

evolutionary dynamics—the BNN dynamic [8], the best response dynamic [17], and the logit dynamic [15]—are of this form.

As a point of comparison, we note that in potential games, dynamics satisfying a mild monotonicity condition called *positive correlation* converge to equilibrium from all initial conditions.[3] The same cannot be said for stable games: in Section 6, we construct a dynamic that satisfies positive correlation but that fails to converge in some stable games.

The reason that convergence need not occur can be explained as follows. Stable games have the following attractive geometric property: if one starts at any nonequilibrium state and then moves in the direction defined by the current vector of payoffs, distance from equilibrium either falls or remains constant.[4] For their part, dynamics that satisfy *positive correlation* always move in a direction that forms an acute angle with the vector of payoffs. It follows that in most cases, many permissible directions of motion lead away from equilibrium; if these directions tend to be followed, convergence will fail to occur.

By introducing an additional restriction on dynamics—namely, *integrability* of the revision protocol—one can eliminate this source of trouble. As we argue in Section 6, integrability implies that on average, the vectors of motion under the dynamic must deviate from the vectors of payoffs in the direction of Nash equilibrium. Since the payoff vectors of a stable game aim towards the equilibrium set, monotonicity and integrability together generate convergence to equilibrium.

We convert this intuition into formal convergence results for stable games in Section 5, where we introduce suitable Lyapunov functions for the dynamics we study. A somewhat similar approach is followed in analyses of potential games, where all dynamics that satisfy positive correlation ascend the game's potential function. However, unlike potential games, stable games do not come equipped with an all-purpose Lyapunov function. To prove convergence results, we must construct a suitable Lyapunov function for each dynamic we wish to consider.

Again, integrability of the revision protocol is essential to completing the argument. By definition, integrability requires that the rates at which agents switch strategies can be described as a gradient of a scalar-valued function, which we call a *revision potential*. For each dynamic we consider, including the BNN, best response, and logit dynamics, this revision potential is the key ingredient in the construction of the Lyapunov function.

The convergence theorems for stable games proved in this paper extend and unify a variety of existing results: Hofbauer [25] proves convergence results for a variety of dynamics in symmetric normal form games with an interior ESS, while Hofbauer and Sandholm [27] establish global convergence of perturbed best response dynamics in all stable games. Building on these analyses, the present paper shows how integrability of revision protocols is a common thread running through these and other convergence results. Between this study and existing analyses of specific dynamics,[5] convergence in stable games has now been established for six basic dynamics from the evolutionary literature. Section 7.3 presents a summary of these results.

Our use of integrability was inspired by the work of Hart and Mas-Colell [21] on heuristic learning in repeated play of normal form games. These authors construct a class of *consistent* repeated game strategies: strategies which ensure that in the long run, and for all possible sequences of opponents' plays, the payoff that a player obtains is as high as the best payoff he

[3] A *potential game* is a game that admits a scalar-valued potential function that encodes the incentives of all players at once. This function serves as a strict Lyapunov function for a wide range of evolutionary dynamics. See [29,39,44,49].

[4] Put differently, every Nash equilibrium of a stable game is a so-called *globally neutrally stable state* (*GNSS*)—see Section 3.

[5] See [1,2,10,28,41,51,55,60], and Section 7.2 below.

could have obtained had he known the empirical frequencies of his opponents' choices in advance.[6] The class of repeated game strategies that [21] consider are based on decision rules to be applied in each period of play. The inputs to these rules are vectors describing the performance of each strategy versus the time average of opponents' past play, normalized by the average payoffs actually obtained; the outputs are vectors of choice probabilities. Ref. [21] shows that decision rules satisfying three conditions—continuity, monotonicity, and integrability—ensure consistent play. In this paper, we show that the integrability property introduced by [21] in an heuristic learning framework is also fruitful in an evolutionary setting, despite substantial differences in the contexts, questions posed, and requisite analytical techniques.

2. Definition, characterization, and examples

2.1. Population games

Let $\mathcal{P} = \{1, \ldots, \wp\}$, be a *society* consisting of $\wp \geqslant 1$ *populations* of agents. Agents in population p form a continuum of *mass* $m^p > 0$. Masses capture the populations' relative sizes; if there is just one population, we assume that its mass is one.

The set of *strategies* available to agents in population p is denoted $S^p = \{1, \ldots, n^p\}$, and has typical elements i and j. We let $n = \sum_{p \in \mathcal{P}} n^p$ equal the total number of pure strategies in all populations.

During game play, each agent in population p selects a (pure) strategy from S^p. The set of *population states* (or *strategy distributions*) for population p is thus $X^p = \{x^p \in \mathbf{R}^{n^p}_+ \colon \sum_{i \in S^p} x^p_i = m^p\}$. The scalar $x^p_i \in \mathbf{R}_+$ represents the mass of players in population p choosing strategy $i \in S^p$. Elements of $X = \prod_{p \in \mathcal{P}} X^p = \{x = (x^1, \ldots, x^{\wp}) \in \mathbf{R}^n_+ \colon x^p \in X^p\}$, the set of *social states*, describe behavior in all $\wp$ populations at once.

The *tangent space* of X^p, denoted TX^p, is the smallest subspace of $\mathbf{R}^{n^p}$ that contains all vectors describing motions between population states in X^p. In other words, if $x^p, y^p \in X^p$, then $y^p - x^p \in TX^p$, and TX^p is the span of all vectors of this form. It is not hard to see that $TX^p = \{z^p \in \mathbf{R}^{n^p} \colon \sum_{i \in S^p} z^p_i = 0\}$ contains exactly those vectors in $\mathbf{R}^{n^p}$ whose components sum to zero; the restriction on the sum embodies the fact that changes in the population state leaves the population's mass constant. Changes in the full social state are elements of the grand tangent space $TX = \prod_{p \in \mathcal{P}} TX^p$.

It will prove useful to specify notation for the orthogonal projections onto the subspaces $TX^p \subset \mathbf{R}^{n^p}$ and $TX \subset \mathbf{R}^n$. The former is given by the matrix $\Phi = I - \frac{1}{n^p}\mathbf{1}\mathbf{1}' \in \mathbf{R}^{n^p \times n^p}$, where $\mathbf{1} \in \mathbf{R}^{n^p}$ is the vector of ones; the latter is given by the block diagonal matrix $\boldsymbol{\Phi} = \mathrm{diag}(\Phi, \ldots, \Phi) \in \mathbf{R}^{n \times n}$.

We generally take the sets of populations and strategies as fixed and identify a game with its payoff function. A *payoff function* $F : X \to \mathbf{R}^n$ is a continuous map that assigns each social state a vector of payoffs, one for each strategy in each population. $F^p_i : X \to \mathbf{R}$ denotes the payoff function for strategy $i \in S^p$ while $F^p : X \to \mathbf{R}^{n^p}$ denotes the payoff functions for all strategies in S^p. When $\wp = 1$, we omit the redundant superscript p from all of our notation.

State $x \in X$ is a *Nash equilibrium*, denoted $x \in NE(F)$, if every strategy in use at x is an optimal strategy. We can express this requirement in a number of equivalent ways:

[6] The original papers on consistency are Hannan [19] and Blackwell [7]. This topic was reintroduced to the learning in games literature by Fudenberg and Levine [14,15]. See [13] for historical notes, and [9,59] for recent textbook treatments.

$$\begin{aligned} x \in NE(F) &\Leftrightarrow \left[x_i^p > 0 \Rightarrow F_i^p(x) \geqslant F_j^p(x)\right] \quad \text{for all } i, j \in S^p \text{ and } p \in \mathcal{P} \\ &\Leftrightarrow \left(x^p\right)' F^p(x) \geqslant \left(y^p\right)' F^p(x) \quad \text{for all } y^p \in X^p \text{ and } p \in \mathcal{P} \\ &\Leftrightarrow (y-x)' F(x) \leqslant 0 \quad \text{for all } y \in X. \end{aligned}$$

2.2. Stable games

We call the population game $F : X \to \mathbf{R}^n$ a *stable game* if

$$(y-x)'\big(F(y) - F(x)\big) \leqslant 0 \quad \text{for all } x, y \in X. \tag{S}$$

If the inequality in condition (S) holds strictly whenever $x \neq y$, we say that F is *strictly stable*, while if this inequality always binds, we say that F is *null stable*.

For a first intuition, imagine for the moment that F is a *full potential game*: in other words, that $F \equiv \nabla f(x)$ for some scalar-valued full potential function $f : \mathbf{R}^n_+ \to \mathbf{R}$.[7] In this case, condition (S) is simply the requirement that the potential function f be concave. Our definition of stable games thus extends the defining property of concave potential games to games whose payoffs are not integrable.

Stable games whose payoffs are differentiable can be characterized in terms of the action of their derivative matrices $DF(x)$ on $TX \times TX$.

Theorem 2.1. *Suppose the population game F is C^1. Then F is a stable game if and only if it satisfies* self-defeating externalities*:*

$$DF(x) \text{ is negative semidefinite with respect to } TX \text{ for all } x \in X. \tag{S$'$}$$

Theorem 2.1 is a direct consequence of the definition of the derivative $DF(x)$ and the Fundamental Theorem of Calculus. For the intuition behind condition (S$'$), note that the condition can be restated as

$$z' DF(x) z \leqslant 0 \quad \text{for all } z \in TX \text{ and } x \in X.$$

This requirement is in turn equivalent to

$$\sum_{p \in \mathcal{P}} \sum_{i \in S^p} z_i^p \frac{\partial F_i^p}{\partial z}(x) \leqslant 0 \quad \text{for all } z \in TX \text{ and all } x \in X.$$

To interpret this expression, recall that the displacement vector $z \in TX$ describes the aggregate effect on the population state of strategy revisions by a small group of agents. The derivative $\frac{\partial F_i^p}{\partial z}(x)$ represents the marginal effect that these revisions have on the payoffs of agents currently choosing strategy $i \in S^p$. Condition (S$'$) considers a weighted sum of these effects, with weights given by the changes in the use of each strategy, and requires that this weighted sum be negative.

Intuitively, a game exhibits self-defeating externalities if the improvements in the payoffs of strategies to which revising players are switching are always exceeded by the improvements in the payoffs of strategies which revising players are abandoning. For example, suppose the tangent vector z takes the form $z = e_j^p - e_i^p$, representing switches by some members of population p

[7] A *potential game* is defined by the weaker requirement that $\boldsymbol{\Phi} F \equiv \nabla f$ for a potential function f defined on domain X; see Example 2.8 below, and especially [49].

from strategy i to strategy j. In this case, the requirement in condition (S′) reduces to $\frac{\partial F_j^p}{\partial z}(x) \leqslant \frac{\partial F_i^p}{\partial z}(x)$: that is, any performance gains that the switches create for the newly chosen strategy j are dominated by the performance gains created for the abandoned strategy i.

2.3. Examples

We now illustrate the definition of stable games through a series of examples. These examples are not needed in the analysis to follow, and could be skipped on a first reading.

Our first four examples consider a single population whose members are randomly matched to play a symmetric two-player normal form game, defined by a strategy set $S = \{1, \dots, n\}$ and a payoff matrix $A \in \mathbf{R}^{n\times n}$. A_{ij} is the payoff a player obtains when he chooses strategy i and his opponent chooses strategy j; this payoff does not depend on whether the player in question is called player 1 or player 2. Random matching in A generates the linear population game $F(x) = Ax$, whose derivative matrices $DF(x) = A$ are independent of x.

Example 2.2 *(Games with an interior evolutionarily or neutrally stable state).* State $x \in X$ is an *evolutionarily stable state* (or an *evolutionarily stable strategy*, or simply an *ESS*) of A [37] if

(i) $x'Ax \geqslant y'Ax$ for all $y \in X$; and
(ii) $x'Ax = y'Ax$ implies that $x'Ay > y'Ay$.

Condition (i) says that x is a symmetric Nash equilibrium of A. Condition (ii) says that x performs better against any alternative best reply y than y performs against itself. If we weaken condition (ii) to

(ii′) if $x'Ax = y'Ax$, then $x'Ay \geqslant y'Ay$,

then a state satisfying conditions (i) and (ii′) is called a *neutrally stable state* (*NSS*) [36].

Suppose that the ESS x lies in the interior of X. Then as x is an interior Nash equilibrium, all pure and mixed strategies are best responses to it: for all $y \in X$, we have that $x'Ax = y'Ax$, or, equivalently, that $(x - y)'Ax = 0$. Next, we can rewrite the inequality in condition (ii) as $(x - y)'Ay > 0$. Subtracting this last expression from the previous one yields $(x - y)'A(x - y) < 0$. But since x is in the interior of X, all tangent vectors $z \in TX$ are proportional to $x - y$ for some choice of $y \in X$. Therefore, $z'DF(x)z = z'Az < 0$ for all $z \in TX$, and so F is a strictly stable game. Similar reasoning shows that if A admits an interior NSS, then F is a stable game.

Example 2.3 *(Rock–Paper–Scissors).* In Rock–Paper–Scissors, Paper covers Rock, Scissors cut Paper, and Rock smashes Scissors. If a win in a match generates a benefit of $w > 0$, a loss imposes a cost of $-l < 0$, and a draw is a neutral outcome, we obtain the symmetric normal form game

$$A = \begin{pmatrix} 0 & -l & w \\ w & 0 & -l \\ -l & w & 0 \end{pmatrix}$$

with $w, l > 0$. When $w = l$, we refer to A as (*standard*) *RPS*; when $w > l$, we refer to A as *good RPS*, and when $w < l$, we refer to A as *bad RPS*. In all cases, the unique symmetric Nash equilibrium of A is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

To determine the parameter values for which this game generates a stable population game, define $d = w - l$. Since $y'Ay = \frac{1}{2}y'(A + A')y$, it is enough to see when the symmetric matrix $\hat{A} = A + A'$, whose diagonal elements equal 0 and whose off-diagonal elements equal d, is negative semidefinite with respect to TX. Now $\hat{A}$ has one eigenvalue of $2d$ corresponding to the eigenvector $\mathbf{1}$, and two eigenvalues of $-d$ corresponding to the orthogonal eigenspace TX. Thus, $z'\hat{A}z = -dz'z$ for each $z \in TX$. Since $z'z > 0$ whenever $z \neq \mathbf{0}$, we conclude that F is stable if and only if $d \geqslant 0$. In particular, good RPS is strictly stable, standard RPS is null stable, and bad RPS is not stable.

Example 2.4 *(Wars of attrition).* In a war of attrition [5], strategies represent amounts of time committed to waiting for a scarce resource. If the two players choose times i and $j > i$, then the j player obtains the resource, worth v, while both players pay a cost of c_i: once the first player leaves, the other seizes the resource immediately. If both players choose time i, the resource is split, so payoffs are $\frac{v}{2} - c_i$ each. We allow the resource value $v \in \mathbf{R}$ to be arbitrary, and require the cost vector $c \in \mathbf{R}^n$ to satisfy $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$. In the online appendix, we show that random matching in a war of attrition generates a stable game.

Example 2.5 *(Symmetric zero sum games).* The symmetric game A is *symmetric zero-sum* if A is skew-symmetric: that is, if $A_{ji} = -A_{ij}$ for all $i, j \in S$. This condition ensures that under single population random matching, the total utility generated in any match is zero. Since payoffs in the resulting single population game are $F(x) = Ax$, we find that $z'DF(x)z = z'Az = 0$ for all vectors $z \in \mathbf{R}^n$, and so F is a null stable game.

Next we consider an example based on random matching across two populations to play a (possibly asymmetric) two-player normal form game. To define a $\mathcal{p}$-player normal form game, let $S^p = \{1, \ldots, n^p\}$ denote player p's strategy set, $S = \prod_{q \in \mathcal{P}} S^p$ the set of pure strategy profiles, and $U^p : S \to \mathbf{R}$ player p's payoff function.

Example 2.6 *(Zero-sum games).* A two-player game $U = (U^1, U^2)$ is *zero-sum* if $U^2 = -U^1$, so that the two players' payoffs always add up to zero. Random matching of two populations to play U generates the population game

$$F\bigl(x^1, x^2\bigr) = \begin{bmatrix} 0 & U^1 \\ (U^2)' & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{bmatrix} 0 & U^1 \\ -(U^1)' & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

If z is a vector in $\mathbf{R}^n = \mathbf{R}^{n^1+n^2}$, then

$$z'DF(x)z = \bigl(\bigl(z^1\bigr)' \quad \bigl(z^2\bigr)'\bigr) \begin{bmatrix} 0 & U^1 \\ -(U^1)' & 0 \end{bmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \bigl(z^1\bigr)'U^1 z^2 - \bigl(z^2\bigr)'\bigl(U^1\bigr)' z^1 = 0,$$

so F is a null stable game.

The previous example shows that random matching across multiple populations can generate a null stable game. Proposition 2.7 reveals that null stable games are the only stable games that can be generated in this way.

Proposition 2.7. *Suppose F is a C^1 stable game without own-population interactions: $F^p(x)$ is independent of x^p for all $p \in \mathcal{P}$. Then F is null stable: $(y-x)'(F(y)-F(x)) = 0$ for all $x, y \in X$.*

Proof. Our proof is by induction on the number of populations. Let F be a two-population stable game without own-population interactions. The absence of such interactions implies that the diagonal blocks $D^1F^1(x)$ and $D^2F^2(x)$ of the derivative matrix $DF(x)$ are identically zero. Now if $z = (z^1, z^2) \in TX$ and $\hat{z} = (-z^1, z^2)$, then

$$z'DF(x)z = \left(z^2\right)'D^1F^2(x)z^1 + \left(z^1\right)'D^2F^1(x)z^2 = -\hat{z}'DF(x)\hat{z}$$

for all $x \in X$. But since F is stable, Theorem 2.1 tells us that $z'DF(x)z \leqslant 0$ and that $\hat{z}'DF(x)\hat{z} \leqslant 0$, so it must be that $z'DF(x)z = 0$. Since $x \in X$ and $z \in TX$ were arbitrary, it follows easily from the Fundamental Theorem of Calculus that F is null stable.

Continuing inductively, we suppose that the theorem is true for games played by $\wp$ populations, and consider games played by $\wp + 1$ populations. Fix a population $p \in \{1, \ldots, \wp + 1\}$, and let $z = (z^p, z^{-p}) \in TX$ and $\hat{z} = (-z^p, z^{-p})$. Then together, the fact that $D^pF^p(x)$ is identically zero and the inductive hypothesis imply that

$$\begin{aligned} z'DF(x)z &= \left(z^{-p}\right)'D^pF^{-p}(x)z^p + \left(z^p\right)'D^{-p}F^p(x)z^{-p} + \left(z^{-p}\right)'D^{-p}F^{-p}(x)z^{-p} \\ &= \left(z^{-p}\right)'D^pF^{-p}(x)z^p + \left(z^p\right)'D^{-p}F^p(x)z^{-p} \\ &= -\hat{z}'DF(x)\hat{z} \end{aligned}$$

for all $x \in X$, allowing us to conclude as before that F is null stable. □

Proposition 2.7 tells us that within-population interactions are required to obtain a strictly stable game. Thus, in random matching contexts, strictly stable games only occur when there is matching within a single population, or when interactions are allowed to occur both across and within populations (see Cressman et al. [10]). But in general population games—for instance, in congestion games—within-population interactions are the norm, and strictly stable games are not uncommon. Our next two examples illustrate this point.

Example 2.8 *(Concave potential games and nearby games).* The population game $F : X \to \mathbf{R}^n$ is a *potential game* if it admits a *potential function* $f : X \to \mathbf{R}$ satisfying $\nabla f \equiv \boldsymbol{\Phi} F$. If f is concave, we say that F is a *concave potential game*. Leading examples of such games include congestion games with increasing cost functions (which provide the basic game-theoretic model of network congestion) and models of variable externality pricing.[8]

It is easy to verify that any concave potential game is a stable game. Let $y - x \in TX$. Since the orthogonal projection matrix $\boldsymbol{\Phi}$ is symmetric, we find that

$$\begin{aligned} (y-x)'\big(F(y)-F(x)\big) &= \big(\boldsymbol{\Phi}(y-x)\big)'\big(F(y)-F(x)\big) \\ &= (y-x)'\big(\boldsymbol{\Phi}F(y) - \boldsymbol{\Phi}F(x)\big) \\ &= (y-x)'\big(\nabla f(y) - \nabla f(x)\big) \\ &\leqslant 0 \end{aligned}$$

and so F is stable.

[8] See [3,29,39,44,47].

Since potential games are characterized by equalities, slightly altering the payoff functions of a potential game often does not result in a new potential game. But since a strictly stable game is defined by strict *inequalities*, nearby games are strictly stable games as well. Combining these observations, we see that while perturbations of strictly concave potential games often fail to be potential games, they remain strictly stable games. This point is noted in the transportation science literature [11,54], where stable games are used to model traffic networks exhibiting asymmetric externalities between drivers on different routes.

Example 2.9 *(Negative dominant diagonal games).* We call the full population game F a *negative dominant diagonal game* if it satisfies

$$\frac{\partial F_i^p}{\partial x_i^p}(x) \leqslant 0 \quad \text{and} \quad \left|\frac{\partial F_i^p}{\partial x_i^p}(x)\right| \geqslant \frac{1}{2}\sum_{(j,q)\neq(i,p)}\left(\left|\frac{\partial F_j^q}{\partial x_i^p}(x)\right| + \left|\frac{\partial F_i^p}{\partial x_j^q}(x)\right|\right)$$

for all $i \in S^p$, $p \in \mathcal{P}$, and $x \in X$. The first condition says that choosing strategy $i \in S^p$ imposes a negative externality on other users of this strategy. The second condition requires that this externality exceed the average of (i) the total externalities that strategy i imposes on other strategies and (ii) the total externalities that other strategies impose on strategy i. These conditions are precisely what is required for the matrix $DF(x) + DF(x)'$ to have a negative dominant diagonal. The dominant diagonal condition implies that all of the eigenvalues of $DF(x) + DF(x)'$ are negative; since $DF(x) + DF(x)'$ is also symmetric, it is negative semidefinite. Therefore, $DF(x)$ is negative semidefinite too, and so F is a stable game.

2.4. Connections with diagonally concave games

The idea of imposing a negative definiteness condition on a game's payoff functions was proposed earlier in an overlapping but distinct context by Rosen [43]. Rosen [43] considers $\wp$-player normal form games in which each player $p \in \mathcal{P} = \{1, \ldots, \wp\}$ chooses strategies from a convex, compact set $C^p \subset \mathbf{R}^{n^p}$. Player p's payoff function $\phi^p : C \equiv \prod_{p\in\mathcal{P}} C^p \to \mathbf{R}$ is assumed to be continuous in x and concave in x^p. These assumptions ensure the existence of Nash equilibrium.

To obtain a uniqueness result, Rosen [43] introduces the function $g : \mathbf{R}^n \to \mathbf{R}^n$ defined by $g^p(x) = \nabla^p \phi^p(x)$, and calls the game with payoffs ϕ *diagonally strictly concave* if

$$(y - x)'\big(g(y) - g(x)\big) < 0 \quad \text{for all } x, y \in C. \tag{1}$$

Rosen [43] proves that any diagonally strictly concave game satisfying mild smoothness and nondegeneracy assumptions admits a unique Nash equilibrium.

To connect these ideas with stable games, note that any $\wp$-population game F without own-population interactions is formally equivalent to a $\wp$-player normal form game in which payoffs are linear in own strategies—namely, the normal form game with strategy sets $C^p = X^p$ and payoff functions

$$\phi^p(x) = \big(x^p\big)' F^p\big(x^{-p}\big).$$

In this case, we have $g \equiv F$, so comparing conditions (S) and (1) reveals that ϕ is diagonally concave (i.e., (1) always holds weakly) if and only if F is stable. However, since F^p is independent of x^p, Proposition 2.7 implies that F must be null stable, or, equivalently, that Rosen's [43] condition (1) must always hold with equality.

Population games which include own-population interactions are not formally equivalent to normal form games. Therefore, stable games with such interactions, including all of the examples from Section 2.3 other than Example 2.6, fall outside of the class that Rosen [43] considers.

3. Equilibrium

We now present equilibrium concepts that are of basic importance for stable games. We call $x \in X$ a *globally neutrally stable state* (GNSS) if

$$(y-x)'F(y) \leqslant 0 \quad \text{for all } y \in X.$$

If this inequality holds strictly whenever $y \neq x$, we call x a *globally evolutionarily stable state* (GESS). We let $GNSS(F)$ and $GESS(F)$ denote the sets of globally neutrally stable strategies and globally evolutionarily stable strategies, respectively.

The inequalities used to define GNSS and GESS are the same ones used to define NSS and ESS in symmetric normal form games (Example 2.2), but they are now required to hold not just at those states y that are optimal against x, but at all $y \in X$. A version of the GESS concept is used by Hamilton [18] in his pioneering analysis of sex-ratio selection, under the name "unbeatable strategy." More recent appearances of this concept in the evolutionary game theory literature can be found in Pohley and Thomas [42] and Joosten [32]. Analogues of the GNSS and GESS concepts are introduced in the variational inequality and transportation science literatures by Minty [38] and Smith [54], respectively.

Proposition 3.1 and Theorem 3.2 relate GNSS and GESS to Nash equilibrium. Versions of these results are known in the variational inequality literature; see John [31].

Proposition 3.1.

(i) *$GNSS(F)$ is convex, and $GNSS(F) \subseteq NE(F)$.*
(ii) *If $x \in GESS(F)$, then $NE(F) = \{x\}$. Hence, if a GESS exists, it is unique.*

Proof. Because $GNSS(F)$ is an intersection of half spaces, it is convex. To prove the second statement in part (i), suppose that $x \in GNSS(F)$ and let $y \neq x$. Define $x_\varepsilon = \varepsilon y + (1-\varepsilon)x$. Since x is a GNSS, $(x - x_\varepsilon)'F(x_\varepsilon) \geqslant 0$ for all $\varepsilon \in (0,1]$. Simplifying and dividing by ε yields $(x-y)'F(x_\varepsilon) \geqslant 0$ for all $\varepsilon \in (0,1]$, so taking ε to zero yields $(y-x)'F(x) \leqslant 0$. In other words, $x \in NE(F)$.

To prove part (ii), it is enough to show that if x is a GESS, then no $y \neq x$ is Nash. But if $x \in GESS(F)$, then $(x-y)'F(y) > 0$, so $y \notin NE(F)$. □

Proposition 3.1 tells us that every GNSS of an arbitrary game F is a Nash equilibrium. Theorem 3.2 shows that more can be said if F is stable: in this case, the (convex) set of globally neutrally stable states is identical to the set of Nash equilibria.

Theorem 3.2.

(i) *If F is a stable game, then $NE(F) = GNSS(F)$, and so is convex.*
(ii) *If in addition F is strictly stable at some $x \in NE(F)$ (that is, if $(y-x)'(F(y) - F(x)) < 0$ for all $y \neq x$), then $NE(F) = GESS(F) = \{x\}$.*

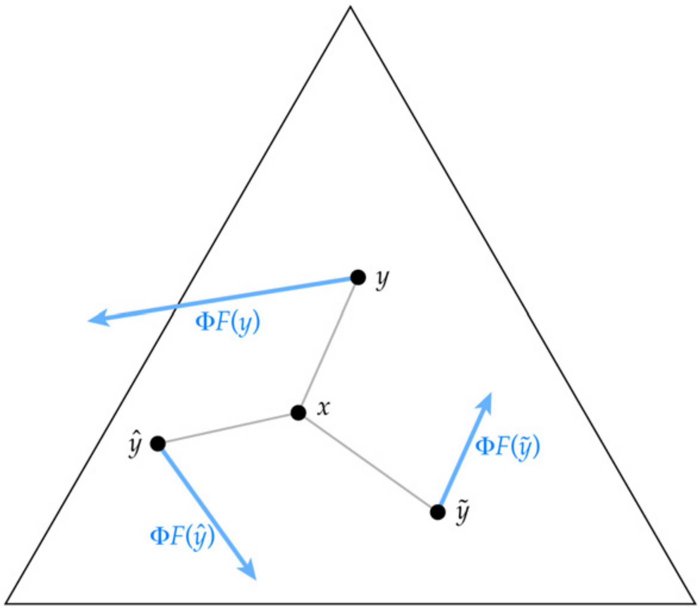

Fig. 1. The geometric definition of GESS.

Proof. Suppose that F is stable, and let $x \in NE(F)$. To establish part (i), it is enough to show that $x \in GNSS(F)$. So fix an arbitrary $y \neq x$. Since F is stable,

$$(y-x)'\bigl(F(y)-F(x)\bigr) \leqslant 0. \tag{2}$$

And since $x \in NE(F)$, $(y-x)'F(x) \leqslant 0$. Adding these inequalities yields

$$(y-x)'F(y) \leqslant 0. \tag{3}$$

As y was arbitrary, x is a GNSS.

Turning to part (ii), suppose that F is strictly stable at x. Then inequality (2) holds strictly, so inequality (3) holds strictly as well. This means that x is a GESS of F, and hence the unique Nash equilibrium of F. $\square$

In geometric terms, population state x is a GESS if a small motion from any state $y \neq x$ in the direction of the payoff vector $F(y)$ (or of the projected payoff vector $\boldsymbol{\Phi} F(y)$) moves the state closer to x (see Fig. 1). If we allow not only these acute motions, but also orthogonal motions, we obtain the weaker notion of GNSS. This geometric interpretation of GESS and GNSS will be important for understanding the behavior of evolutionary dynamics in stable games.

4. Target dynamics and EPT dynamics

The set of Nash equilibria of any stable game is geometrically simple: it is convex, and typically a singleton. If a population of myopic agents recurrently play a stable game, will they learn to behave in accordance with Nash equilibrium?

Before pursuing this question, we should emphasize that uniqueness of equilibrium, while suggestive, does not imply any sort of stability under evolutionary dynamics. Indeed, most of the games used to illustrate the possibility of nonconvergence are games with a unique Nash equilibrium: see [16,26,30,33,52]. If convergence results can be established for stable games, it is not just a consequence of uniqueness of equilibrium; rather, the results must depend on the global structure of payoffs in these games.

The analysis to follow requires a few additional definitions. The *average payoff* in population p is given by $\bar{F}^p(x) = \frac{1}{m^p}\sum_{i\in S^p} x_i^p F_i^p(x)$. The *excess payoff* to strategy $i \in S^p$, defined by $\hat{F}_i^p(x) = F_i^p(x) - \bar{F}^p(x)$, is the difference between the strategy's payoff and the average payoff earned in population p. The *excess payoff vector* for population p is thus $\hat{F}^p(x) = F^p(x) - \mathbf{1}\bar{F}^p(x)$, where $\mathbf{1} \in \mathbf{R}^{n^p}$ is the vector of ones.

Also, let $\Delta^p = \{y \in \mathbf{R}_+^{n^p}: \sum_{i\in S^p} y_i^p = 1\}$ be the set of mixed strategies for population p. Then the map $B^p : X \Rightarrow \Delta^p$, defined by

$$B^p(x) = \operatorname*{argmax}_{y^p\in\Delta^p} \big(y^p\big)' F^p(x), \tag{4}$$

is population p's *best response correspondence*.

4.1. Revision protocols and evolutionary dynamics

We consider evolutionary dynamics derived from an explicit model of individual choice. This model is defined in terms of *revision protocols* $\rho^p : \mathbf{R}^{n^p} \times X^p \to \mathbf{R}_+^{n^p\times n^p}$, which describe the process through which agents in each population p make decisions. As time passes, agents are chosen at random from the population and granted opportunities to switch strategies. When a strategy $i \in S^p$ player receives an opportunity, he switches to strategy $j \in S^p$ with probability proportional to the *conditional switch rate* $\rho_{ij}^p(F^p(x), x^p)$, a rate that may depend on the payoff vector $F^p(x)$ and the population state x^p.

Aggregate behavior in population game F under protocol ρ is described by the dynamic

$$\dot{x}_i^p = \sum_{j\in S^p} x_j^p \rho_{ji}^p\big(F^p(x), x^p\big) - x_i^p \sum_{j\in S^p} \rho_{ij}^p\big(F^p(x), x^p\big). \tag{5}$$

The first term captures the inflow of agents into strategy i from other strategies, while the second term captures the outflow of agents from strategy i to other strategies. We sometimes write (5) as $\dot{x} = V_F(x)$ to emphasize the dependence of the law of motion on the underlying game.[9]

4.2. Target dynamics

Until the final section of the paper, we focus on revision protocols of the form

$$\rho_{ij}^p\big(\pi^p, x^p\big) = \tau_j^p\big(\pi^p, x^p\big).$$

In this formulation, the conditional switch rate from i to j is independent of the current strategy i. In this case, Eq. (5) takes the simpler form

$$\dot{x}_i^p = m^p \tau_i^p\big(F^p(x), x^p\big) - x_i^p \sum_{j\in S^p} \tau_j^p\big(F^p(x), x^p\big). \tag{T}$$

For reasons we explain next, we call evolutionary dynamics of this form *target dynamics*.

Target dynamics admit a simple geometric interpretation. If $\tau^p(F^p(x), x^p) \in \mathbf{R}_+^{n^p}$ is not the zero vector, we can let

$$\lambda^p\big(F^p(x), x^p\big) = \sum_{i\in S} \tau_i^p\big(F^p(x), x^p\big) \quad \text{and} \quad \sigma_i^p\big(F^p(x), x^p\big) = \frac{\tau_i^p(F^p(x), x^p)}{\lambda^p(F^p(x), x^p)},$$

[9] For more on the foundations of deterministic evolutionary dynamics, see [4,6,45,50].

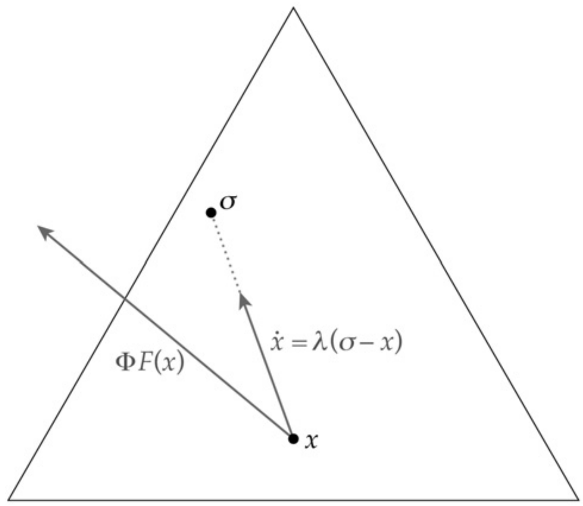

Fig. 2. Target dynamics and positive correlation.

and rewrite Eq. (T) as

$$\dot{x}^p = \begin{cases} \lambda^p(F^p(x), x^p)(m^p\sigma^p(F^p(x), x^p) - x^p) & \text{if } \tau^p(F^p(x), x^p) \neq \mathbf{0}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{6}$$

Eq. (6) tells us that the population state x^p always moves in the direction of the target state $m^p\sigma^p(F^p(x), x^p)$, with motion toward the latter state proceeding at rate $\lambda^p(F^p(x), x^p)$. Fig. 2 illustrates this idea in the single population case.

4.3. *EPT dynamics*

We further restrict the class of dynamics under consideration by only allowing conditional switch rates to depend on the vector of excess payoffs:

$$\tau_j^p(\pi^p, x^p) = \tau_j^p(\hat{\pi}^p),$$

where $\hat{\pi}_i^p = \pi_i^p - \frac{1}{m^p}(x^p)'\pi^p$ represents the excess payoff to strategy $i \in S^p$. The target dynamic (T) then take the simpler form

$$\dot{x}_i^p = m^p\tau_i^p(\hat{F}^p(x)) - x_i^p \sum_{j \in S^p} \tau_j^p(\hat{F}^p(x)). \tag{E}$$

We call dynamics of this form *excess payoff/target dynamics*, or *EPT dynamics* for short.

It is worth observing that three of the most-studied dynamics in the literature are EPT dynamics. If the protocol τ^p is of the form

$$\tau_j^p(\hat{\pi}^p) = [\hat{\pi}_j^p]_+,$$

then agents only switch to strategies with positive excess payoffs, doing so at rates proportional to the magnitudes of excess payoffs. In this case, Eq. (E) becomes the *Brown–von Neumann–Nash dynamic* [8].[10] If instead that the protocol is of the form $\tau^p \equiv M^p$, where M^p is the *maximizer correspondence*

$$M^p(\hat{\pi}^p) = \operatorname*{argmax}_{y^p \in \Delta^p} (y^p)'\hat{\pi}^p,$$

[10] See also [25,53,56,58].

Table 1
Three EPT dynamics and their revision protocols.

Revision protocol	Evolutionary dynamic	Name of dynamic
$\tau_j^p(\hat{\pi}^p) = [\hat{\pi}_j^p]_+$	$\dot{x}_i^p = m^p[\hat{F}_i^p(x)]_+ - x_i^p \sum_{j\in S^p}[\hat{F}_j^p(x)]_+$	BNN
$\tau^p(\hat{\pi}^p) \in M^p(\hat{\pi}^p)$	$\dot{x}^p \in m^p B^p(x) - x^p$	best response
$\tau^p(\hat{\pi}^p) = L^p(\hat{\pi}^p)$	$\dot{x}_i^p = m^p \dfrac{\exp(\eta^{-1}F_i^p(x))}{\sum_{j\in S^p}\exp(\eta^{-1}F_j^p(x))} - x_i^p$	logit

Eq. (E) becomes the *best response dynamic* [17]. Finally, if $\tau^p \equiv L^p$ the *logit choice function* with *noise level* $\eta > 0$,

$$L_i^p\big(\hat{\pi}^p\big) = \frac{\exp(\eta^{-1}\hat{\pi}_i^p)}{\sum_{j\in S^p}\exp(\eta^{-1}\hat{\pi}_j^p)},$$

then Eq. (E) becomes the *logit dynamic* [15]. We summarize these three examples in Table 1.[11]

4.4. Nash stationarity and positive correlation

In order to obtain convergence results for classes of evolutionary dynamics, one needs to introduce properties that connect the behavior of the dynamics with payoffs in the underlying game. If we let $\dot{x} = V_F(x)$ denote the law of motion associated with game F, we can define two standard properties of evolutionary dynamics as follows:

(NS) *Nash stationarity*: $V_F(x) = \mathbf{0}$ if and only if $x \in NE(F)$.
(PC) *Positive correlation*: $V_F^p(x) \neq \mathbf{0}$ implies that $V_F^p(x)'F^p(x) > 0$.

In words, Nash stationarity (NS) states that the rest points of the dynamic are always the Nash equilibria of the underlying game. For its part, positive correlation (PC) is a mild monotonicity condition. It requires that whenever a population is not at rest, there is always an acute angle between the current direction of motion and the current payoff vector. We illustrate property (PC) in the single population case in Fig. 2.[12]

Sandholm [46] identifies conditions under which EPT dynamics exhibit these two properties.

Proposition 4.1 *(Sandholm [46]). Suppose the EPT dynamic* (E) *is derived from a revision protocol τ^p that is Lipschitz continuous and* acute:

$$\tau^p\big(\hat{\pi}^p\big)'\hat{\pi}^p > 0 \quad \textit{whenever } \hat{\pi}^p \in \mathbf{R}^{n^p} - \mathbf{R}_-^{n^p}. \tag{A}$$

Then (E) *satisfies Nash stationarity* (NS) *and positive correlation* (PC).

[11] The formulas for the best response and logit dynamics are easy to derive once one observes that $M^p(\hat{\pi}^p) = M^p(\pi^p)$ and that $L^p(\hat{\pi}^p) = L^p(\pi^p)$: subtracting the same constant from each strategy's payoff affects neither the optimal strategies nor the logit choice probabilities.

[12] Because we are representing a payoff vector of a three-strategy game in a two-dimensional picture, we draw the projected payoff vector $\Phi F(x) \in TX$ in place of the actual payoff vector $F(x)$. Since $V(x) \in TX$, including the projection Φ does not affect the inner product in condition (PC).

Sandholm [46] calls EPT dynamics that satisfy the conditions of Proposition 4.1 *excess payoff dynamics*; the BNN dynamic is the leading example of this class. The best response and logit dynamics are not excess payoff dynamics: the protocol that defines the former dynamic is discontinuous, while the protocol for the latter is not acute.[13]

In potential games, property (PC) is enough to guarantee that an evolutionary dynamic converges to equilibrium from all initial conditions, as it ensures that the game's potential function serves as a Lyapunov function for the dynamic at issue. One might hope that in like fashion, a general convergence result for stable games could be proved based on monotonicity of the dynamic alone. Unfortunately, this is not the case: in Section 6, we construct an example of an excess payoff dynamic that does not converge in a stable game. Additional structure is needed to establish the desired convergence results.

4.5. Integrable revision protocols

We obtain this structure by imposing an additional condition on revision protocols: *integrability*.

$$\text{There exists a } C^1 \text{ function } \gamma^p : \mathbf{R}^{n^p} \to \mathbf{R} \text{ such that } \tau^p \equiv \nabla\gamma^p. \tag{I}$$

We call the functions γ^p introduced in this condition *revision potentials*.

To give this condition a behavioral interpretation, it is useful to compare it to *separability*:

$$\tau_i^p\left(\hat{\pi}^p\right) \text{ is independent of } \hat{\pi}_{-i}^p. \tag{S}$$

The latter condition is stronger than the former: if τ^p satisfies (S), then it satisfies (I) with

$$\gamma^p\left(\hat{\pi}^p\right) = \sum_{i \in S^p} \int_0^{\hat{\pi}_i^p} \tau_i^p(s)\,\mathrm{d}s. \tag{7}$$

Building on this motivation, Sandholm [48] provides a game-theoretic interpretation of integrability. Roughly speaking, integrability (I) is equivalent to a requirement that in expectation, learning the weight placed on strategy j does not convey information about other strategies' excess payoffs. It thus generalizes separability (S), which requires that learning the weight placed on strategy j conveys no information at all about other strategies' excess payoffs.

The results in the next section show that in combination, monotonicity and integrability are sufficient to ensure global convergence in stable games. In Section 6, we provide examples that explain the role played by integrability in establishing convergence results.

5. Global convergence of EPT dynamics

We noted earlier that in potential games, the potential function serves as a global Lyapunov function for many evolutionary dynamics. Stable games, in contrast, do not come equipped with candidate Lyapunov functions. But if the revision protocol agents follow is integrable, then the revision potential of this protocol provides a building block for constructing a suitable Lyapunov function. Evidently, this Lyapunov function will vary with the dynamic under study, even when the game under consideration is fixed.

[13] Still, the logit dynamic satisfies an analogue of positive correlation (PC) called *virtual positive correlation*; see Hofbauer and Sandholm [27] and Section 5.2 below.

Recall that a *Lyapunov function* $\Lambda : X \to \mathbf{R}$ for the closed set $A \subseteq X$ is a continuous function whose set of minimizers is A and that is nonincreasing along solutions of (5). If the value of Λ is decreasing outside of A, Λ is called a *strict Lyapunov function*. A variety of classical results from dynamical systems show that the existence of a suitable Lyapunov function implies various forms of stability for the set A. Definitions of the relevant notions of stability as well as precise statements of the results we need are offered in Appendix A.1.

5.1. *Integrable excess payoff dynamics*

Our first result concerns *integrable excess payoff dynamics*: that is, EPT dynamics (E) whose protocols τ^p are Lipschitz continuous, acute, and integrable. The prototype for this class is the BNN dynamic: its protocol $\tau_i^p(\hat{\pi}^p) = [\hat{\pi}_i^p]_+$ is not only acute and integrable, but also separable, and so admits potential function $\gamma^p(\hat{\pi}^p) = \frac{1}{2}\sum_{i \in S^p}[\hat{\pi}_i^p]_+^2$ (cf. Eq. (7)).

Theorem 5.1. *Let F be a C^1 stable game, and let V_F be the evolutionary dynamic for F defined by Eq.* (E) *from protocols τ^p that are Lipschitz continuous, acute* (A)*, and integrable* (I)*. Define the C^1 function $\Gamma : X \to \mathbf{R}$ by*

$$\Gamma(x) = \sum_{p \in \mathcal{P}} m^p \gamma^p\big(\hat{F}^p(x)\big).$$

Then $\dot{\Gamma}(x) \leqslant 0$ for all $x \in X$, with equality if and only if $x \in NE(F)$. Thus $NE(F)$ is globally attracting, and if $NE(F)$ is a singleton it is globally asymptotically stable. If each τ^p is also separable (S)*, then Γ is nonnegative with $\Gamma^{-1}(0) = NE(F)$, and so $NE(F)$ is globally asymptotically stable.*

For future reference, observe that the value of the Lyapunov function Γ at state x is the (m^p-weighted) sum of the values of the revision potentials γ^p evaluated at the excess payoff vectors $\hat{F}^p(x)$. The proof of Theorem 5.1 can be found in Appendix A.2.

5.2. *Perturbed best response dynamics*

Next, we revisit a convergence result of Hofbauer and Sandholm [27] for perturbed best response dynamics in the light of the present analysis. Call the function $v^p : \mathrm{int}(\Delta^p) \to \mathbf{R}$ an *admissible deterministic perturbation* if it is differentiably strictly convex and infinitely steep at the boundary of Δ^p. The perturbation v^p induces a *perturbed maximizer function* $\tilde{M}^p : \mathbf{R}^n \to \mathrm{int}(\Delta^p)$ defined by

$$\tilde{M}^p\big(\pi^p\big) = \operatorname*{argmax}_{y^p \in \mathrm{int}(\Delta^p)} \big(y^p\big)'\pi^p - v^p\big(y^p\big).$$

The *perturbed best response dynamic* [15] associated with these functions is

$$\dot{x}^p = m^p \tilde{M}^p\big(\hat{F}^p(x)\big) - x^p. \tag{PBR}$$

The prototype for this class of dynamics, the logit dynamic, is obtained when v^p is the negated entropy function $v^p(y^p) = \eta \sum_{j \in S^p} y_j^p \log y_j^p$.

Following the previous logic, we can assess the possibilities for convergence in stable games by checking monotonicity and integrability. For the former, we note that while perturbed best response dynamics (PBR) do not satisfy positive correlation (PC), they do satisfy an analogue called *virtual positive correlation*.[14] Moreover, the protocol $\tau^p = \tilde{M}^p$ is integrable; its revision potential,

$$\tilde{\mu}^p(\pi^p) = \max_{y^p \in \operatorname{int}(\Delta^p)} (y^p)'\pi^p - v^p(y^p),$$

is the *perturbed maximum function* induced by v^p.

Mimicking Theorem 5.1, we can attempt to construct Lyapunov functions for (PBR) by composing the revision potentials $\tilde{\mu}^p$ with the excess payoff functions $\hat{F}^p$. After some manipulation, the Lyapunov function derived in [27] can be shown to be of this form, modulo the addition of perturbation terms.

Theorem 5.2 *(Hofbauer and Sandholm [27]). Let F be a C^1 stable game, and consider a perturbed best response dynamic* (PBR) *for F. Define the C^1 function $\tilde{G} : X \to \mathbf{R}$ by*

$$\tilde{G}(x) = \sum_{p \in \mathcal{P}} m^p \left(\tilde{\mu}^p\big(\hat{F}^p(x)\big) + v^p\left(\frac{1}{m^p} x^p\right)\right).$$

Then $\dot{\tilde{G}}(x) \leqslant 0$ for all $x \in X$, with equality if and only if $x = x^*$, the unique rest point of (PBR). *Thus, x^* is globally asymptotically stable.*

5.3. The best response dynamic

Finally, we consider the best response dynamic. As we saw in Section 4.3, (BR) is the EPT dynamic obtained by using the maximizer correspondence

$$M^p(\hat{\pi}^p) = \operatorname*{argmax}_{y^p \in \Delta^p} (y^p)'\hat{\pi}^p$$

as the revision protocol. Following Hofbauer [24], we formulate this dynamic as the differential inclusion

$$\dot{x}^p \in m^p M^p\big(\hat{F}^p(x)\big) - x^p. \tag{BR}$$

Let us now check the two conditions for convergence. Concerning monotonicity, one can show that (BR) satisfies a version of positive correlation (PC) appropriate for differential inclusions.[15] Moreover, the protocol M^p, despite being multivalued, is integrable in a suitably defined sense, with its "potential function" being given by the *maximum function*

$$\mu^p(\pi^p) = \max_{y^p \in \Delta^p} (y^p)'\pi^p = \max_{i \in S^p} \pi_i^p.$$

Note that if the payoff vector π^p, and hence the excess payoff vector $\hat{\pi}^p$, have a unique maximizing component $i \in S^p$, then the gradient of μ^p at $\hat{\pi}^p$ is the standard basis vector e_i^p. But this vector corresponds to the unique mixed best response to $\hat{\pi}^p$, and so

$$\nabla \mu^p(\hat{\pi}^p) = e_i^p = M^p(\hat{\pi}^p).$$

[14] *Virtual positive correlation* requires that $V_F^p(x)'\tilde{F}^p(x) > 0$ whenever $V_F^p(x) \neq \mathbf{0}$, where the *virtual payoff* $\tilde{F}^p$ is defined by $\tilde{F}^p(x) = F^p(x) - \nabla v^p(\frac{1}{m^p}x^p)$.

[15] See the last three lines of display (A.3) in Appendix A.3.

One can account for multiple optimal components using a broader notion of differentiation: for all $\hat{\pi}^p \in \mathbf{R}^n$, $M^p(\hat{\pi}^p)$ is the subdifferential of the convex function μ^p at $\hat{\pi}^p$.[16]

Having verified monotonicity and integrability, we again construct our candidate Lyapunov function by plugging the excess payoff vectors into the revision potentials μ^p. The resulting function G is very simple: it measures the difference between the payoffs agents could obtain by choosing optimal strategies and their actual aggregate payoffs.

Theorem 5.3. *Let F be a C^1 stable game, and consider the best response dynamic* (BR) *for F. Define the Lipschitz continuous function $G : X \to \mathbf{R}$ by*

$$G(x) = \sum_{p \in \mathcal{P}} m^p \mu^p\big(\hat{F}^p(x)\big) = \max_{y \in X} (y - x)' F(x).$$

Then G is nonnegative with $G^{-1}(0) = NE(F)$. Moreover, if $\{x_t\}_{t \geqslant 0}$ is a solution to (BR)*, then for almost all $t \geqslant 0$ we have that $\dot{G}(x_t) \leqslant -G(x_t)$. Therefore, $NE(F)$ is globally asymptotically stable under* (BR).

Because (BR) is a discontinuous differential inclusion, the proof of Theorem 5.3 requires a subtle technical argument. This proof is presented in Appendix A.3.

6. How integrability fosters convergence

In this section, we explore the role of integrability in establishing global convergence in stable games. We begin by introducing an excess payoff dynamic that is not integrable and that leads to cycling in some stable games. This example allows us to identify what property of dynamics beyond monotonicity is needed to prove a general convergence result. The remainder of the section explains how integrability gives us this missing property.

Example 6.1. To keep the analysis of cycling as simple as possible, we focus on an elementary game: standard Rock–Paper–Scissors (cf. Example 2.3).[17] When the benefit of winning a match and the cost of losing a match both equal 1, random matching in standard RPS generates the population game

$$F(x) = \begin{pmatrix} F_R(x) \\ F_P(x) \\ F_S(x) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \end{pmatrix} = \begin{pmatrix} x_S - x_P \\ x_R - x_S \\ x_P - x_R \end{pmatrix}, \tag{8}$$

whose unique Nash equilibrium is $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Fig. 3 plots a selection of payoff vectors of standard RPS, along with the Nash equilibrium x^*.[18] At each state y, the payoff vector $F(y)$ is orthogonal to the segment from y to x^*, reflecting the facts that F is null stable and that x^* is a GNSS.

[16] See [22], especially Example D.3.4.

[17] A similar analysis can be used to establish cycling in bad RPS games: see the online appendix.

[18] Since the columns of the payoff matrix of standard RPS sum to zero, projected payoffs $\Phi F(x) = (I - \frac{1}{n}\mathbf{1}\mathbf{1}')F(x)$ equal payoffs $F(x)$ in this game.

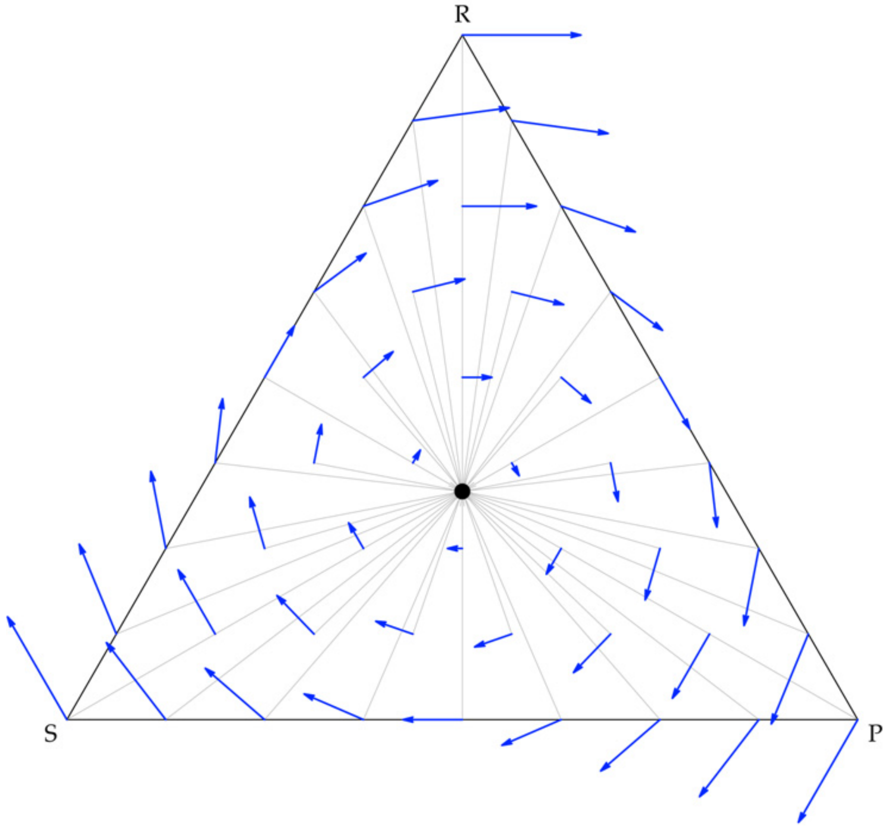

Fig. 3. Payoff vectors and GNSS in standard RPS.

Fix $\varepsilon > 0$, and let $g^\varepsilon : \mathbf{R} \to \mathbf{R}$ be a continuous decreasing function that equals 1 on $(-\infty, 0]$, equals ε^2 on $[\varepsilon, \infty)$, and is linear on $[0, \varepsilon]$. Then define the revision protocol τ by

$$\begin{pmatrix} \tau_R(\hat{\pi}) \\ \tau_P(\hat{\pi}) \\ \tau_S(\hat{\pi}) \end{pmatrix} = \begin{pmatrix} [\hat{\pi}_R]_+ g^\varepsilon(\hat{\pi}_S) \\ [\hat{\pi}_P]_+ g^\varepsilon(\hat{\pi}_R) \\ [\hat{\pi}_S]_+ g^\varepsilon(\hat{\pi}_P) \end{pmatrix}. \tag{9}$$

Under this protocol, the weight placed on a strategy is proportional to positive part of the strategy's excess payoff, as in the protocol for the BNN dynamic; however, this weight is only of order ε^2 if the strategy it beats in RPS has an excess payoff greater than ε.

It is easy to verify that protocol (9) is acute:

$$\tau(\hat{\pi})'\hat{\pi} = [\hat{\pi}_R]_+^2 g^\varepsilon(\hat{\pi}_S) + [\hat{\pi}_P]_+^2 g^\varepsilon(\hat{\pi}_R) + [\hat{\pi}_S]_+^2 g^\varepsilon(\hat{\pi}_P),$$

which is positive when $\hat{\pi} \in \mathbf{R}^n - \mathbf{R}^n_-$. Therefore, Proposition 4.1 implies that the corresponding EPT dynamic (E) satisfies Nash stationarity (NS) and positive correlation (PC). Nevertheless, we show in the online appendix that in the Rock–Paper–Scissors game (8), this dynamic enters limit cycles from many initial conditions when ε is less than $\frac{1}{10}$.

Protocol (9) has this noteworthy feature: the weights agents place on each strategy depend systematically on the payoffs of the next strategy in the best response cycle. Of course, this could not be if the protocol were separable in the sense of condition (S). The results in the previous section verify that supplementing monotonicity with integrability (I), a generalization of separability, is enough to ensure that cycling does not occur.

To provide a deeper understanding why cycling occurs in Example 6.1, it will be helpful to review some earlier definitions and results. In a stable game, every Nash equilibrium x^* is a

GNSS. Geometrically, this means that at every nonequilibrium state x, the projected payoff vector $\Phi F(x)$ forms an acute or right angle with the segment leading back to x^* (Figs. 1 and 3). Meanwhile, our monotonicity condition for dynamics, positive correlation (PC), requires that away from equilibrium, the direction of motion forms an acute angle with the projected payoff vector (Fig. 2). Combining these observations, we see that if the law of motion $\dot{x} = V_F(x)$ tends to deviate from the projected payoffs ΦF in an "outward" direction—that is, in a direction heading away from equilibrium—then cycling will occur. On the other hand, if the deviations of V_F from ΦF tend to be "inward," then solutions should converge to equilibrium.

By this logic, we should be able to guarantee convergence of EPT dynamics in stable games by ensuring that the deviations of V_F from ΦF are toward the equilibrium, at least in some average sense. To make this link, let us recall a well-known characterization of integrability: the map $\tau : \mathbf{R}^n \to \mathbf{R}^n$ is integrable if and only if its line integral over any piecewise smooth closed curve $C \subset \mathbf{R}^n$ evaluates to zero:

$$\oint_C \tau(\hat{\pi}) \cdot \mathrm{d}\hat{\pi} = 0. \tag{10}$$

Example 6.2. As in Example 6.1, let the population game F be generated by random matching in standard RPS, as defined in Eq. (8). The unique Nash equilibrium of F is the GNSS $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Game F has the convenient property that at each state $x \in X$, the payoff vector $F(x)$, the projected payoff vector $\Phi F(x)$, and the excess payoff vector $\hat{F}(x)$ are all the same, a fact that will simplify the notation in the argument to follow.

Since F is null stable, we know that at each state $x \neq x^*$, the payoff vector $F(x)$ is orthogonal to the vector $x^* - x$. In Fig. 3, these payoff vectors point clockwise relative to x^*. Since positive correlation (PC) requires that the direction of motion $V_F(x)$ form an acute angle with $F(x)$, dynamics satisfying (PC) also travel clockwise around the equilibrium.

To address whether the deviations of V_F from F tend to be inward or outward, let $C \subset X$ be a circle of radius $c \in (0, \frac{1}{\sqrt{6}}]$ centered at the equilibrium x^*. This circle is parameterized by the function $\xi : [0, 2\pi] \to X$, where

$$\xi_\alpha = \frac{c}{\sqrt{6}} \begin{pmatrix} -2\sin\alpha \\ \sqrt{3}\cos\alpha + \sin\alpha \\ -\sqrt{3}\cos\alpha + \sin\alpha \end{pmatrix} + x^*. \tag{11}$$

Here α is the clockwise angle between the vector $\xi_\alpha - x^*$ and a rightward horizontal vector (see Fig. 4).

Since state ξ_α lies on the circle C, the vector $x^* - \xi_\alpha$ can be drawn as a radius of C; thus, the payoff vector $\pi_\alpha \equiv F(\xi_\alpha)$, which is orthogonal to $x^* - \xi_\alpha$, must be tangent to C at ξ_α, as shown in Fig. 4. This observation is easy to verify analytically:

$$\pi_\alpha = F(\xi_\alpha) = \frac{c}{\sqrt{6}} \begin{pmatrix} -2\sqrt{3}\cos\alpha \\ -3\sin\alpha + \sqrt{3}\cos\alpha \\ 3\sin\alpha + \sqrt{3}\cos\alpha \end{pmatrix} = \sqrt{3}\frac{\mathrm{d}}{\mathrm{d}\alpha}\xi_\alpha. \tag{12}$$

If we differentiate both sides of identity (12) with respect to the angle α, and note that $\frac{\mathrm{d}^2}{(\mathrm{d}\alpha)^2}\xi_\alpha = -(\xi_\alpha - x^*)$, we can link the rate of change of the payoff vector $\pi_\alpha = F(\xi_\alpha)$ to the displacement of state ξ_α from x^*:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\pi_\alpha = \sqrt{3}\frac{\mathrm{d}^2}{(\mathrm{d}\alpha)^2}\xi_\alpha = -\sqrt{3}\big(\xi_\alpha - x^*\big). \tag{13}$$

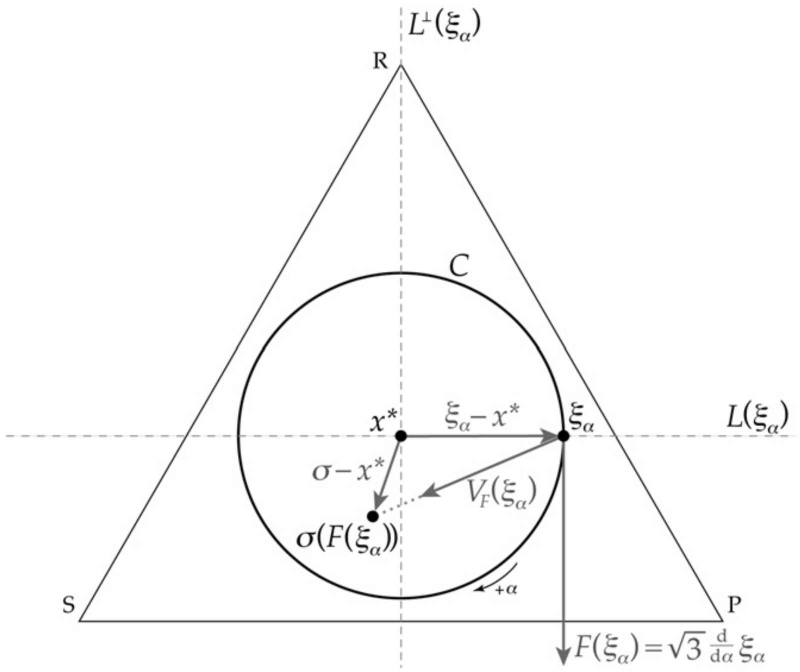

Fig. 4. Integrability and inward motion of EPT dynamics in standard RPS.

Now introduce an acute, integrable revision protocol τ. By combining integrability condition (10) with Eq. (13), we obtain

$$0 = \oint_C \tau(\pi) \cdot \mathrm{d}\pi \equiv \int_0^{2\pi} \tau(\pi_\alpha)' \left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\pi_\alpha\right) \mathrm{d}\alpha = -\sqrt{3} \int_0^{2\pi} \tau(\pi_\alpha)' \left(\xi_\alpha - x^*\right) \mathrm{d}\alpha. \tag{14}$$

If we write $\lambda(\pi) = \sum_{i \in S} \tau_i(\pi)$ and $\sigma_i(\pi) = \frac{\tau_i(\pi)}{\lambda(\pi)}$ as in Section 4.2, then because $\xi_\alpha - x^* \in TX$ is orthogonal to $x^* = \frac{1}{3}\mathbf{1}$, we can conclude from Eq. (14) that

$$\int_0^{2\pi} \lambda\left(F(\xi_\alpha)\right)\left(\sigma\left(F(\xi_\alpha)\right) - x^*\right)'\left(\xi_\alpha - x^*\right) \mathrm{d}\alpha = 0. \tag{15}$$

Eq. (15) is a form of the requirement described at the start of this section: it asks that at states on the circle C, the vector of motion under the EPT dynamic

$$\dot{x} = V_F(x) = \lambda\left(F(x)\right)\left(\sigma\left(F(x)\right) - x\right) \tag{16}$$

typically deviates from the payoff vector $F(x)$ in an inward direction—that is, in the direction of the equilibrium x^*.

To reach this interpretation of Eq. (15), note first that if the target state $\sigma(F(\xi_\alpha))$ lies on or even near line $L^{\perp}(\xi_\alpha)$, then motion from ξ_α toward $\sigma(F(\xi_\alpha))$ is initially inward, as shown in Fig. 4.[19] Now, the integrand in (15) contains the inner product of the vectors $\sigma(F(\xi_\alpha)) - x^*$ and $\xi_\alpha - x^*$. This inner product is zero precisely when then the two vectors are orthogonal, or, equivalently, when target state $\sigma(F(\xi_\alpha))$ lies on $L^{\perp}(\xi_\alpha)$. While Eq. (15) does not require the

19 Target state $\sigma(F(\xi_\alpha))$ lies below $L(\xi_\alpha)$ by virtue of positive correlation (PC), which in turn follows from the acuteness of τ—see Proposition 4.1.

two vectors to be orthogonal, it asks that this be true on average, where the average is taken over states $\xi_\alpha \in C$, and weighted by the rates $\lambda(F(\xi_\alpha))$ at which ξ_α approaches $\sigma(F(\xi_\alpha))$. Thus, in the presence of acuteness, integrability implies that on average, the dynamic (16) tends to point inward, toward the equilibrium x^*.

7. Other convergence results

In this final section of the paper, we present convergence results for stable games for dynamics that are not of the target form (T).

7.1. Pairwise comparison dynamics

Let us consider revision protocols based on pairwise comparisons of payoffs:

$$\rho_{ij}^p\left(\pi^p, x^p\right) = \phi_{ij}^p\left(\pi_j^p - \pi_i^p\right), \quad \phi_{ij}^p : \mathbf{R} \to \mathbf{R}_+ \text{ Lipschitz continuous.}$$

Note that unlike the target protocols considered in the previous sections, the protocol considered here conditions on the agent's current strategy. Substituting this protocol into Eq. (5) yields dynamics of the form

$$\dot{x}_i^p = \sum_{j \in S^p} x_j^p \phi_{ji}^p\left(F_i^p(x) - F_j^p(x)\right) - x_i^p \sum_{j \in S^p} \phi_{ij}^p\left(F_j^p(x) - F_i^p(x)\right). \tag{17}$$

If one sets $\phi_{ij}^p(\pi_j^p - \pi_i^p) = [\pi_j^p - \pi_i^p]_+$, then Eq. (17) becomes the Smith dynamic [55]. More generally, Sandholm [50] calls (17) a *pairwise comparison dynamic* if the protocols ϕ^p satisfy *sign preservation*:

$$\operatorname{sgn}\left(\phi_{ij}^p\left(\pi_j^p - \pi_i^p\right)\right) = \operatorname{sgn}\left(\left[\pi_j^p - \pi_i^p\right]_+\right). \tag{SP}$$

In words: the conditional switch rate from $i \in S^p$ to $j \in S^p$ is positive if and only if j earns a higher payoff than i. Sandholm [50] shows that like excess payoff dynamics, pairwise comparison dynamics satisfy Nash stationarity (NS) and positive correlation (PC), and so converge in potential games.

Smith [55] proves that his dynamic converges to Nash equilibrium from all initial conditions in every stable game. We now show that while global convergence in stable games does need not occur under all pairwise comparison dynamics, it does obtain if an additional condition is satisfied. We call this condition *impartiality*:

$$\phi_{ij}^p\left(\pi_j^p - \pi_i^p\right) = \phi_j^p\left(\pi_j^p - \pi_i^p\right) \quad \text{for some functions } \phi_j^p : \mathbf{R} \to \mathbf{R}_+. \tag{18}$$

Impartiality requires that the function of the payoff difference that describes the conditional switch rate from i to j does not depend on an agent's current strategy i. This condition introduces at least a superficial connection with EPT dynamics (E), as both restrict the dependence of agents' decisions on their current choices of strategy.

Theorem 7.1 shows that when paired with the monotonicity condition (SP), incumbent independence ensures global convergence to Nash equilibrium in stable games.

Theorem 7.1. *Let F be a C^1 stable game, and let* (17) *be an impartial pairwise comparison dynamic for F. Define the C^1 function $\Psi : X \to \mathbf{R}_+$ by*

$$\Psi(x) = \sum_{p \in \mathcal{P}} \sum_{i \in S^p} \sum_{j \in S^p} x_i^p \psi_j^p\left(F_j^p(x) - F_i^p(x)\right), \quad \textit{where } \psi_k^p(d) = \int_0^d \phi_k^p(s)\,\mathrm{d}s$$

is the definite integral of ϕ_k^p. Then Ψ is nonnegative with $\Psi^{-1}(0) = NE(F)$. Moreover, $\dot{\Psi}(x) \leqslant 0$ for all $x \in X$, with equality if and only if $x \in NE(F)$, and so $NE(F)$ is globally asymptotically stable under (17).

The proof of Theorem 7.1 is presented in Appendix A.4.

To understand the role played by impartiality, notice that according to Eq. (17), the rate of outflow from strategy i under a pairwise comparison dynamic is $x_i^p \sum_{k\in S^p} \phi_{ik}^p(\pi_k^p - \pi_i^p)$. Thus, the *percentage* rate of outflow from i, $\sum_{k\in S^p} \phi_{ik}^p(\pi_k^p - \pi_i^p)$, varies with i.[20] It follows that strategies with high payoffs can nevertheless have high percentage outflow rates: even if $\pi_i^p > \pi_j^p$, one can still have $\phi_{ik}^p > \phi_{jk}^p$ for $k \neq i, j$. Having good strategies lose players more quickly than bad strategies is an obvious impediment to convergence to Nash equilibrium.

Impartiality places controls on these percentage outflow rates. If the conditional switch rates ϕ_j^p are monotone in payoffs, then impartiality ensures that better strategies have lower percentage outflow rates. If the conditional switch rates are not monotone, but merely sign-preserving (SP), impartiality still implies that the *integrated* conditional switch rates ψ_k^p are ordered by payoffs. According to our analysis, this control is enough to ensure convergence of pairwise comparison dynamics to Nash equilibrium in stable games.

7.2. The replicator and projection dynamics

Convergence results for stable games are also available for two other evolutionary dynamics. A simple extension of existing results[21] shows that in strictly stable games, interior solutions of the replicator dynamic (Taylor and Jonker [57]) reduce "distance" from and eventually converge to the unique Nash equilibrium x^*, where the notion of "distance" from equilibrium is defined in terms of relative entropies (see Table 2 below). Under the projection dynamic of Nagurney and Zhang [41], Euclidean distance from x^* decreases along each solution trajectory, again ensuring convergence. However, unlike the convergence results developed here, the convergence results for the replicator and projection dynamics do not seem robust to changes in the underlying revision protocols. Moreover, the stability results are weaker for these two dynamics: asymptotic stability is guaranteed only for strictly stable games, whereas for null stable games only Lyapunov stability holds; indeed, closed orbits are known to occur in the latter case. See Lahkar and Sandholm [35] and Sandholm et al. [51] for further discussion.

7.3. Summary

In Table 2, we summarize the convergence results for six basic evolutionary dynamics in stable games by presenting the Lyapunov function for each dynamic for the single population case. Each of the dynamics listed in the table is also known to converge in potential games. While convergence in potential games follows immediately from the monotonicity of the dynamics, this paper has shown that additional structure is beyond monotonicity is needed to prove convergence in stable games.

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20 By contrast, under excess payoff dynamics all strategies' percentage outflow rates are the same and equal to $\sum_{k\in S^p} \tau_k^p(\hat{F}^p(x))$, while under the best response dynamic these rates are all fixed at 1.

21 See [28,60], [1, Theorem 6.4], [2, Section 1.4], and [10].

Table 2
Lyapunov functions for six basic deterministic dynamics in stable games.

Dynamic	Lyapunov function ($p = 1$)
BNN	$\Gamma(x) = \frac{1}{2}\sum_{i\in S}[\hat{F}_i(x)]_+^2$
Best response	$G(x) = \max_{i\in S} \hat{F}_i(x)$
Logit	$\tilde{G}(x) = \max_{y\in \operatorname{int}(X)}(y'\hat{F}(x) - \eta\sum_{i\in S} y_i \log y_i) + \eta\sum_{i\in S} x_i \log x_i$
Smith	$\Psi(x) = \frac{1}{2}\sum_{i\in S}\sum_{j\in S} x_i[F_j(x) - F_i(x)]_+^2$
Replicator	$H_{x^*}(x) = \sum_{i\in S(x^*)} x_i^* \log \frac{x_i^*}{x_i}$
Projection	$E_{x^*}(x) = \lvert x - x^*\rvert^2$

Acknowledgments

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Appendix A

A.1. Stability for dynamical systems via Lyapunov functions

We call the closed set $A \subseteq X$ *Lyapunov stable* if for every neighborhood O of A there exists a neighborhood O' of A such that every solution $\{x_t\}_{t\geqslant 0}$ to (5) that starts in O' is contained in O: that is, $x_0 \in O'$ implies that $x_t \in O$ for all $t \geqslant 0$. A is *attracting* if there is a neighborhood Q of A such that every solution to (5) that starts in Q converges to A: that is, $x_0 \in Q$ implies that $\lim_{t\geqslant 0} \operatorname{dist}(x_t, A) = 0$. A is *globally attracting* if it is attracting with $Q = X$. Finally, the set A is *asymptotically stable* if it is Lyapunov stable and attracting, and it is *globally asymptotically stable* if it is Lyapunov stable and globally attracting.

We now state three auxiliary theorems that establish global and local stability results for dynamics that are forward invariant on the compact set X and that admit Lyapunov functions. We consider dynamics satisfying one of these two regularity conditions:

(D1) $\dot{x} = V(x)$, where V is Lipschitz continuous;
(D2) $\dot{x} \in V(x)$, where V is upper hemicontinuous, nonempty, compact valued, and convex valued.

Theorem A.1. *Let $\tilde{X} \subseteq X$, and let the C^1 function $L : \tilde{X} \to \mathbf{R}_+$ be such that $L(x)$ approaches infinity whenever x approaches $X - \tilde{X}$. Let $\{x_t\}_{t\geqslant 0}$ be a solution of* (D1) *with $x_0 \in \tilde{X}$. If $\dot{L}(x_t) \leqslant 0$ for all $t \in [0, \infty)$, then $\{x_t\}_{t\geqslant 0}$ converges to the set $\{x \in \tilde{X}\colon\ \dot{L}(x) = 0\}$.*

Theorem A.2. *Let $L : X \to \mathbf{R}_+$ be Lipschitz continuous. Let $\{x_t\}_{t\geqslant 0}$ be a solution of* (D2)*. If $\dot{L}(x_t) \leqslant -L(x_t)$ for almost all $t \in [0, \infty)$, then $\{x_t\}_{t\geqslant 0}$ converges to the set $\{x \in X\colon\ L(x) = 0\}$.*

Theorem A.3. *Let $A \subseteq X$ be closed, and let $O \subseteq X$ be a neighborhood of A. Let $L : O \to \mathbf{R}_+$ be Lipschitz continuous and satisfy $L^{-1}(0) = A$. If each solution $\{x_t\}_{t\geqslant 0}$ of* (D1) *(or of* (D2)*) satisfies $\dot{L}(x_t) \leqslant 0$ for almost all $t \in [0, \infty)$, then A is Lyapunov stable.*

Each of these theorems is an easy extension of standard results. Theorems A.1 and A.2 are versions of Theorem 7.6 of Hofbauer and Sigmund [29], and Theorem A.3 is a version of Theorem 9.3.1 of Hirsch and Smale [23]. (To prove Theorem A.2, note that the conditions of the theorem imply that

$$L(x_t) = L(x_0) + \int_0^t \dot{L}(x_u)\,\mathrm{d}u \leqslant L(x_0) + \int_0^t \bigl(-L(x_u)\bigr)\,\mathrm{d}u = L(x_0)\mathrm{e}^{-t},$$

where the final equality follows from the fact that $\alpha_0 + \int_0^t (-\alpha_u)\,\mathrm{d}u$ is the value at time t of the solution to the linear ODE $\dot{\alpha}_t = -\alpha_t$ with initial condition $\alpha_0 \in \mathbf{R}$.)

A.2. *Proof of Theorem 5.1*

In the proofs in this section and the next two, we keep the notation manageable by focusing on the single population case ($\not{p} = 1$); the proofs for multipopulation cases are similar.

To prove Theorem 5.1, assume first that protocol τ satisfies integrability (I), so that $\tau \equiv \nabla\gamma$. Using this fact and Eq. (A.2) below, we can compute the time derivative of Γ over a solution to (E):

$$\begin{aligned}
\dot{\Gamma}(x) &= \nabla\Gamma(x)'\dot{x} \\
&= \nabla\gamma\bigl(\hat{F}(x)\bigr)' D\hat{F}(x)\dot{x} \\
&= \tau\bigl(\hat{F}(x)\bigr)'\bigl(DF(x) - \mathbf{1}\bigl(x'DF(x) + F(x)'\bigr)\bigr)\dot{x} \\
&= \bigl(\tau\bigl(\hat{F}(x)\bigr) - \tau\bigl(\hat{F}(x)\bigr)'\mathbf{1}x\bigr)'DF(x)\dot{x} - \tau\bigl(\hat{F}(x)\bigr)'\mathbf{1}F(x)'\dot{x} \\
&= \dot{x}'DF(x)\dot{x} - \bigl(\tau\bigl(\hat{F}(x)\bigr)'\mathbf{1}\bigr)\bigl(F(x)'\dot{x}\bigr).
\end{aligned}$$

The first term in the final expression is nonpositive since F is stable. The second term is nonpositive because $\tau(\hat{F}(x))$ is nonnegative and because V_F satisfies positive correlation (PC)—see Proposition 4.1. Therefore, $\dot{\Gamma}(x) \leqslant 0$ for all $x \in X$.

We now show that this inequality binds precisely on the set $NE(F)$. To begin, note that if $x \in RP(V_F)$ (i.e., if $\dot{x} = \mathbf{0}$), then $\dot{\Gamma}(x) = 0$. On the other hand, if $x \notin RP(V_F)$, then $F(x)'\dot{x} > 0$ (by condition (PC)) and $\tau(\hat{F}(x))'\mathbf{1} > 0$, implying that $\dot{\Gamma}(x) < 0$. Since $NE(F) = RP(V_F)$ by Proposition 4.1, the claim is proved. That $NE(F)$ is globally attracting then follows from Theorem A.1.

If in addition τ satisfies separability (S), then it is easy to verify (using also the continuity and acuteness of τ) that τ is sign preserving, in the sense that $\mathrm{sgn}(\tau_i(\pi_i)) = \mathrm{sgn}([\pi_i]_+)$. It then follows that Γ is nonnegative, and that $\Gamma(x) = 0$ if and only if $\hat{F}(x) \in \mathrm{bd}(\mathbf{R}^n - \mathbf{R}^n_-)$. By Proposition 3.4 of Sandholm [46], the latter statement is true if and only if $x \in NE(F)$. Therefore, the global asymptotic stability of $NE(F)$ follows from Theorem A.3.

A.3. *Proof of Theorem 5.3*

We begin by deriving a version of Danskin's [12] Envelope Theorem.

Theorem A.4. *For each $i \in S$, let $g_i : [0, \infty) \to \mathbf{R}$ be Lipschitz continuous. Let*

$$g^*(t) = \max_{i\in S} g_i(t) \quad \textit{and} \quad S^*(t) = \operatorname*{argmax}_{i\in S} g_i(t).$$

Then g^* is Lipschitz continuous, and for almost all $t \in [0, \infty)$, we have that

$$\dot{g}^*(t) = \dot{g}_i(t) \quad \text{for all } i \in S^*(t).$$

Proof. The Lipschitz continuity of g^* is immediate and implies that g^* and g_i are differentiable almost everywhere. If $i \in S^*(t)$, then

$$g^*(t) - g^*(s) = g_i(t) - g^*(s) \leqslant g_i(t) - g_i(s).$$

Suppose that g^* and g_i are differentiable at t. Dividing the previous inequality by $t - s$ and letting s approach t from below shows that $\dot{g}^*(t) \leqslant \dot{g}_i(t)$; if we instead let s approach t from above, we obtain $\dot{g}^*(t) \geqslant \dot{g}_i(t)$. Hence, $\dot{g}^*(t) = \dot{g}_i(t)$. □

We now proceed with the proof of Theorem 5.3. It is easy to verify that G is nonnegative with $G^{-1}(0) = NE(F)$ (see, e.g., Proposition 3.4 of Sandholm [46]). To prove the second claim, let $\{x_t\}_{t \geqslant 0}$ be a solution to V_F, and let $S^*(t) \subseteq S$ be the set of pure best responses to state x_t. Since $\{x_t\}_{t \geqslant 0}$ is clearly Lipschitz continuous, and since $G(x) = \max_{y \in X}(y - x)'F(x) = \max_{i \in S} \hat{F}_i(x)$, Theorem A.4 shows that the map $t \mapsto G(x_t)$ is Lipschitz continuous, and that at almost all $t \in [0, \infty)$,

$$\dot{G}(x_t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} \max_{i \in S} \hat{F}_i(x_t) = \frac{\mathrm{d}}{\mathrm{d}t} \hat{F}_{i^*}(x_t) \quad \text{for all } i^* \in S^*(t). \tag{A.1}$$

Now observe that the derivative of the excess payoff function $\hat{F}(x) = F(x) - \mathbf{1}\overline{F}(x)$ is

$$D\hat{F}(x) = DF(x) - \mathbf{1}\big(x'DF(x) + F(x)'\big). \tag{A.2}$$

Thus, for t satisfying Eq. (A.1) and at which $\dot{x}_t$ exists, we have that

$$\begin{aligned}
\dot{G}(x_t) &= \frac{\mathrm{d}}{\mathrm{d}t} \hat{F}_{i^*}(x_t) \quad \text{for all } i^* \in S^*(t) \\
&= \big(e_{i^*}'DF(x_t) - x_t'DF(x_t) - F(x_t)'\big)\dot{x}_t \quad \text{for all } i^* \in S^*(t) \\
&= (y^* - x_t)'DF(x_t)\dot{x}_t - F(x_t)'\dot{x}_t \quad \text{for all } y^* \in \operatorname*{argmax}_{y \in \Delta} y'\hat{F}(x_t) \\
&= \dot{x}_t'DF(x_t)\dot{x}_t - F(x_t)'\dot{x}_t \\
&\leqslant -F(x_t)'\dot{x}_t \\
&= -\max_{y \in X} F(x_t)'(y - x_t) \\
&= -G(x_t),
\end{aligned}$$

where the inequality follows from the fact that F is a stable game. The global asymptotic stability of $NE(F)$ then follows from Theorems A.2 and A.3.

A.4. Proof of Theorem 7.1

The first claim is proved as follows:

$$\begin{aligned}
\Psi(x) = 0 &\Leftrightarrow \big[x_i = 0 \text{ or } \psi_j\big(F_j(x) - F_i(x)\big) = 0\big] \quad \text{for all } i, j \in S \\
&\Leftrightarrow \big[x_i = 0 \text{ or } F_i(x) \geqslant F_j(x)\big] \quad \text{for all } i, j \in S \\
&\Leftrightarrow \Big[x_i = 0 \text{ or } F_i(x) \geqslant \max_{j \in S} F_j(x)\Big] \quad \text{for all } i, j \in S \\
&\Leftrightarrow x \in NE(F).
\end{aligned}$$

To begin the proof of the second claim, we compute the partial derivatives of Ψ:

$$\begin{aligned}\frac{\partial \Psi}{\partial x_l}(x) &= \sum_{i\in S}\sum_{j\in S} x_i\rho_{ij}\left(\frac{\partial F_j}{\partial x_l}(x) - \frac{\partial F_i}{\partial x_l}(x)\right) + \sum_{k\in S}\psi_k\big(F_k(x) - F_l(x)\big)\\ &= \sum_{i\in S}\sum_{j\in S}(x_i\rho_{ij} - x_j\rho_{ji})\frac{\partial F_j}{\partial x_l}(x) + \sum_{k\in S}\psi_k\big(F_k(x) - F_l(x)\big)\\ &= \sum_{j\in S}\dot{x}_j\frac{\partial F_j}{\partial x_l}(x) + \sum_{k\in S}\psi_k\big(F_k(x) - F_l(x)\big).\end{aligned}$$

Using this expression, we find the rate of change of Ψ over time along solutions to (17):

$$\begin{aligned}\dot{\Psi}(x) &= \nabla\Psi(x)'\dot{x}\\ &= \dot{x}'DF(x)\dot{x} + \sum_{i\in S}\dot{x}_i\sum_{k\in S}\psi_k(F_k - F_i)\\ &= \dot{x}'DF(x)\dot{x} + \sum_{i\in S}\sum_{j\in S}(x_j\rho_{ji} - x_i\rho_{ij})\sum_{k\in S}\psi_k(F_k - F_i)\\ &= \dot{x}'DF(x)\dot{x} + \sum_{i\in S}\sum_{j\in S}\left(x_j\rho_{ji}\sum_{k\in S}\big(\psi_k(F_k - F_i) - \psi_k(F_k - F_j)\big)\right).\end{aligned}$$

To evaluate the summation, first observe that if $F_i(x) > F_j(x)$, then $\rho_{ji}(F(x)) \equiv \phi_i(F_i(x) - F_j(x)) > 0$ and $F_k(x) - F_i(x) < F_k(x) - F_j(x)$; since each ψ_k is nondecreasing, it follows that $\psi_k(F_k - F_i) - \psi_k(F_k - F_j) \leqslant 0$. In fact, when $k = i$, the comparison between payoff differences becomes $0 < F_i(x) - F_j(x)$; since each ψ_i is increasing on $[0, \infty)$, it follows that $\psi_i(0) - \psi_i(F_i - F_j) < 0$. We therefore conclude that if $F_i(x) > F_j(x)$, then $\rho_{ji}(F(x)) > 0$ and $\sum_{k\in S}(\psi_k(F_k - F_i) - \psi_k(F_k - F_j)) < 0$. On the other hand, if $F_j(x) \geqslant F_i(x)$, we have immediately that $\rho_{ji}(F(x)) = 0$. And of course, $\dot{x}'DF(x)\dot{x} \leqslant 0$ since F is stable.

Marshaling these facts, we find that $\dot{\Psi}(x) \leqslant 0$, and that

$$\dot{\Psi}(x) = 0 \quad \text{if and only if} \quad x_j\rho_{ji}\big(F(x)\big) = 0 \quad \text{for all } i, j \in S. \tag{A.3}$$

Using an inductive argument, Sandholm [50] shows that condition (A.3) is equivalent to the requirement that $x \in RP(V_F)$, and also to the requirement that $x \in NE(F)$. This proves the second claim. The global asymptotic stability of $NE(F)$ follows from the two claims, Theorem A.1, and Theorem A.3.

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Online appendix to "Stable games and their dynamics"

O.1. Analysis of the war of attrition

In this section, we prove that random matching of a single population to play a war of attrition generates a stable game. Recalling the description in Example 2.4, we see that the payoff matrix for the war of attrition is

$$A = \begin{pmatrix} \frac{v}{2} - c_1 & -c_1 & \cdots & -c_1 \\ v - c_1 & \frac{v}{2} - c_2 & \cdots & -c_2 \\ \vdots & \vdots & \ddots & \vdots \\ v - c_1 & v - c_2 & \cdots & \frac{v}{2} - c_n \end{pmatrix}.$$

Reasoning as in Example 2.3, we consider the symmetric matrix

$$\hat{A} = A + A' = v\mathbf{1}\mathbf{1}' - 2\begin{pmatrix} c_1 & c_1 & \cdots & c_1 \\ c_1 & c_2 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_n \end{pmatrix} = v\mathbf{1}\mathbf{1}' - 2C,$$

where the matrix C can be decomposed as

$$C = \begin{pmatrix} c_1 & c_1 & \cdots & c_1 \\ c_1 & c_1 & \cdots & c_1 \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_1 & \cdots & c_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & c_2 - c_1 & \cdots & c_2 - c_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_2 - c_1 & \cdots & c_2 - c_1 \end{pmatrix} + \cdots$$
$$+ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n - c_{n-1} \end{pmatrix}.$$

Thus, if $z \in TX$, then

$$\begin{aligned} z'\hat{A}z &= vz'\mathbf{1}\mathbf{1}'z - 2z'Cz \\ &= (v - 2c_1)z'\mathbf{1}\mathbf{1}'z - 2\sum_{k=2}^{n}\sum_{i=k}^{n}\sum_{j=k}^{n}(c_k - c_{k-1})z_i z_j \\ &= -2\sum_{k=2}^{n}(c_k - c_{k-1})\left(\sum_{i=k}^{n} z_i\right)^2 \\ &\leqslant 0, \end{aligned}$$

so $F(x) = Ax$ is a stable game.

O.2. Cycling in stable games

Proposition O.2.1. *Consider the EFT dynamic* (E) *generated by revision protocol* (9) *in standard Rock–Paper–Scissors.*

(i) *When $\varepsilon < .1094$, there are initial conditions from which solutions to* (E) *converge to periodic orbits.*

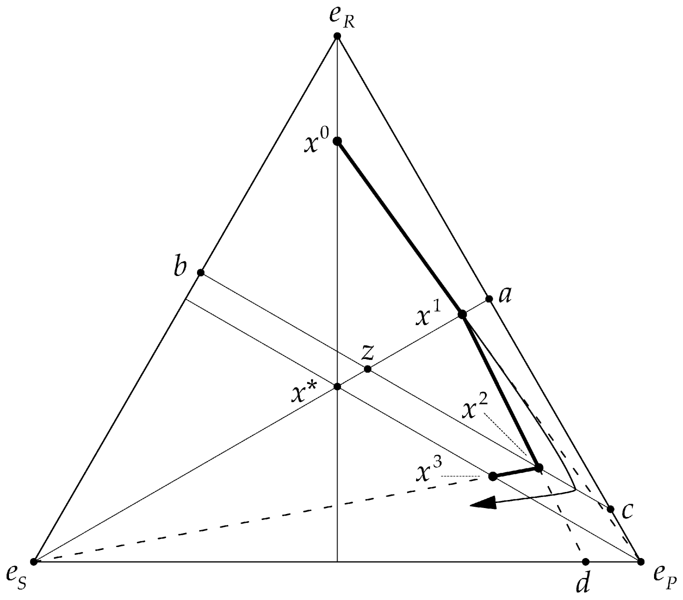

Fig. O.1. The proof of Proposition O.2.1.

(ii) *Fix* $\delta > 0$. *When s is sufficiently small, solutions to* (E) *from all initial conditions that are not within* δ *of the equilibrium* x^* *converge to periodic orbits.*

For intuition, consider Fig. O.1, which presents a portion of a solution to the dynamic (E) generated by (9) in standard RPS when $\varepsilon = \frac{1}{10}$. Scissors earns a positive payoff as soon as this trajectory crosses segment ax^*, and becomes the sole strategy that does so once segment $e_P x^*$ is reached. However, protocol (9) puts very little probability on Scissors until Paper, the strategy it beats, yields a payoff close to zero. As a result, the solution heads almost directly towards state e_P until Scissors becomes the sole strategy earning a payoff of ε. This extends the phase during which the solution approaches the vertex e_P before turning towards e_S. By symmetry, the same phenomenon occurs near the other two vertices, and as a result, the solution never strays far from the boundary of the simplex.

Considering a zero-sum game simplifies the proof of the existence of cycles, but is not necessary for the result to hold: cycles occur under this dynamic even in strictly stable games. In Fig. O.2, we present solutions to the dynamic generated by protocol (9) with $\varepsilon = \frac{1}{10}$ in both standard RPS ($w = 1$, $l = 1$) and good RPS ($w = 3$, $l = 2$). In each case, convergence to a periodic orbit occurs from most initial conditions.

Proof of Proposition O.2.1. Since standard RPS is zero-sum, we have that $\hat{F}(x) = F(x) - \mathbf{1}x'F(x) = F(x)$: excess payoffs and original payoffs are always the same. This fact simplifies the analysis below.

Consider the trajectory that starts from some initial state $x^0 = (\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ that lies on segment $e_R x^*$ and satisfies $\alpha > \underline{\alpha} = \frac{1+\varepsilon}{3-3\varepsilon}$ (see Fig. O.1). This trajectory travels clockwise around the simplex. Our main task is to obtain a lower bound on the distance of this solution from state x^* when the solution crosses segment $e_P x^*$. Doing so enables us to bound the action of the Poincaré map of the dynamic on $e_R x^*$, which in turn lets us use the Poincaré–Bendixson Theorem to demonstrate the existence of a periodic orbit.

When the current state lies in the triangle with vertices e_R, x^*, and $a = (0, \frac{1}{2}, \frac{1}{2})$, as it does at x^0, only strategy P has a positive payoff, so the target state under dynamic V is $\tau(F(x)) = e_P$.

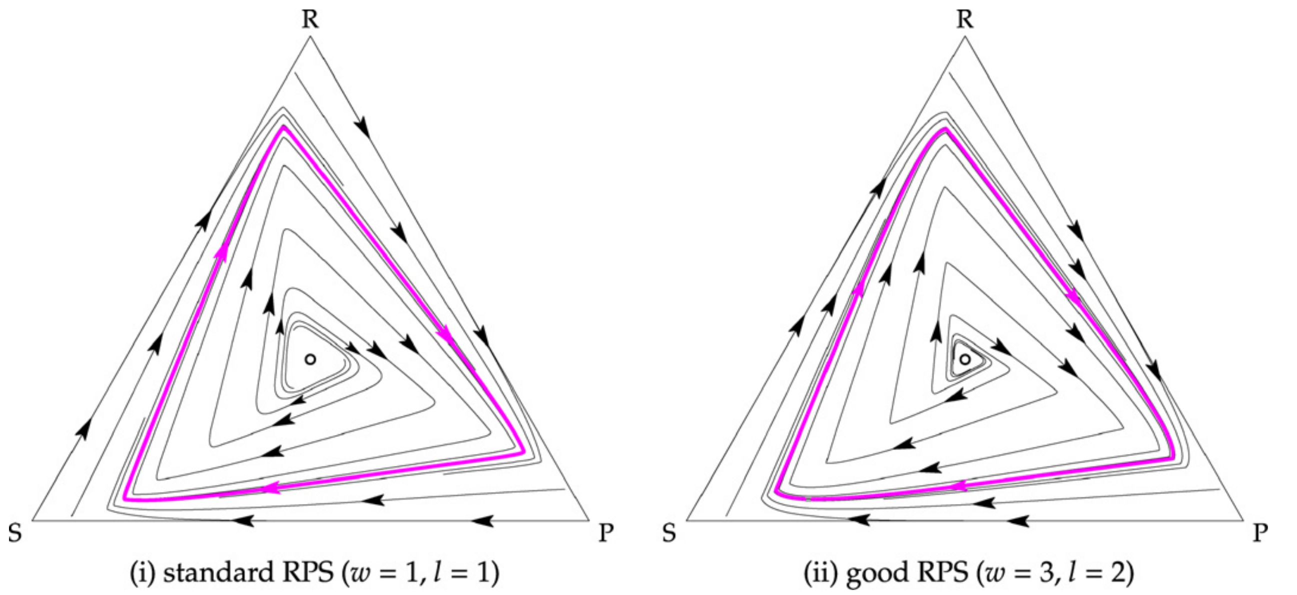

Fig. O.2. Cycling in standard and good Rock–Paper–Scissors games.

Therefore, the trajectory from x^0 leaves triangle $e_R x^* a$ at state $x^1 = (\frac{2\alpha}{1+3\alpha}, \frac{1-\alpha}{1+3\alpha}, \frac{2\alpha}{1+3\alpha})$. Since $\alpha > \underline{\alpha} = \frac{1+\varepsilon}{2-2\varepsilon}$, x^1 lies on the interior of segment az, where $z = (\frac{1+\varepsilon}{3}, \frac{1-2\varepsilon}{3}, \frac{1+\varepsilon}{3})$. For future reference, we observe that z is the intersection of segments ax^* and bc, where $b = (\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2}, 0)$ and $c = (\varepsilon, 0, 1-\varepsilon)$.

In triangle $e_P x^* a$, only strategies P and S earn positive payoffs. By construction, $\tau_S(F(x)) = \varepsilon^2[F_S(x)]_+$ as long as the payoff to P is at least ε, which is the case in triangle $e_R bc$. The intersection of these two triangles is the triangle azc. When the current state x is in this region, the target state is always a point $(0, \tau_P(F(x)), \tau_R(F(x)))$ at which

$$\begin{aligned}
\tau_S\big(F(x)\big) &= \frac{\tau_S(F(x))}{\tau_S(F(x)) + \tau_P(F(x))} \\
&= \frac{[F_S(x)]_+ g^\varepsilon(F_P(x))}{[F_S(x)]_+ g^\varepsilon(F_P(x)) + [F_P(x)]_+ g^\varepsilon(F_R(x))} \\
&\leqslant \frac{1 \times \varepsilon^2}{(1 \times \varepsilon^2) + (\varepsilon \times 1)} \\
&= \frac{\varepsilon}{\varepsilon + 1}.
\end{aligned}$$

Now the ray from point x^1 through point $d = (0, \frac{\varepsilon}{1+\varepsilon}, \frac{1}{1+\varepsilon})$ intersects segment bc at $x^2 = (\frac{2\alpha\varepsilon(2+\varepsilon)}{3\alpha(1+2\varepsilon)-1}, \frac{\varepsilon(1+\alpha-4\alpha\varepsilon)}{3\alpha(1+2\varepsilon)-1}, \frac{\alpha(3+\varepsilon+2\varepsilon^2)-\varepsilon-1}{3\alpha(1+2\varepsilon)-1})$. Hence, the inequality above implies that the solution trajectory from x^1 (and hence the one from x^0) hits segment zc at a point between x^2 and c.

Finally, consider the behavior of solution trajectories passing through the polygon $ce_P x^* z$. In this region, the target point is always on segment $e_S e_P$. In fact, once the solution hits segment $e_P x^*$, strategy S becomes the sole strategy earning a positive payoff, so the target point must be e_S. Thus, the solution starting from x^2 must hit $e_P x^*$ no closer to x^* than $x^3 = (\frac{2\alpha\varepsilon(2+\varepsilon)}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1}, \frac{2\alpha\varepsilon(2+\varepsilon)}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1}, \frac{\alpha(3+\varepsilon+2\varepsilon^2)-\varepsilon-1}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1})$, the point where a ray from x^2 through e_S crosses segment $e_P x^*$. Since the solution starting from x^0 hits segment zc to the right of x^2, it too must hit $e_P x^*$ to the right of x^3. We have thus established a lower bound of $\beta(\alpha) = \frac{\alpha(3+\varepsilon+2\varepsilon^2)-\varepsilon-1}{(1+\varepsilon)3\alpha(1+2\varepsilon)-1}$ on the value of x_P at the point where the solution starting from $x^0 = (\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ intersects segment $e_P x$.

The function β is an increasing hyperbola whose asymptotes lie at $\alpha = \frac{1}{3+9\varepsilon+6\varepsilon^2}$ and $\beta = \frac{3+\varepsilon+2\varepsilon^2}{3+9\varepsilon+6\varepsilon^2}$. It intersects the 45° line at

$$\alpha_\pm = \frac{2+\varepsilon+\varepsilon^2 \pm \sqrt{1-8\varepsilon-10\varepsilon^2-4\varepsilon^3+\varepsilon^4}}{3+9\varepsilon+6\varepsilon^2}$$

whenever the expression under the square root is positive. This is true whenever $\varepsilon < .1094$. In this case, $(\alpha_-, \alpha_+) \subset (\frac{1}{3}, 1)$, and β is above the 45° line on the former interval. Hence, any solution that begins at a point $x^0 = (\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ with $\alpha > \max\{\underline{\alpha}, \alpha_-\}$ will hit segment $e_P x^*$ at some point y with $y_P > \beta(\alpha) \in (\alpha, \alpha_+)$. It then follows from the symmetry of the game and of the choice rule that the region bounded on the inside by the solution from x^0 to y, its 120° and 240° rotations about x^*, and the pieces of $e_P x^*$, $e_S x^*$, and $e_R x^*$ that connect the three solutions, and on the outside by the boundary of X is a trapping region for the dynamic V. By Proposition 4.1, the only rest point of the dynamic is the Nash equilibrium x^*, which lies outside of this region. Therefore, the Poincaré–Bendixson Theorem [61, Theorem 11.4] implies that every solution with an initial condition in the region converges to a periodic orbit. If we take ε to zero, $\underline{\alpha}$ and α_- approach $\frac{1}{3}$, which implies that the radius of the ball around x^* from which convergence to a periodic orbit is not guaranteed vanishes. This completes the proof of the proposition. □

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