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Stable Games

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***Abstract*— We introduce a new class of population games called *stable games*. These games are characterized by *self-defeating externalities*: when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. Stable games subsume many well-known classes of examples, including zero-sum games, games with an interior ESS, wars of attrition, and concave potential games. We prove that the set of Nash equilibria of any stable game is convex. Finally, we show that the set of Nash equilibria of a stable game is globally asymptotically stable under various classes of evolutionary dynamics, classes that include the best response dynamic, the Brown-von Neumann-Nash dynamic, and the Smith dynamic.**

I. INTRODUCTION

One basic aim of game theory is to find general classes of games that are easy to describe, arise frequently in applications, and have attractive theoretical properties—most importantly, the convergence of evolutionary and learning processes to Nash equilibrium. The class of *supermodular games* fulfills all three of these criteria. The defining attribute of these games is the presence of strategic complementarities, an attribute exhibited by models of arms races, Bertand competition, and macroeconomic spillovers ([36], [37], [21], [38], [6]). Moreover, certain learning procedures based on myopic optimization converge to Nash equilibrium in supermodular games, justifying the prediction of Nash equilibrium play ([17], [11], [13]). The class of *potential games* also meets our three criteria. These games are defined by a symmetry in the externalities that different agents impose on one another through their choices, a symmetry that appears in models of genetic natural selection, Cournot competition, network congestion, and externality pricing. Nash equilibria of potential games admit a simple characterization in terms of a scalar-valued potential function, a function that also serves as a strict Lyapunov function for a wide range of evolutionary dynamics ([23], [14], [26], [32]).

In this paper, we introduce a new class of population games called *stable games*. These games are characterized by a condition we call *self-defeating externalities*, which requires that when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. Stable games subsume many well-known classes of examples, including zero-sum games, games with an interior ESS, wars of attrition, and concave potential games; the last of these includes models of network congestion. Turning to theoretical properties, we prove that the set of Nash equilibria of any stable game is convex, and offer an elementary proof of existence of equilibrium. Finally, we show that the set of Nash equilibria of any stable game is globally asymptotically stable under various classes of evolutionary dynamics, classes that include the best response dynamic, the Brown-von Neumann-Nash (BNN) dynamic, and the Smith dynamic as special cases.

Our treatment of stable games builds on ideas from a variety of fields. From the point of view of mathematical biology, one can view stable games as a generalization of the class of symmetric normal form games with an interior ESS ([20], [19]) to settings with multiple populations and nonlinear payoffs. Stable games can also be found in the transportation science literature ([34], [35], [7]), where they are used to extend the standard network congestion model ([1]) to allow for asymmetric externalities between drivers on different routes. Alternatively, stable games can be understood as a class of games that preserves many attractive properties of concave potential games: in a sense to be made explicit soon, stable games preserve the concavity of these games without requiring the existence of a potential function at all. Finally, stable games can be viewed as examples of objects called monotone operators from the theory of variational inequalities ([22], [18], [9], [24]).

II. DEFINITION AND CHARACTERIZATION

A. Population Games

Let $\mathcal{P} = \{1, \ldots, p\}$, be a *society* consisting of $p \geq 1$ *populations* of agents. Agents in population p form a continuum of *mass* $m^p > 0$. Masses capture the populations' relative sizes; if there is just one population, we assume that its mass is one.

The set of *strategies* available to agents in population p is denoted $S^p = \{1, \ldots, n^p\}$, and has typical elements i, j, and (in the context of normal form games) s^p. We let $n = \sum_{p \in \mathcal{P}} n^p$ equal the total number of pure strategies in all populations.

During game play, each agent in population p selects a (pure) strategy from S^p. The set of *population states* (or *strategy distributions*) for population p is thus $X^p = \{x^p \in \mathbf{R}^{n^p}_+ : \sum_{i \in S^p} x^p_i = m^p\}$. The scalar $x^p_i \in \mathbf{R}_+$ represents the mass of players in population p choosing strategy $i \in S^p$. Elements of $X = \prod_{p \in \mathcal{P}} X^p = \{x = (x^1, \ldots, x^p) \in \mathbf{R}^n_+ :$

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$x^p \in X^p\}$, the set of *social states*, describe behavior in all p populations at once.

The *tangent space* of X^p, denoted TX^p, is the smallest subspace of $\mathbf{R}^{n^p}$ that contains all vectors describing motions between population states in X^p. In other words, if $x^p, y^p \in X^p$, then $y^p - x^p \in TX^p$, and TX^p is the span of all vectors of this form. It is not hard to see that $TX^p = \{z^p \in \mathbf{R}^{n^p} : \sum_{i \in S^p} z_i^p = 0\}$ contains exactly those vectors in $\mathbf{R}^{n^p}$ whose components sum to zero; the restriction on the sum embodies the fact that changes in the population state leaves the population's mass constant. Changes in the full social state are elements of the grand tangent space $TX = \prod_{p \in \mathcal{P}} TX^p$.

It will prove useful to specify notation for the orthogonal projections onto the subspaces $TX^p \subset \mathbf{R}^{n^p}$ and $TX \subset \mathbf{R}^n$. The former is given by the matrix $\Phi = I - \frac{1}{n^p}\mathbf{1}\mathbf{1}' \in \mathbf{R}^{n^p \times n^p}$, where $\mathbf{1} \in \mathbf{R}^{n^p}$ is the vector of ones; the latter is given by the block diagonal matrix $\mathbf{\Phi} = \mathrm{diag}(\Phi, \ldots, \Phi) \in \mathbf{R}^{n \times n}$.

We generally take the sets of populations and strategies as fixed and identify a game with its payoff function. A *payoff function* $F : X \to \mathbf{R}^n$ is a continuous map that assigns each social state a vector of payoffs, one for each strategy in each population. $F_i^p : X \to \mathbf{R}$ denotes the payoff function for strategy $i \in S^p$ while $F^p : X \to \mathbf{R}^{n^p}$ denotes the payoff functions for all strategies in S^p. When $p = 1$, we omit the redundant superscript p from all of our notation.

B. Stable Games

The population game $F : X \to \mathbf{R}^n$ is a *stable game* if

$$(y - x)'(F(y) - F(x)) \leq 0 \text{ for all } x, y \in X. \qquad \text{(S)}$$

If the inequality in condition (S) always holds strictly, we say that F is *strictly stable*.[1]

For a first intuition, imagine for the moment that F is a *full potential game*: in other words, that $F \equiv \nabla f(x)$ for some scalar-valued full potential function $f : \mathbf{R}^n_+ \to \mathbf{R}$.[2] In this case, condition (S) is simply the requirement that the potential function f be concave. Our definition of stable games thus extends the defining property of concave potential games to games whose payoffs are not integrable.

Stable games whose payoffs are differentiable can be characterized in terms of the action of their derivative matrices $DF(x)$ on $TX \times TX$.

Theorem 1. *Suppose the population game F is C^1. Then F is a stable game if and only if it satisfies* self-defeating externalities*: for all $x \in X$,*

$$DF(x) \textit{ is negative semidefinite with respect to } TX. \qquad \text{(S}'\text{)}$$

Let us provide some intuition for condition (S$'$). This condition asks that

$$z'DF(x)z \leq 0 \text{ for all } z \in TX \text{ and } x \in X.$$

[1] [25] and [5] consider finite player games with continuous strategy sets in which derivatives of payoffs with respect to own actions satisfy analogues of condition (S).

[2] A *potential game* is defined by the weaker requirement that $\mathbf{\Phi}F \equiv \nabla f$ for a potential function f defined on domain X; see Example 4 and [32].

This requirement is in turn equivalent to

$$\sum_{p \in \mathcal{P}} \sum_{i \in S^p} z_i^p \frac{\partial F_i^p}{\partial z}(x) \leq 0 \text{ for all } z \in TX, x \in X.$$

To interpret this expression, recall that the displacement vector $z \in TX$ describes the aggregate effect on the population state of strategy revisions by a small group of agents. The derivative $\frac{\partial F_i^p}{\partial z}(x)$ represents the marginal effect that these revisions have on the payoffs of agents currently choosing strategy $i \in S^p$. Condition (S$'$) considers a weighted sum of these effects, with weights given by the changes in the use of each strategy, and requires that this weighted sum be negative.

Intuitively, a game exhibits self-defeating externalities if the improvements in the payoffs of strategies to which revising players are switching are always exceeded by the improvements in the payoffs of strategies which revising players are abandoning. For example, suppose the tangent vector z takes the form $z = e_j^p - e_i^p$, representing switches by some members of population p from strategy i to strategy j. In this case, the requirement in (S$'$) reduces to $\frac{\partial F_j^p}{\partial z}(x) \leq \frac{\partial F_i^p}{\partial z}(x)$: that is, any performance gains that the switches create for the newly chosen strategy j are dominated by the performance gains created for the abandoned strategy i.

III. Examples

Example 1. Zero sum games. A symmetric two-player normal form game is defined by a strategy set $S = \{1, \ldots, n\}$ and a payoff matrix $A \in \mathbf{R}^{n \times n}$. A_{ij} is the payoff a player obtains when he chooses strategy i and his opponent chooses strategy j; this payoff does not depend on whether the player in question is called player 1 or player 2. A symmetric two player normal form game A is *symmetric zero-sum* if A is skew-symmetric: that is, if $A_{ji} = -A_{ij}$ for all $i, j \in S$. This condition ensures that under single population random matching, the total utility generated in any match is zero. Since payoffs in the resulting single population game are $F(x) = Ax$, we find that $z'DF(x)z = z'Az = 0$ for all vectors $z \in \mathbf{R}^n$, and so F is a stable game. §

Example 2. Games with an interior neutrally stable state. Let A be symmetric normal form game. State $x \in X$ is an *evolutionarily stable state* (or an *evolutionarily stable strategy*, or simply an *ESS*) of A ([20]) if

$$x'Ax \geq y'Ax \text{ for all } y \in X; \text{ and} \qquad (i)$$
$$x'Ax = y'Ax \text{ implies that } x'Ay > y'Ay. \qquad (ii)$$

Condition (*i*) says that x is a symmetric Nash equilibrium of A. Condition (*ii*) says that x performs better against any alternative best reply y than y performs against itself. (Alternatively, (*i*) says that no $y \in X$ can strictly invade x, and (*i*) and (*ii*) together say that if y can weakly invade x, then x can strictly invade y—see Section IV below.) If we weaken condition (*ii*) to

$$\text{If } x'Ax = y'Ax, \text{ then } x'Ay \geq y'Ay, \qquad (ii')$$

then a state satisfying conditions (i) and (ii') is called a *neutrally stable state* (*NSS*) ([19]). It is not difficult to show that if A admits an NSS in the interior of X, then F is a stable game. §

Example 3. Wars of attrition. A war of attrition ([3]) is a two player symmetric normal form game. Strategies represent amounts of time committed to waiting for a scarce resource. If the two players choose times i and $j > i$, then the j player obtains the resource, worth v, while both players pay a cost of c_i: once the first player leaves, the other seizes the resource immediately. If both players choose time i, the resource is split, so payoffs are $\frac{v}{2} - c_i$ each. We allow the resource value $v \in \mathbf{R}$ to be arbitrary, and require the cost vector $c \in \mathbf{R}^n$ to satisfy $c_1 \leq c_2 \leq \ldots \leq c_n$. One can show that random matching in a war of attrition generates a stable game. §

Example 4. Concave potential games. The population game $F : X \to \mathbf{R}^n$ is a *potential game* if it admits a *potential function* $f : X \to \mathbf{R}$ satisfying $\nabla f \equiv \Phi F$. If f is concave, we say that F is a *concave potential game*. Leading examples of such games include congestion games with increasing cost functions, the basic game-theoretic model of network congestion, and models of variable externality pricing ([1], [23], [14], [26], [29], [32]). It is easy to verify that any concave potential game is a stable game. §

IV. Equilibrium

Below we introduce new equilibrium concepts that are of basic importance for stable games: global neutral stability and global evolutionary stability. These concepts are best understood in terms of the notion of invasion to be presented now. If $F : X \to \mathbf{R}^n$ is a population game and $x, y \in X$ are two social states, we say that y can *weakly invade* x ($y \in \overline{I}_F(x)$) if $(y-x)'F(x) \geq 0$. Similarly, y can *strictly invade* x ($y \in I_F(x)$) if $(y-x)'F(x) > 0$.

The intuition behind these definitions is simple. Consider a single population of agents who play the game F, and whose initial behavior is described by the state $x \in X$. Now imagine that a very small group of agents decide to switch strategies. After these agents select their new strategies, the distribution of choices within their group is described by some $y \in X$, but since the group is so small the impact of its behavior on the overall population state is negligible. Thus, the average payoff in the invading group is at least as high as that in the incumbent population if $y'F(x) \geq x'F(x)$, or equivalently, if $y \in \overline{I}_F(x)$. Similarly, the average payoff in the invading group exceeds that in the incumbent population if $y \in I_F(x)$.

The interpretation of invasion does not change much when there are multiple populations. If we write $(y-x)'F(x)$ as $\sum_p (y^p - x^p)'F^p(x)$, we see that if $y \in I_F(x)$, there must be some population p for which the small group switching to y^p outperforms the incumbent population playing x^p at social state x.

Before introducing our new solution concepts, we first characterize Nash equilibrium in terms of invasion: a Nash equilibrium is a state that no other state can strictly invade.

Proposition 2. *$x \in NE(F)$ if and only if $I_F(x) = \varnothing$.*

With this background at hand, we call $x \in X$ a *globally neutrally stable state* (GNSS) if

$$(y-x)'F(y) \leq 0 \text{ for all } y \in X.$$

Similarly, we call x a *globally evolutionarily stable state* (*GESS*) if

$$(y-x)'F(y) < 0 \text{ for all } y \in X - \{x\}.$$

We let $GNSS(F)$ and $GESS(F)$ denote the sets of globally neutrally stable strategies and globally evolutionarily stable strategies, respectively. Analogues of these concepts are introduced in the variational inequality and the transportation science literatures by [22] and [34], respectively.

To see the reason for our nomenclature, note that the inequalities used to define GNSS and GESS are the same ones used to define NSS and ESS in symmetric normal form games (Example 2), but that they are now required to hold not just at those states y that are optimal against x, but at all $y \in X$. NSS and ESS also require a state to be a Nash equilibrium, but our new solution concepts implicitly require this as well—see Proposition 4 below.

It is easy to describe both of these concepts in terms of the notion of invasion.

Observation 3. (i) *$GNSS(F) = \bigcap_{y \in X} \overline{I}_F(y)$, and so is convex.*
(ii) *$x \in GESS(F)$ if and only if $x \in \bigcap_{y \in X - \{x\}} I_F(y)$.*

In words: a GNSS is a state that can weakly invade every state (or, equivalently, every other state), while a GESS is a state that can strictly invade every other state.

Our new solution concepts can also be described in geometric terms. For example, x is a GESS if a small motion from any state $y \neq x$ in the direction $F(y)$ moves the state closer to x. If we allow not only these acute motions, but also orthogonal motions, we obtain the weaker notion of GNSS.

We conclude this section by relating our new solution concepts to Nash equilibrium.

Proposition 4. (i) *If $x \in GNSS(F)$, then $x \in NE(F)$.*
(ii) *If $x \in GESS(F)$, then $NE(F) = \{x\}$. Hence, if a GESS exists, it is unique.*

Evidently, this proposition implies that every GNSS is an NSS, and that every GESS is an ESS.

Proposition 4 tells us that every GNSS of an arbitrary game F is a Nash equilibrium. Theorem 5 shows that much more can be said if F is stable: in these case, the sets of globally neutrally stable states and Nash equilibria coincide. Together, this fact and Observation 3 imply that the Nash equilibria of any stable game form a convex set. In fact, if we can replace certain of the weak inequalities that define stable games with strict ones, then the Nash equilibrium is actually unique.

Theorem 5. (i) *If F is a stable game, then $NE(F) = GNSS(F)$, and so is convex.*

(ii) If in addition F is strictly stable at some $x \in NE(F)$ (that is, if $(y-x)'(F(y)-F(x)) < 0$ for all $y \neq x$, then $NE(F) = GESS(F) = \{x\}$.

V. Global Convergence of Evolutionary Dynamics

The set of Nash equilibria of any stable game is geometrically simple: it is convex, and it is typically a singleton. If a population of myopic agents recurrently play a stable game, will they learn to play in accordance with Nash equilibrium?

It is worth emphasizing that uniqueness of equilibrium, while suggestive, does not imply any sort of stability under evolutionary dynamics. Indeed, most of the games used to illustrate the possibility of nonconvergence are games with a unique Nash equilibrium: see [33], [16], [8], [15], and [12]. If convergence results can be established for stable games, it is not just a consequence of uniqueness of equilibrium; rather, the results must depend on the global structure of payoffs in these games.

A. Revision Protocols and Evolutionary Dynamics

We consider evolutionary dynamics derived from an explicit model of individual choice. This model is defined in terms of *revision protocols* $\rho^p : \mathbf{R}^{n^p} \times X^p \to \mathbf{R}_+^{n^p \times n^p}$, which describe the process through which agents in each population p make decisions. (We will also refer to the collection $\rho = (\rho^1, \ldots, \rho^p)$ as a revision protocol when no confusion will arise.) As time passes, agents are chosen at random from the population and granted opportunities to switch strategies. When a strategy $i \in S^p$ player receives an opportunity, he switches to strategy $j \in S^p$ with probability proportional to the *conditional switch rate* $\rho_{ij}^p(F^p(x), x^p)$, a rate that may depend on the payoff vector $F^p(x)$ and the population state x^p.

If we fix a Lipschitz continuous population game F, then aggregate behavior in F under protocol ρ is described by the dynamic

$$\dot{x}_i^p = \sum_{j \in S^p} x_j^p \rho_{ji}^p(F^p(x), x^p) - x_i^p \sum_{j \in S^p} \rho_{ij}^p(F^p(x), x^p). \quad \text{(D)}$$

The first term captures the inflow of agents into strategy i from other strategies, while the second term captures the outflow of agents from strategy i to other strategies. We sometimes write (D) as $\dot{x} = V_F(x)$ to emphasize the dependence of the law of motion on the underlying game.[3]

To describe some basic revision protocols and dynamics, we require additional definitions. Let $\Delta^p = \{y \in \mathbf{R}_+^{n^p} : \sum_{i \in S^p} y_i^p = 1\}$ be the set of mixed strategies for population p. Then the map $B^p : X \Rightarrow \Delta^p$, defined by

$$B^p(x) = \underset{y^p \in \Delta^p}{\operatorname{argmax}}\, (y^p)' F^p(x),$$

is population p's *best response correspondence*.

The *average payoff* in population p is given by $\overline{F}^p(x) = \frac{1}{m^p} \sum_{i \in S^p} x_i^p F_i^p(x)$. The *excess payoff* to strategy $i \in S^p$, defined by $\hat{F}_i^p(x) = F_i^p(x) - \overline{F}^p(x)$, is the difference between the strategy's payoff and the average payoff earned in population p. The *excess payoff vector* for population p is thus $\hat{F}^p(x) = F^p(x) - \mathbf{1}\overline{F}^p(x)$, where $\mathbf{1} \in \mathbf{R}^{n^p}$ is the vector of ones.

[3] For more on the foundations of deterministic evolutionary dynamics, see [2], [27], and [31].

B. Lyapunov Functions and Gap Functions

In the sections to come, we prove that in stable games, under the dynamics described above and other qualitatively similar dynamics, evolution leads to Nash equilibrium play from all initial conditions. Our analyses rely on the construction of Lyapunov functions. A *Lyapunov function* $L : X \to \mathbf{R}$ for the closed set $A \subseteq X$ is a continuous function that achieves its minimum throughout the set A and is nonincreasing along solutions of (D). If the value of L is decreasing outside of A, L is called a *strict Lyapunov function*. A variety of classical results from dynamical systems show that the existence of a suitable Lyapunov function implies various forms of stability for the set A.

Most of the Lyapunov functions introduced below take an especially attractive form. We call the function $G : X \to \mathbf{R}$ a *gap function* for the game F if it is continuous, nonnegative, and satisfies $G^{-1}(0) = NE(F)$. Thus, a gap function's minimum value of 0 is attained on the set of Nash equilibria of the underlying game.

C. The Best Response Dynamic

Our first stability result for stable games shows that a very simple gap function serves as a Lyapunov function for the best response dynamic. Following [10], we formulate the best response dynamic as the differential inclusion

$$\dot{x}^p \in m^p B^p(x) - x^p. \quad \text{(BR)}$$

Theorem 6. *Let F be a C^1 stable game, and let V_F be the best response dynamic for F. Define the Lipschitz continuous function $G : X \to \mathbf{R}$ by*

$$G(x) = \max_{y \in X} (y-x)' F(x).$$

Then G is a gap function for F. Moreover, if $\{x_t\}_{t \geq 0}$ is a solution to V_F, then for almost all $t \geq 0$ we have that $\dot{G}(x_t) \leq -G(x_t)$. Therefore, $NE(F)$ is globally asymptotically stable under V_F.

The gap function G measures the difference between the payoffs agents could obtain by choosing optimal strategies and their actual aggregate payoffs.

D. Excess Payoff Dynamics

Next we consider evolutionary dynamics based on revision protocols of the form

$$\rho_{ij}^p(F^p(x), x^p) = \beta_j^p(\hat{F}^p(x)),$$

where $\beta^p : \mathbf{R}^{n^p} \to \mathbf{R}_+^{n^p}$ is Lipschitz continuous. In this formulation, the conditional switch rate from i to j is independent of the current strategy i, and only depends on payoffs and the population state through the excess payoff vector

$\hat{F}^p(x)$. Evolutionary dynamics based on such protocols take the form

$$\dot{x}_i^p = m^p \beta_i^p(\hat{F}^p(x)) - x_i^p \sum_{j \in S^p} \beta_j^p(\hat{F}^p(x)). \qquad (1)$$

If we let $\beta_i^p(\hat{F}^p(x)) = [\hat{F}_i^p(x)]_+$ equal the positive part of strategy i's excess payoff, then (1) becomes the BNN dynamic of [4]. More generally, [28] calls (1) an *excess payoff dynamic* if the protocols β^p satisfy *acuteness*:

$$\beta^p(\pi^p)'\pi^p > 0 \text{ whenever } \pi^p \in \mathbf{R}^{n^p} - \mathbf{R}_-^{n^p}. \qquad \text{(A)}$$

When an agent facing excess payoff vector π^p plays a mixed strategy proportional to $\beta^p(\pi^p)$, the monotonicity condition (A) ensures that the agent's expected excess payoff is positive. [28] shows that every excess payoff dynamic satisfies *Nash stationarity*: if V is an excess payoff dynamic, then for any population game F, the set of rest points of V_F is identical to the set of Nash equilibria of F.

While monotonicity condition (A) is enough to ensure convergence to Nash equilibrium in potential games, it is not enough in stable games. We therefore consider these two restrictions on revision protocols:

$$\beta^p \equiv \nabla\gamma^p \text{ for some } C^1 \text{ function } \gamma^p : \mathbf{R}^{n^p} \to \mathbf{R}. \qquad \text{(I)}$$

$$\beta_i^p(\pi^p) \text{ is independent of } \pi_{-i}^p. \qquad \text{(S)}$$

The former condition, *integrability*, is weaker than the latter, *separability*: if (S) holds, then (I) is satisfied with

$$\gamma^p(\pi^p) = \sum_{i \in S^p} \int_0^{\pi_i^p} \beta_i^p(s)\,\mathrm{d}s.$$

In particular, the protocol for the BNN dynamic satisfies both (S) and (I): since $\beta_i^p(\pi^p) = [\pi_i^p]_+$, we have that $\gamma^p(\pi^p) = \frac{1}{2}\sum_{i \in S^p}[\pi_i^p]_+^2$.

Continuing our analogy with the best response dynamic leads to the following result.

Theorem 7. *Let F be a C^1 stable game, and let V_F be an excess payoff dynamic for F based on revision protocols β^p. Define the C^1 function $\Gamma : X \to \mathbf{R}$ by*

$$\Gamma(x) = \sum_{p \in \mathcal{P}} m^p \gamma^p(\hat{F}^p(x)).$$

(i) *If the protocols β^p satisfy integrability* (I)*, then $\dot{\Gamma}(x) \leq 0$ for all $x \in X$, with equality if and only if $x \in NE(F)$. Thus $NE(F)$ is globally attracting, and if $NE(F)$ is a singleton it is globally asymptotically stable.*

(ii) *If the protocols β^p also satisfy separability* (S)*, then Γ is also a gap function for F, and so $NE(F)$ is globally asymptotically stable.*

[30] offers an example of a stable game (a good RPS game) in which an excess payoff dynamic whose revision protocol fails condition (I) exhibits a limit cycle far from any Nash equilibrium. The revision protocol has the feature that agents' probabilities of choosing each strategy depend systematically on the payoffs of the next strategy in the best response cycle. Building on this motivation, [30] provides a game-theoretic interpretation of integrability: roughly speaking, condition (I) is equivalent to a requirement that in expectation, learning the conditional switch rate to strategy j conveys no information about the excess payoffs of other strategies $i \neq j$. This is a natural generalization of separability condition (S), which requires that learning the conditional switch rate to strategy j conveys no information at all about the excess payoffs of other strategies $i \neq j$.

E. Pairwise Comparison Dynamics

We now consider revision protocols based on pairwise comparisons of payoffs:

$$\rho_{ij}^p(\pi^p) = \phi_{ij}^p(\pi_j^p - \pi_i^p),$$

where $\phi_{ij}^p : \mathbf{R} \to \mathbf{R}_+$ is Lipschitz continuous. Substituting this revision protocol into (D) yields dynamics of the form

$$\begin{aligned}\dot{x}_i^p &= \sum_{j \in S^p} x_j^p \phi_{ji}^p(F_i^p(x) - F_j^p(x)) \\ &\quad - x_i^p \sum_{j \in S^p} \phi_{ij}^p(F_j^p(x) - F_i^p(x)).\end{aligned} \qquad (2)$$

While these dynamics look more complicated than those of the excess payoff form (1), they may nevertheless provide a more realistic model of behavior: instead of requiring comparisons involving (not directly observable) population average payoffs, they are based on direct comparisons of individual strategies' payoffs.

If one sets $\phi_{ij}^p(\pi_j^p - \pi_i^p) = [\pi_j^p - \pi_i^p]_+$, then equation (2) becomes the Smith dynamic ([35]). More generally, [31] calls (2) a *pairwise comparison dynamic* if the protocols ϕ^p satisfy *sign preservation*:

$$\operatorname{sgn}\left(\phi_{ij}^p(\pi_j^p - \pi_i^p)\right) = \operatorname{sgn}\left([\pi_j^p - \pi_i^p]_+\right). \qquad \text{(SP)}$$

In words: the conditional switch rate from $i \in S^p$ to $j \in S^p$ is positive if and only if j earns a higher payoff than i. Like excess payoff dynamics, pairwise comparison dynamics satisfy Nash stationarity: Nash equilibria of F and rest points of (2) coincide.

[35] proves that in every stable game, the Smith dynamic converges to Nash equilibrium from all initial conditions. We now show that global convergence in stable games obtains much more generally. Our result requires a new condition called *incumbent independence*:

$$\phi_{ij}^p(\pi_j^p - \pi_i^p) = \phi_j^p(\pi_j^p - \pi_i^p) \qquad \text{(II)}$$

for some functions $\phi_j^p : \mathbf{R} \to \mathbf{R}_+$. Under condition (II), the function of the payoff difference $\pi_j^p - \pi_i^p$ that describes the conditional switch rate from i to j is independent of the "incumbent" strategy i. Theorem 8 shows that when paired with the monotonicity condition (SP), incumbent independence ensures global convergence to Nash equilibrium in stable games.

Theorem 8. *Let F be a C^1 stable game, and let V_F be a pairwise comparison dynamic for F whose revision protocol* *satisfies incumbent independence* (II). *Define the* C^1 *function* $\Psi : X \to \mathbf{R}_+$ *by*

$$\Psi(x) = \sum_{p\in\mathcal{P}} \sum_{i\in S^p} \sum_{j\in S^p} x_i^p \psi_j^p (F_j^p(x) - F_i^p(x)),$$
$$\text{where } \psi_k^p(d) = \int_0^d \phi_k^p(s)\,\mathrm{d}s.$$

Then Ψ *is a gap function for* F. *Moreover,* $\dot{\Psi}(x) \le 0$ *for all* $x \in X$, *with equality if and only if* $x \in NE(F)$, *and so* $NE(F)$ *is globally asymptotically stable.*

To understand the role played by condition (II), examine the expression (2) for a pairwise comparison dynamic. According to the second term of this expression, the rate of outflow from strategy i is $x_i^p \sum_{k\in S^p} \phi_{ik}^p(\pi_k^p - \pi_i^p)$; thus, the *percentage* rate of outflow from i, $\sum_{k\in S^p} \phi_{ik}^p(\pi_k^p - \pi_i^p)$, varies with i.[4] It follows that strategies with high payoffs can nevertheless have high percentage outflow rates: even if $\pi_i^p > \pi_j^p$, one can still have $\phi_{ik}^p > \phi_{jk}^p$ for $k \neq i, j$. Having good strategies lose players more quickly than bad strategies is an obvious impediment to convergence to Nash equilibrium.

Incumbent independence (II) places controls on these percentage outflow rates. If the conditional switch rates ϕ_j^p are monotone in payoffs, then condition (II) ensures that better strategies have lower percentage outflow rates. If the conditional switch rates are not monotone, but merely sign-preserving (SP), condition (II) still implies that the *integrated* conditional switch rates ψ_k^p are ordered by payoffs. According to our analysis, this control is enough to ensure convergence of pairwise comparison dynamics to Nash equilibrium in stable games.

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[4] By contrast, under excess payoff dynamics all strategies' percentage outflow rates are the same and equal to $\sum_{k\in S^p} \beta_k^p(\hat{F}^p(x))$, while under the best response dynamic these rates are all fixed at 1.

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