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DIAGONAL DOMINANCE AND HARMLESS OFF-DIAGONAL DELAYS

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Abstract. Systems of linear differential equations with constant coefficients, as well as Lotka–Volterra equations, with delays in the off–diagonal terms are considered. Such systems are shown to be asymptotically stable for any choice of delays if and only if the matrix has a negative weakly dominant diagonal.

1. Introduction

Consider a system of retarded linear differential equations with constant coefficients of the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}), \qquad \text{for } i = 1, \ldots, n, \tag{1.1}$$

with

$$\tau_{ij} \geq 0 \qquad \text{for } 1 \leq i \neq j \leq n \qquad \text{and} \qquad \tau_{ii} = 0 \qquad \text{for } i = 1, \ldots, n. \tag{1.2}$$

This paper deals with the following question:

(*) For which matrices $A = (a_{ij})$ is the trivial solution $x = 0$ of (1.1) asymptotically stable for any choice of delays satisfying (1.2)?

Such type of stability has been referred to as 'absolute stability' in the literature (cf. El'sgol'ts and Norkin [3, p. 175]). In the context of population modelling, the term 'harmless' delays has also been used (see Gopalsamy [5]). For $n = 2$, this question was answered by Lu and Wang [12]. They showed that (*) holds iff $a_{11}, a_{22} < 0$ and $a_{11}a_{22} > |a_{12}a_{21}|$ (with equality allowed in the latter if $a_{12}a_{21} < 0$). Actually, their result was for Lotka–Volterra equations but the proof applies to (1.1) with minor modifications. We extend this result to arbitrary n, both for the linear case (1.1) and the Lotka–Volterra case (3.1).

Definition. Let $\tilde{A} = (\tilde{a}_{ij})$ be the matrix with entries $\tilde{a}_{ii} = a_{ii}$ and $\tilde{a}_{ij} = |a_{ij}|$ for $i \neq j$. A is said to be *weakly diagonally dominant* if all the principal minors of $-\tilde{A}$ are non–negative.

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Our main result is the following.

Theorem 1. *(1.1) is asymptotically stable for all choices of delays of the form (1.2) if and only if* $a_{ii} < 0$ *for* $i = 1, \ldots, n$, $\det A \neq 0$ *and* A *is weakly diagonally dominant.*

For quasimonotone matrices A (i.e., $a_{ij} \geq 0$ for $i \neq j$), a similar result was obtained by Győri (1992).

2. Proof of Theorem 1

The characteristic equation of the delay differential equation (1.1) is given by

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12}e^{-\lambda\tau_{12}} & \cdots \\ a_{21}e^{-\lambda\tau_{21}} & a_{22} - \lambda & \cdots \\ & \cdots & \end{pmatrix} = 0. \tag{2.1}$$

Since $x(t) = ce^{\lambda t}$ is a solution of (1.1) for suitable $c \neq 0$ iff λ satisfies (2.1), by Corollary 6.1 of Hale and Verduyn Lunel [7, p. 215], the trivial solution $x = 0$ of (1.1) is asymptotically stable iff all the roots of (2.1) have negative real part.

2.1. Sufficiency part.

Lemma 1. *If* A *is weakly diagonally dominant, then all roots of (2.1) have negative real part, with the possible exception of* $\lambda = 0$.

Proof. We first consider irreducible matrices A. By Theorem 5.9 of Fiedler [4, p. 124], for an irreducible weakly diagonally dominant matrix A, there is a $c > 0$ such that $\tilde{A}c \leq 0$, i.e., there exist $c_i > 0$ such that

$$a_{ii}c_i + \sum_{j \neq i} |a_{ij}|c_j \leq 0 \qquad \text{for all } i = 1, \ldots, n. \tag{2.2}$$

Suppose, for some set of delays τ_{ij} satisfying (1.2), there exists a root λ of (2.1) with $\operatorname{Re}\lambda \geq 0$. Then λ is an eigenvalue of the matrix $B = (b_{ij})$, where $b_{ij} = a_{ij}e^{-\lambda\tau_{ij}}$. Since $b_{ii} = a_{ii} \leq 0$ and $|b_{ij}| \leq |a_{ij}|$, (2.2) implies

$$b_{ii}c_i + \sum_{j \neq i} |b_{ij}|c_j \leq 0 \qquad \text{for } i = 1, \ldots, n.$$

Applying Geršgorin's theorem (cf. Lancaster and Tismenetsky [10, p. 371]) to the matrix $\hat{B} = (c_i^{-1}b_{ij}c_j)$, which is similar to B, we know that the eigenvalue λ of $\hat{B}$ is contained in a circle with center $b_{ii} \leq 0$ and radius at most $|b_{ii}|$ (for some i). Hence either $\operatorname{Re}\lambda < 0$ or $\lambda = 0$.

In the case of a reducible matrix A, we can (by suitably relabeling the indices) turn A into an upper block triangular matrix with irreducible (or zero) blocks along the diagonal (cf. (3.6) of Berman and Plemmons [2, p. 39]). Since index relabeling is done via a permutation matrix and does not affect the principal minors of A, each diagonal block is itself weakly diagonally dominant. The result now follows by applying the previous argument to each irreducible diagonal block. □

The solution $\lambda = 0$ of (2.1) is possible only if $\det A = 0$ which is excluded. This concludes the proof of the sufficiency part of Theorem 1.

2.2. Necessity part. We start with three lemmas.

Lemma 2. *If $a_{ii} < 0$ for all $i = 1, \dots, n$ and $\det(-\tilde{A}) < 0$, then there exist delays τ_{ij} satisfying (1.2) such that (2.1) has a root λ with $\operatorname{Re}\lambda > 0$.*

Proof. Consider the function

$$F_\varepsilon(z) = \det\begin{pmatrix} a_{11} - z\varepsilon & a_{12}e^{-z\eta_{12}} & \cdots \\ a_{21}e^{-z\eta_{21}} & a_{22} - z\varepsilon & \cdots \\ & \cdots & \end{pmatrix}$$

where

$$\eta_{ij} = \begin{cases} \dfrac{1}{2}, & \text{for } a_{ij} < 0, \\ 1, & \text{for } a_{ij} \geq 0. \end{cases}$$

For $z = x + 2\pi i$, where x is real, $F_0(z)$ becomes

$$D(x) = \det\begin{pmatrix} a_{11} & |a_{12}|e^{-x\eta_{12}} & \cdots \\ |a_{21}|e^{-x\eta_{21}} & a_{22} & \cdots \\ & \cdots & \end{pmatrix}.$$

By assumption, $(-1)^n D(0) = \det(-\tilde{A}) < 0$, and $(-1)^n D(\infty) = (-1)^n a_{11} \dots a_{nn} > 0$. Hence, by intermediate value theorem, there exists $\hat{x} > 0$ such that $D(\hat{x}) = 0$ and $\hat{z} = \hat{x} + 2\pi i$ is a zero of F_0. It follows from Rouché's theorem (cf. Ahlfors [1, p. 153]) that the analytic function F_ε has a zero $\hat{z}(\varepsilon)$ near $\hat{z}$ for small $\varepsilon > 0$. Clearly, $\lambda = \hat{z}(\varepsilon)\varepsilon$ and $\tau_{ij} = \frac{\eta_{ij}}{\varepsilon}$ satisfy (2.1), with $\operatorname{Re}\lambda > 0$. ☐

Lemma 3. *Let $f(\lambda)$ be a real polynomial of degree n with n real zeros, including 0. Then the equation*

$$f(\lambda) = e^{-\lambda\tau} \tag{2.3}$$

has a solution λ with $\operatorname{Re}\lambda > 0$ for some $\tau > 0$.

Proof. Write $f(\lambda)$ as $\alpha\prod_{i=1}^n(\lambda - \lambda_i)$, where $0 \neq \alpha \in \mathbb{R}$ and $\lambda_i \in \mathbb{R}$ with $\lambda_1 = 0$. We first show (2.3) has a purely imaginary root $\lambda = i\omega$, where $\omega > 0$, for some $\tau > 0$. By suitably choosing $\tau > 0$, $f(i\omega) = e^{-i\omega\tau}$ holds iff $|f(i\omega)| = 1$. Since $|f(0)| = 0$ and $\lim_{\omega\to\infty}|f(i\omega)| = \infty$, the existence of $\omega > 0$ follows from the intermediate value theorem. Let $f(\hat{\lambda}) = e^{-\hat{\lambda}\hat{\tau}}$, where $\hat{\lambda} = i\hat{\omega}$ and $\hat{\omega}, \hat{\tau} > 0$.

Next, we show that by slightly increasing τ past $\hat{\tau}$, the solution λ crosses the imaginary axis. Indeed, let $g(\lambda, \tau) = f(\lambda) - e^{-\lambda\tau}$. Then

$$g_\lambda(\hat{\lambda}, \hat{\tau}) = e^{-\hat{\lambda}\hat{\tau}}\left[\sum_{i=1}^n \frac{1}{\hat{\lambda} - \lambda_i} + \hat{\tau}\right] \neq 0,$$

since

$$\operatorname{Im}\left(\sum_{i=1}^n \frac{1}{\hat{\lambda} - \lambda_i}\right) = -\hat{\omega}\sum_{i=1}^n \frac{1}{\lambda_i^2 + \hat{\omega}^2} < 0.$$

The implicit function theorem applied to (2.3) at $(\hat{\lambda}, \hat{\tau})$ yields the solution

$$\lambda(\tau) = \hat{\lambda} + c(\tau - \hat{\tau}) + O((\tau - \hat{\tau})^2) \tag{2.4}$$

where $c = -\frac{g_\tau(\hat{\lambda}, \hat{\tau})}{g_\lambda(\hat{\lambda}, \hat{\tau})}$. It is easy to see that $\operatorname{Re} c > 0$, and hence $\operatorname{Re}\lambda(\tau) > 0$ for $\tau > \hat{\tau}$. ☐

Lemma 4. *If $a_{ii} = 0$ for some i and $\det A \neq 0$, then there exist delays τ_{ij} satisfying (1.2) such that (2.1) has a root λ with $\operatorname{Re}\lambda > 0$.*

Proof. Assume $a_{11} = 0$. Since $\det A \neq 0$, there exists a non–zero term in the expansion of $\det A$, i.e. there is a permutation σ of $\{1, \ldots, n\}$ such that $\prod_{i=1}^n a_{i\sigma_i} \neq 0$. The corresponding term in the expansion of (2.1) together with the product of the diagonal entries yields the 'truncated' equation

$$\prod_{i=1}^{n}(a_{ii} - \lambda) + \operatorname{sgn}(\sigma)\prod_{i=1}^{n}(a_{i\sigma_i} - \lambda\delta_{i\sigma_i}) \cdot e^{-\lambda\sum_{i=1}^{n}\tau_{i\sigma_i}} = 0. \tag{2.5}$$

After cancelling the factor $a_{ii} - \lambda$ (for each i with $\sigma_i = i$), there remains an equation of the form (2.3), where $\tau = \sum_{i=1}^n \tau_{i\sigma_i}$. By Lemma 3, there is a solution $\hat\lambda$ of (2.5) with $\operatorname{Re}\hat\lambda > 0$ for a suitable $\tau > 0$. By letting $\tau_{ij} \to \infty$ whenever $j \neq i$ and $j \neq \sigma_i$, the remaining $n! - 2$ terms in the expansion of (2.1) left out in (2.5) can be made arbitrarily small (in a fixed neighbourhood of $\hat\lambda$). Hence Rouché's theorem again shows the existence of a root λ of (2.1) near $\hat\lambda$. This shows the lemma.

Returning to the proof of the sufficiency part, the asymptotic stability assumption on $x = 0$ precludes $\lambda = 0$ from being a root of (2.1). Hence $\det A \neq 0$.

If $a_{kk} > 0$ for some k, then Rouché's theorem shows that (2.1) has a root near a_{kk} for large τ_{ij}. This contradiction together with Lemma 4 shows that $a_{ii} < 0$ for all $i = 1, \ldots, n$.

Now suppose that some principal minor of $-\tilde A$ is negative. Without loss of generality, we may assume $\det(-\tilde A_k) < 0$, where $A_k = (a_{ij})_{1\le i,j\le k}$. Applying Lemma 2 to this principal submatrix yields a root $\hat\lambda$ with $\operatorname{Re}\hat\lambda > 0$ for the characteristic equation (2.1) corresponding to A_k. By letting $\tau_{ij} \to \infty$ whenever $i > k$ or $j > k$, a similar perturbation argument as above shows that the full characteristic equation (2.1) (corresponding to A) has a root near $\hat\lambda$. In particular, (2.1) will have a root with positive real part, which contradicts the assumption of stability of $x = 0$. This completes the proof of Theorem 1. □

3. Lotka–Volterra equations

In this section we prove a similar result for a class of nonlinear differential equations, widely used in population dynamics, the Lotka–Volterra systems

$$\dot y_i = y_i\left(r_i + \sum_{j=1}^{n} a_{ij} y_j(t - \tau_{ij})\right), \qquad i = 1, \ldots, n. \tag{3.1}$$

We assume in the following that there exists a positive vector $\hat y$ with

$$r + A\hat y = 0. \tag{3.2}$$

This $\hat y$ is then an equilibrium for (3.1), i.e., $y(t) = \hat y$ is a (constant) solution of (3.1).

We are interested in the absolute stability of $\hat y$; and we will show that this is the case under the same conditions on A as in the linear case (1.1). That the strict version of diagonal dominance (all principal minors of $-\tilde A$ are positive) implies the global stability of the unique saturated equilibrium has been discussed in detail in Hofbauer and Sigmund [8] for ordinary Lotka–Volterra equations and in Kuang [9] for delayed versions like (3.1). That weak diagonal dominance is already sufficient is more subtle, and its proof uses ideas from Lu and Takeuchi [11], Lu and Wang [12] and Redheffer [13].

Theorem 2. *Suppose there exists a positive vector $\hat{y}$ satisfying (3.2). Then $\hat{y}$ is globally asymptotically stable for (3.1) (for initial conditions $y_i(0) > 0$) for all delays τ_{ij} satisfying (1.2) if and only if $a_{ii} < 0$ for $i = 1, \dots, n$, $\det A \neq 0$ and A is weakly diagonally dominant.*

Proof. For the necessity part, first we observe that in the course of proving Theorem 1, we have shown that, under the assumption $\det(A) \neq 0$, the characteristic equation (2.1) has a root with positive real part (for a certain choice of τ_{ij} satisfying (1.2)), provided $a_{ii} \geq 0$ for some i or A is not weakly diagonally dominant. Next, let $B = (\hat{y}_i a_{ij})$ be the linearization of (3.1) at $\hat{y}$. Then $\det(B) \neq 0$, because otherwise $\det(A) = 0$ and (3.1) has many equilibrium solutions near $\hat{y}$ (implying $\hat{y}$ is not asymptotically stable). Now, since $\hat{y}$ is asymptotically stable, none of the roots of (2.1), with A replaced by B, can have positive real parts (cf. Theorem 2.1 of Hale and Verduyn Lunel [7, p. 314]). Hence, $b_{ii} < 0$ for all i and B is weakly diagonally dominant. This implies $a_{ii} < 0$ for all i and A is weakly diagonally dominant as well.

For the sufficiency part, the proof in section 2 only gives (local) asymptotic stability. When A is irreducible, it turns out that one can choose $\alpha_i > 0$ and $\beta_{ij} > 0$ appropriately (see later) so that

$$V(y(.), t) = \sum_{i=1}^{n} \alpha_i \big(y_i(t) - \hat{y}_i \log y_i(t)\big) + \sum_{i,j=1}^{n} \beta_{ij} \int_{t-\tau_{ij}}^{t} (y_j(s) - \hat{y}_j)^2 ds \tag{3.3}$$

becomes a Lyapunov functional. For the case of a general A, we will use induction to show that every solution $y(t)$ of (3.1) with positive initial conditions converges to $\hat{y}$.

Denoting $y - \hat{y}$ by x, (3.1) can be written as

$$\dot{y}_i(t) = y_i(t) \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}). \tag{3.4}$$

The derivative of V along a solution $y(t)$ of (3.1) is then given by

$$\dot{V} = \sum_{i,j=1}^{n} \alpha_i x_i(t) a_{ij} x_j(t - \tau_{ij}) + \sum_{i,j=1}^{n} \beta_{ij} \left[x_j^2(t) - x_j^2(t - \tau_{ij}) \right].$$

First, we assume that A is an irreducible weakly diagonally dominant matrix. Then there exist $c_i > 0$ such that (2.2) holds. Moreover, there also exist $d_i > 0$ such that

$$d_i a_{ii} + \sum_{j \neq i} d_j |a_{ji}| \leq 0 \qquad \text{for } i = 1, \dots, n. \tag{3.5}$$

(The vectors c and d may be chosen as right and left eigenvectors of $\tilde{A}$.)

By choosing $\alpha_i = 2\frac{d_i}{c_i}$ and $\beta_{ij} = \frac{d_i}{c_j}|a_{ij}|$, we have

$$\dot{V} = \sum_{i \neq j} \frac{d_i}{c_j} |a_{ij}| x_j^2(t) + 2 \sum_{i,j=1}^{n} \frac{d_i}{c_i} a_{ij} x_i(t) x_j(t - \tau_{ij}) - \sum_{i \neq j} \frac{d_i}{c_j} |a_{ij}| x_j^2(t - \tau_{ij}). \tag{3.6}$$

After collecting terms, the coefficient of $x_i^2(t)$ (for i fixed) is given by

$$\sum_{j=1,\ j\neq i}^{n} \frac{d_j}{c_i}|a_{ji}| + 2\frac{d_i}{c_i}a_{ii}.$$

By adding $\frac{1}{c_i}$ times (3.5) to $\frac{d_i}{c_i^2}$ times (2.2), we see that this coefficient is no larger than $-\sum_{j=1,\ j\neq i}^{n} \frac{d_i c_j}{c_i^2}|a_{ij}|$. Hence (3.6) reduces to

$$\dot{V} \leq -\sum_{i\neq j} \frac{d_i}{c_j}|a_{ij}| \left(\frac{c_j}{c_i}x_i(t)\operatorname{sgn} a_{ij} - x_j(t-\tau_{ij})\right)^2 \leq 0. \tag{3.7}$$

The boundedness of V along a positive solution $y(t)$ implies the existence of constants $m, M > 0$ such that $m \leq y_i(t) \leq M$ for all i and for all $t \geq 0$. Hence forward orbits are precompact and the ω–limit set exists and is nonempty, compact and invariant (cf. Lemma 1.3 of Hale and Verduyn Lunel [7, p. 103]). By LaSalle's theorem, $\omega(y(.))$ is contained in the maximal compact invariant subset of $\dot{V} = 0$.

Equality $\dot{V} \equiv 0$ is possible for $y(.) \not\equiv \hat{y}$ only if there is equality in (2.2) and (3.5) (i.e., $\det \tilde{A} = 0$) and $x_j(t-\tau_{ij}) = \frac{c_j}{c_i}x_i(t)\operatorname{sgn} a_{ij}$ holds for all t and all i, j with $i \neq j$ and $a_{ij} \neq 0$. Inserting this into (3.1), we obtain

$$\dot{y}_i = y_i \left[a_{ii}x_i(t) + \sum_{j=1,\ j\neq i}^{n} a_{ij}x_j(t-\tau_{ij})\right] = y_i \left[a_{ii}x_i(t) + \sum_{j=1,\ j\neq i}^{n} |a_{ij}|\frac{c_j}{c_i}x_i(t)\right] = 0.$$

Hence $y(t)$ must be a constant solution of the differential equation (3.1). Since $\det A \neq 0$ by assumption, $x(t) \equiv 0$, i.e. $y(t) \equiv \hat{y}$ is the only positive constant solution of (3.1). Thus $\{\hat{y}\}$ is the global attractor and hence the maximal compact invariant subset in $C([-\tau, 0], \operatorname{int}\mathbb{R}_+^n)$, the space of *positive* continuous functions on $[-\tau, 0]$, where $\tau = \max_{i,j}\{\tau_{i,j}\}$. In particular, every positive solution $y(t)$ of (3.1) converges to $\hat{y}$ as $t \to \infty$, when A is irreducible.

Next, we consider the case of a general A. We will use induction on n to show that (i) for any positive solution $y(t)$ $(t \geq 0)$ of (3.1), there exist $0 < m \leq M$ such that $m \leq y_i(t) \leq M$ for all $t \geq 0$ and i, and that $\lim_{t\to\infty} y(t) = \hat{y}$ and (ii) if $y(t)$ is a positive solution (3.1) defined for all $t \in \mathbb{R}$ such that $m \leq y_i(t) \leq M$ for all t, where $0 < m \leq M$, then $y(t) \equiv \hat{y}$.

If A is irreducible, then the assertions hold by the above Lyapunov functional argument. So we assume A is reducible. By a renumbering of indices, A can be written in block triangular form $A = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$, with A_{11} irreducible of order k. The delay equation (3.1) can then be written schematically as a 2×2 system

$$\begin{aligned} \dot{Y}_1 &= Y_1(A_{11}X_1 + A_{12}X_2), \\ \dot{Y}_2 &= Y_2 A_{22} X_2. \end{aligned} \tag{3.8}$$

Since the order $n-k$ of A_{22} is strictly less than that of A, by the induction hypothesis on the second half of (3.8), we can assume that $y_i(t)$ remains bounded (both from 0 and ∞) for all $i = k+1, \ldots, n$ and $t \geq 0$ and $Y_2(t) \to \hat{Y}_2 = (\hat{y}_{k+1}, \ldots, \hat{y}_n)$ as $t \to \infty$. From the linear analysis of section 2, we can further infer that $|X_2(t)| \leq Ce^{-\varepsilon t}$ $(t \geq 0)$ for suitable constants $C, \varepsilon > 0$.

Now consider again the function V from (3.3), with n replaced by k, the size of the irreducible block A_{11}. Then a similar computation, taking into account (3.7) for the terms arising from A_{11}, leads to (with $|X_1|$ denoting the norm of the vector $X_1 = (x_1, \ldots x_k)$)

$$\begin{aligned}\dot{V} &\leq \sum_{i=1}^{k} \alpha_i x_i(t) \left[\sum_{j=k+1}^{n} a_{ij} x_j(t-\tau_{ij}) \right] \\ &\leq C_1 |X_1(t)| e^{-\varepsilon t} \leq C_2 (V(t) + C_3) e^{-\varepsilon t}.\end{aligned}$$

This differential inequality shows that $V(t)$ stays bounded as $t \to \infty$; and hence the first k components of $y(t)$ stay bounded (both from 0 and from ∞). This shows that there exist $0 < m \leq M$ such that $m \leq y_i(t) \leq M$ for all $i = 1, \ldots, n$ and $t \geq 0$.

Now consider a (full) solution $y(t) = (Y_1(t), Y_2(t))$ of (3.1) defined for all $t \in \mathbb{R}$ such that $m \leq y_i(t) \leq M$ for all i and t, where $0 < m \leq M$. Applying the induction hypothesis to the second part of (3.8), we deduce that $Y_2(t) \equiv \hat{Y}_2$. Thus $X_2(t) \equiv 0$. Hence $Y_1(t)$ satisfies $\dot{Y}_1 = Y_1 A_{11} X_1$. Since A_{11} is irreducble, we conclude $Y_1(t) \equiv \hat{Y}_1$. Hence $y(t) \equiv \hat{y}$.

Lastly, let $y(t)$ $(t \geq 0)$ be a positive solution of (3.1). Pick any z_t, a full orbit in $\omega(y_t)$, the ω–limit set of the orbit y_t. Then $z(t)$ is a solution of (3.1) defined for all $t \in \mathbb{R}$ which is bounded (from 0 and ∞). As was shown in the previous paragraph, $z(t) \equiv \hat{y}$. Hence $\omega(y_t) = \{\hat{y}\}$ and $\lim_{t\to\infty} y(t) = \hat{y}$, which completes the induction proof. □

Remark. We note that the sufficiency part for the linear case (Theorem 1) can also be shown by basically the same argument as above, using the Lyapunov function $V(x(.), t) = \sum_{i=1}^{n} \alpha_i x_i^2(t) + \sum_{i,j=1}^{n} \beta_{ij} \int_{t-\tau_{ij}}^{t} x_j^2(s)\, ds$ instead of (3.3).

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