

POST-LIE ALGEBRA STRUCTURES FOR PERFECT LIE ALGEBRAS

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ABSTRACT. We study the existence of post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where one of the algebras is perfect non-semisimple, and the other one is abelian, nilpotent non-abelian, solvable non-nilpotent, simple, semisimple non-simple, reductive non-semisimple or complete non-perfect. We prove several non-existence results, but also provide examples in some cases for the existence of a post-Lie algebra structure. Among other results we show that there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is perfect non-semisimple, and \mathfrak{n} is $\mathfrak{sl}_3(\mathbb{C})$. We also show that there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is perfect and \mathfrak{n} is reductive with a 1-dimensional center.

1. INTRODUCTION

Post-Lie algebras and Post-Lie algebra structures (or PA-structures) on pairs of Lie algebras have been studied in many areas of mathematics during the last years. PA-structures are a natural generalization of pre-Lie algebra structures on Lie algebras, which arise among other things from affine manifolds and affine actions on Lie groups, crystallographic groups, étale affine representations of Lie algebras, quantum field theory, operad theory, Rota-Baxter operators, and deformation theory of rings and algebras. There is a large literature on pre-Lie and post-Lie algebras, see for example [5, 6, 7, 8, 9, 12, 14, 18] and the references therein. For a survey on pre-Lie algebra respectively post-Lie algebra structures see [2, 11].

In the present article we study the existence question of post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where one Lie algebra is perfect and the other one is abelian, nilpotent, solvable, simple, semisimple, reductive, complete or perfect. In all but three cases we can solve the existence question and generalize our previous results for semisimple Lie algebras to perfect Lie algebras. However, for these three cases of $(\mathfrak{g}, \mathfrak{n})$, namely where \mathfrak{g} is perfect non-semisimple, and \mathfrak{n} is either nilpotent, simple, or semisimple, we are not able solve the existence question in general. We conjecture that there do not exist post-Lie algebra structures in these cases. For some special families of examples we can prove this conjecture.

The outline of this paper is as follows. In the second section we provide basic results on perfect Lie algebras, including a classification of complex perfect Lie algebras of dimension $n \leq 9$. We also recall the basic notions for post-Lie algebra structures. In the third section we study the existence question for pairs $(\mathfrak{g}, \mathfrak{n})$ where \mathfrak{g} is perfect. If \mathfrak{n} is nilpotent, then we show for perfect Lie algebras \mathfrak{g} of dimension 6, that there exist no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$. We also prove, using the classification of complex perfect Lie algebras in low dimension, that there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. We find post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ for examples of reductive Lie algebras \mathfrak{n} , and show that such structures do

Date: March 8, 2024.

2000 Mathematics Subject Classification. Primary 17B30, 17D25.

Key words and phrases. Post-Lie algebra structure, perfect Lie algebra.

not exist for reductive Lie algebras \mathfrak{n} with a 1-dimensional center. In the fourth section we study the existence question for pairs $(\mathfrak{g}, \mathfrak{n})$ where \mathfrak{n} is perfect. Here we often find post-Lie algebra structures and are able to solve the existence problem in all cases.

2. PRELIMINARIES

Let \mathfrak{g} be a finite-dimensional Lie algebra over a field K . Denote by $Z(\mathfrak{g})$ the center of \mathfrak{g} , by $\text{rad}(\mathfrak{g})$ the solvable radical of \mathfrak{g} , and by $\text{nil}(\mathfrak{g})$ the nilradical of \mathfrak{g} . A Lie algebra \mathfrak{g} is called *perfect*, if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Every semisimple Lie algebra over a field of characteristic zero is perfect. The converse does not hold. It is well known that the solvable radical of a perfect Lie algebra is nilpotent. One can also give a necessary and sufficient condition for a Levi decomposition of \mathfrak{g} , so that \mathfrak{g} is perfect.

Proposition 2.1. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \text{rad}(\mathfrak{g})$ be a Levi decomposition of a Lie algebra \mathfrak{g} and consider $V = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ with $\mathfrak{r} = \text{rad}(\mathfrak{g})$ as an \mathfrak{s} -module. Then \mathfrak{g} is perfect if and only if V does not contain the trivial 1-dimensional \mathfrak{s} -module.*

The lowest-dimensional example of a complex perfect non-semisimple Lie algebra is $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ of dimension 5. Here $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e_1, e_2, e_3\}$ has Lie brackets given by $[e_1, e_2] = e_3$, $[e_1, e_3] = -2e_1$, $[e_2, e_3] = 2e_2$, and $V(2) = \text{span}\{e_4, e_5\}$ is the natural representation of $\mathfrak{sl}_2(\mathbb{C})$. The Lie brackets of $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ are given by

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_2, e_3] &= 2e_2, & [e_3, e_5] &= -e_5. \\ [e_1, e_3] &= -2e_1, & [e_2, e_4] &= e_5, & & \\ [e_1, e_5] &= e_4, & [e_3, e_4] &= e_4, & & \end{aligned}$$

This Lie algebra is perfect and has a non-trivial solvable radical. Its center is trivial. We also give an example of a non-semisimple perfect Lie algebra with non-trivial center. For this, let $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{n}_3(\mathbb{C}) = \text{span}\{e_4, e_5, e_6\}$ be the Heisenberg Lie algebra, with $[e_4, e_5] = e_6$. Consider the following semidirect sum of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{n}_3(\mathbb{C})$, given by the following Lie brackets in the basis (e_1, \dots, e_6) :

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_2, e_3] &= 2e_2, & [e_3, e_5] &= -e_5, \\ [e_1, e_3] &= -2e_1, & [e_2, e_4] &= e_5, & [e_4, e_5] &= e_6. \\ [e_1, e_5] &= e_4, & [e_3, e_4] &= e_4, & & \end{aligned}$$

We denote this Lie algebra by $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$.

Example 2.2. *The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$ is perfect, but not semisimple. It has a 1-dimensional center.*

Indeed, we have $Z(\mathfrak{g}) = \text{span}\{e_6\}$, and \mathfrak{g} is not semisimple. The nilradical $\mathfrak{n}_3(\mathbb{C})$ is isomorphic to $V(2) \oplus V(1)$ as $\mathfrak{sl}_2(\mathbb{C})$ -module, where $V(n)$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{C})$ -module of dimension $n \geq 1$. It follows from the Lie brackets that \mathfrak{g} is perfect. We also can derive this from Proposition 2.1. The quotient $V = \mathfrak{n}_3/[\mathfrak{n}_3, \mathfrak{n}_3]$ is isomorphic to $V(2)$ and does not contain the trivial 1-dimensional module $V(1)$. Hence \mathfrak{g} is perfect.

In the study of PA-structures for perfect Lie algebras we are also interested in a classification of perfect Lie algebras in low dimensions. Turkowski has classified Lie algebras with non-trivial Levi decomposition up to dimension 8 over the real numbers in [16], where he lists explicit Lie brackets for all algebras. From this work it is not difficult to derive a classification of complex perfect Lie algebras of dimension $n \leq 8$. We need to add one Lie algebra though, which Turkowski has not in his list. It is the complexification of the algebra $L_{8,13}^\varepsilon$ for $\varepsilon = 0$, isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus \mathfrak{n}_3(\mathbb{C}))$. Turkowski only allows $\varepsilon = \pm 1$.

There is another classification given by Alev, Ooms and Van den Bergh in [1], namely the classification of non-solvable algebraic Lie algebras of dimension $n \leq 8$ over an algebraically closed field of characteristic zero. It also contains explicit Lie brackets for all algebras. Since perfect Lie algebras are algebraic, we obtain again a list of complex perfect non-semisimple Lie algebras of dimension $n \leq 8$. This list coincides with the (corrected) one by Turkowski. Let $\mathfrak{n}_5(\mathbb{C})$ denote the 5-dimensional Heisenberg Lie algebra, with basis (e_1, \dots, e_5) and Lie brackets

$$[e_1, e_2] = [e_3, e_4] = e_5.$$

Denote by $\mathfrak{f}_{2,3}(\mathbb{C})$ the free-nilpotent Lie algebra with 2 generators and nilpotency class 3, with basis (e_1, \dots, e_5) and Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5.$$

The classification result is as follows.

Proposition 2.3. *Every complex perfect non-semisimple Lie algebra of dimension $n \leq 8$ is isomorphic to one of the following Lie algebras:*

\mathfrak{g}	$\dim \mathfrak{g}$	$\dim Z(\mathfrak{g})$	Turkowski	Alev et al.
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$	5	0	$L_{5,1}$	L_5
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3)$	6	0	$L_{6,4} \cong L_{6,1}$	$L_{6,1}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$	6	1	$L_{6,2}$	$L_{6,3}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(4)$	7	0	$L_{7,6}$	$L_{7,1}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(2))$	7	0	$L_{7,7}$	$L_{7,2}$
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$	8	0	$\mathfrak{sl}_2 \oplus L_{5,1}$	$\mathfrak{sl}_2 \oplus L_5$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(5)$	8	0	$L_{8,21}$	$L_{8,1}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(3))$	8	0	$L_{8,22}$	$L_{8,2}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus \mathfrak{n}_3(\mathbb{C}))$	8	1	$L_{8,13}^{\varepsilon=0}$	$L_{8,13}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{f}_{2,3}(\mathbb{C})$	8	0	$L_{8,15}$	$L_{8,18}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes_{\varphi} \mathfrak{n}_5(\mathbb{C})$	8	1	$L_{8,13}^1 \cong L_{8,13}^{-1}$	$L_{8,15}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes_{\psi} \mathfrak{n}_5(\mathbb{C})$	8	1	$L_{8,19}$	$L_{8,16}$

Turkowski has also classified in [17] real and complex Lie algebras with non-trivial Levi decomposition in dimension 9. This yields a list of complex perfect Lie algebras of dimension 9. Denote by $\mathfrak{f}_{3,2}(\mathbb{C})$ the free-nilpotent Lie algebra with 3 generators and nilpotency class 2,

and by $\mathcal{A}_{6,4}(\mathbb{C})$ the 2-step nilpotent Lie algebra with basis (e_1, \dots, e_6) and Lie brackets

$$[e_1, e_4] = e_5, [e_2, e_3] = -e_5, [e_3, e_4] = e_6.$$

Then we have the following result.

Proposition 2.4. *Every complex perfect non-semisimple Lie algebra of dimension 9 is isomorphic to one of the following Lie algebras:*

\mathfrak{g}	$\dim \mathfrak{g}$	$\dim Z(\mathfrak{g})$	<i>Turkowski</i>
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3))$	9	0	$\mathfrak{sl}_2 \oplus L_{6,1}$
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C}))$	9	1	$\mathfrak{sl}_2 \oplus L_{6,2}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(6)$	9	0	$L_{9,59}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(4))$	9	0	$L_{9,60}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(3) \oplus V(3))$	9	0	$L_{9,61}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(2) \oplus V(2))$	9	0	$L_{9,63}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(3) \oplus \mathfrak{n}_3(\mathbb{C}))$	9	1	$L_{9,58}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (\mathfrak{n}_3(\mathbb{C}) \oplus \mathfrak{n}_3(\mathbb{C}))$	9	2	$L_{9,37} \cong L_{9,42}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{f}_{3,2}(\mathbb{C})$	9	0	$L_{9,62}$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathcal{A}_{6,4}(\mathbb{C})$	9	2	$L_{9,41}$

In particular the nilradical of such an algebra always has nilpotency class $c \leq 2$.

We recall the definition of a post-Lie algebra structure on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ over a field K , see [5]:

Definition 2.5. Let $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{, \})$ be two Lie brackets on a vector space V over K . A *post-Lie algebra structure*, or *PA-structure* on the pair $(\mathfrak{g}, \mathfrak{n})$ is a K -bilinear product $x \cdot y$ satisfying the identities:

- (1) $x \cdot y - y \cdot x = [x, y] - \{x, y\}$
- (2) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$
- (3) $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$

for all $x, y, z \in V$.

Define by $L(x)(y) = x \cdot y$ the left multiplication operators of the algebra $A = (V, \cdot)$. By (3), all $L(x)$ are derivations of the Lie algebra $(V, \{, \})$. Moreover, by (2), the left multiplication

$$L: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{n}) \subseteq \text{End}(V), \quad x \mapsto L(x)$$

is a linear representation of \mathfrak{g} .

If \mathfrak{n} is abelian, then a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ corresponds to a *pre-Lie algebra structure*, or left-symmetric structure on \mathfrak{g} . In other words, if $\{x, y\} = 0$ for all $x, y \in V$, then the conditions reduce to

- (4) $x \cdot y - y \cdot x = [x, y]$
- (5) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z),$

i.e., $x \cdot y$ is a *pre-Lie algebra structure* on the Lie algebra \mathfrak{g} .

For semisimple Lie algebras \mathfrak{n} we have the following result on PA-structures on pairs $(\mathfrak{g}, \mathfrak{n})$, see Proposition 2.14 in [5].

Proposition 2.6. *Let \mathfrak{n} be a semisimple Lie algebra. Then a pair $(\mathfrak{g}, \mathfrak{n})$ admits a PA-structure if and only if there is an injective Lie algebra homomorphism $\varphi: \mathfrak{g} \hookrightarrow \mathfrak{n} \oplus \mathfrak{n}$ such that the map $(p_1 - p_2)|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathfrak{n}$ is bijective. Here $p_i: \mathfrak{n} \oplus \mathfrak{n} \rightarrow \mathfrak{n}$ denotes the projection onto the i -th factor for $i = 1, 2$.*

Let us denote the composition of p_i and φ by $j_i = \varphi \circ p_i: \mathfrak{g} \rightarrow \mathfrak{n}$ for $i = 1, 2$.

We also recall the following results, see Propositions 2.14 and 2.21 in [9].

Proposition 2.7. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{n} is complete. Then there is a bijection between PA-structures on $(\mathfrak{g}, \mathfrak{n})$ and Rota-Baxter operators R of weight 1 on \mathfrak{n} . Every such PA-structure is then of the form $x \cdot y = \{R(x), y\}$. If \mathfrak{g} and \mathfrak{n} are not isomorphic, then both $\ker(R)$ and $\ker(R + \text{id})$ are nonzero ideals in \mathfrak{g} .*

Here a Rota-Baxter algebra operator, for a nonassociative algebra over a field K , of weight $\lambda \in K$ is a linear operator $R: A \rightarrow A$ satisfying

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

for all $x, y \in A$.

3. PA-STRUCTURES WITH \mathfrak{g} PERFECT

In this section we study the existence question of PA-structures on pairs of complex Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is perfect non-semisimple. We consider 7 different cases for \mathfrak{n} , namely (a) \mathfrak{n} is abelian, (b) \mathfrak{n} is nilpotent non-abelian, (c) \mathfrak{n} is solvable non-nilpotent, (d) \mathfrak{n} is simple, (e) \mathfrak{n} is semisimple non-simple, (f) \mathfrak{n} is reductive non-semisimple, and (g) \mathfrak{n} is complete non-perfect.

We start with case (a).

Proposition 3.1. *There is no PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is perfect and \mathfrak{n} is abelian.*

Proof. A PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} is abelian, corresponds to a left-symmetric (or pre-Lie algebra) structure on \mathfrak{g} . By Corollary 21 of [15] there is no such structure on a perfect Lie algebra. \square

For case (b) we only have partial results so far. In Proposition 3.6 of [7] we have proved that there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is perfect of dimension 5, namely $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$, and \mathfrak{n} is nilpotent. We can generalize this result to perfect Lie algebras \mathfrak{g} of dimension 6. According to Proposition 2.3, the non-semisimple perfect Lie algebras of dimension 6 are given by $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3)$ and $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$.

Proposition 3.2. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is either $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3)$ or $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$, and \mathfrak{n} is nilpotent. Then there is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{a}$ and assume that there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, with the homomorphism $L: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{n})$ given as in Definition 2.5. Let $\varphi: \mathfrak{g} \hookrightarrow \mathfrak{n} \times \text{Der}(\mathfrak{n})$ be the embedding defined by $x \mapsto (x, L(x))$. We claim that $\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = 0$. Suppose that this

intersection is nonzero. Since it is an ideal in the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, this implies that $\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$, so that $\mathfrak{sl}_2(\mathbb{C}) \subseteq \ker(L)$. In particular, we have $L(s) = 0$ for all $s \in \mathfrak{sl}_n(\mathbb{C})$. By axiom (1) in Definition 2.5 it follows that

$$[s, t] - \{s, t\} = s \cdot t - t \cdot s = 0$$

for all $s, t \in \mathfrak{sl}_2(\mathbb{C})$. Hence \mathfrak{n} has a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. This is impossible, because \mathfrak{n} is nilpotent, so that the claim follows.

We have shown in the proof of Theorem 3.3 in [12] that the Lie algebra $\mathfrak{n} \rtimes \mathfrak{h}$, with

$$\mathfrak{h} = L(\mathfrak{g}) \subseteq \text{Der}(\mathfrak{n}),$$

has a direct vector space sum decomposition

$$\mathfrak{n} \rtimes \mathfrak{h} = \varphi(\mathfrak{g}) \dot{+} \mathfrak{h}.$$

Since $\varphi(\mathfrak{g})$ and \mathfrak{h} are homomorphic images of a perfect Lie algebra, they are perfect. Hence $\mathfrak{n} \rtimes \mathfrak{h}$ is perfect. Let $\mathfrak{s} = L(\mathfrak{sl}_2(\mathbb{C}))$ and $\mathfrak{r} = L(\mathfrak{a})$. Then we have $\mathfrak{h} = \mathfrak{s} \rtimes \mathfrak{r}$. Since $\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = 0$, \mathfrak{s} is nonzero and semisimple. Hence $\dim(\mathfrak{s}) = 3$ and $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C})$. So

$$\mathfrak{h} \rtimes \mathfrak{n} = (\mathfrak{s} \rtimes \mathfrak{r}) \rtimes \mathfrak{n}$$

is perfect and has nilradical $\mathfrak{m} = \mathfrak{r} \dot{+} \mathfrak{n}$. By Proposition 2.1, $\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}]$ does not contain the trivial 1-dimensional \mathfrak{s} -module $V(1)$. Since $\mathfrak{n} \subseteq \mathfrak{m}$, also $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{m}/[\mathfrak{m}, \mathfrak{m}]$ does not contain the trivial 1-dimensional \mathfrak{s} -module $V(1)$. Hence $\mathfrak{s} \rtimes \mathfrak{n}$ is a perfect Lie algebra of dimension 9. By Proposition 2.4, the nilpotency class $c(\mathfrak{n})$ of \mathfrak{n} is at most 2. Proposition 4.2 of [5] says, that if $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure, and $c(\mathfrak{n}) \leq 2$, then \mathfrak{g} admits a pre-Lie algebra structure. Since \mathfrak{g} is perfect, this is impossible by Corollary 21 of [15]. \square

Let us again state the last result used in the proof, see also Proposition 3.3 in [7].

Proposition 3.3. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is perfect and \mathfrak{n} is 2-step nilpotent. Then there exists no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

In case (c), we have proved the following result in Proposition 4.4 of [5].

Proposition 3.4. *There is no PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is perfect and \mathfrak{n} is solvable non-nilpotent.*

For case (d) we start with low-dimensional simple Lie algebras \mathfrak{n} . There is no pair $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} \cong \mathfrak{sl}_2(\mathbb{C})$ and \mathfrak{g} perfect non-semisimple, since the only perfect Lie algebra in dimension 3 is simple. The next case is to consider pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n} \cong \mathfrak{sl}_3(\mathbb{C})$ and \mathfrak{g} is a perfect non-semisimple Lie algebra of dimension 8. We start with the following result.

Lemma 3.5. *Let $i: \mathfrak{sl}_2(\mathbb{C}) \rtimes V(2) \hookrightarrow \mathfrak{sl}_3(\mathbb{C})$ be an injective Lie algebra homomorphism. By conjugating with a matrix in $GL_3(\mathbb{C})$ we may assume that the image of i is of the form*

$$\text{im}(i) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & -a_1 & a_5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\}, \text{ or } \text{im}(i) = \left\{ \begin{pmatrix} b_1 & b_2 & 0 \\ b_3 & -b_1 & 0 \\ b_4 & b_5 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\}.$$

Proof. The vector space \mathbb{C}^3 becomes an $\mathfrak{sl}_2(\mathbb{C})$ -module via the embedding i restricted to $\mathfrak{sl}_2(\mathbb{C})$. Hence either $\mathbb{C}^3 \cong V(2) \oplus V(1)$, or $\mathbb{C}^3 \cong V(3)$ as $\mathfrak{sl}_2(\mathbb{C})$ -module. In the first case we can choose

a basis of \mathbb{C}^3 such that

$$i(\mathfrak{sl}_2(\mathbb{C})) = \left\{ \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & -a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\},$$

by using the natural representation $i(e_1) = E_{12}$, $i(e_2) = E_{21}$ and $i(e_3) = E_{11} - E_{22}$ for the basis (e_1, e_2, e_3) of $\mathfrak{sl}_2(\mathbb{C})$. A short computation shows that when such a representation extends to $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$, we obtain one of the forms for $\text{im}(i)$ as described above.

In the second case we may assume that $i(e_1) = E_{12} + 2E_{23}$, $i(e_2) = E_{21} + 2E_{32}$ and $i(e_3) = 2E_{11} - 2E_{33}$. It is easy to see that this representation does not extend to one of $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$. \square

Lemma 3.6. *Let $j: \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{sl}_3(\mathbb{C})$ be an injective Lie algebra homomorphism. Denote by $c_i: \mathfrak{gl}_3(\mathbb{C}) \rightarrow \mathbb{C}^3$ the projection of a matrix in $\mathfrak{gl}_3(\mathbb{C})$ to its i -th column, and by $r_i: \mathfrak{gl}_3(\mathbb{C}) \rightarrow \mathbb{C}^3$ the projection to its i -th row. Then none of the linear maps $c_i \circ j, r_i \circ j: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathbb{C}^3$ for $i = 1, 2, 3$ is bijective.*

Proof. It is enough to show the claim for columns. We obtain the result for rows by applying the isomorphism of Lie algebras $\mathfrak{sl}_3(\mathbb{C}) \rightarrow \mathfrak{sl}_3(\mathbb{C})$, given by $X \mapsto -X^T$, to the result for columns. We will give the proof for $c_3 \circ j$. The other two cases are similar. Note that \mathbb{C}^3 is an $\mathfrak{sl}_2(\mathbb{C})$ -module via j . Hence the map $c_i \circ j$ is actually the map

$$\mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathbb{C}^3, x \mapsto x \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Because of $\dim \mathfrak{sl}_2(\mathbb{C}) = \dim \mathbb{C}^3$ the annihilator of any vector is non-trivial by Lemma 4.1 in [6], so that the map $c_3 \circ j$ is not injective. \square

Lemma 3.7. *Let \mathfrak{s}_1 and \mathfrak{s}_2 be Lie algebras isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Then the ideals of the Lie algebra $\mathfrak{g} = \mathfrak{s}_1 \oplus (\mathfrak{s}_2 \ltimes V(2))$ are given by*

$$0, \mathfrak{s}_1, \mathfrak{s}_2 \ltimes V(2), \mathfrak{s}_1 \oplus V(2), V(2), \mathfrak{g}.$$

Proof. It is clear that all of these subspaces are ideals in \mathfrak{g} . Conversely, assume that \mathfrak{a} is an ideal in \mathfrak{g} . If $\mathfrak{a} \cap \mathfrak{s}_1 \neq 0$, then $\mathfrak{s}_1 \subseteq \mathfrak{a}$ and $\mathfrak{a}/\mathfrak{s}_1$ is an ideal of $\mathfrak{g}/\mathfrak{s}_1 \cong \mathfrak{s}_2 \ltimes V(2)$. But the only ideals of $\mathfrak{s}_2 \ltimes V(2)$ are $0, V(2)$ and $\mathfrak{s}_2 \ltimes V(2)$. So, if $\mathfrak{a} \cap \mathfrak{s}_1 \neq 0$ then all ideals are given by $\mathfrak{s}_1, \mathfrak{s}_1 \oplus V(2)$ and \mathfrak{g} .

Now suppose that $\mathfrak{a} \cap \mathfrak{s}_1 = 0$. We claim that then $\mathfrak{a} \subseteq \mathfrak{s}_2 \ltimes V(2)$. Indeed, suppose that there exists an element x in $\mathfrak{a} \setminus (\mathfrak{s}_2 \ltimes V(2))$. We can write $x = x_1 + x_2$ with $x_1 \in \mathfrak{s}_1$ and $x_2 \in \mathfrak{s}_2 \ltimes V(2)$, where $x_1 \neq 0$. There exists a $y \in \mathfrak{s}_1$ such that $[y, x_1] \neq 0$, so that $0 \neq [y, x] = [y, x_1 + x_2] = [y, x_1] \in \mathfrak{s}_1 \cap \mathfrak{a}$, which is a contradiction. Hence we have $\mathfrak{a} \subseteq \mathfrak{s}_2 \ltimes V(2)$, which leads to the ideals $0, V(2)$ and $\mathfrak{s}_2 \ltimes V(2)$. \square

Lemma 3.8. *There is no direct vector space decomposition $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{a} \dot{+} \mathfrak{b}$ with subalgebras \mathfrak{a} and \mathfrak{b} of $\mathfrak{sl}_3(\mathbb{C})$ satisfying $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ and $\mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{C})$.*

Proof. Assume that there is such a decomposition $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{a} \dot{+} \mathfrak{b}$. Then after applying a base change we may assume by Lemma 3.5 that

$$\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & -a_1 & a_5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\}, \text{ or } \mathfrak{a} = \left\{ \begin{pmatrix} b_1 & b_2 & 0 \\ b_3 & -b_1 & 0 \\ b_4 & b_5 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\}$$

As $\mathfrak{sl}_3(\mathbb{C})$ is a direct vector space sum of \mathfrak{a} and \mathfrak{b} we must have that the row projection map $r_3: \mathfrak{b} \rightarrow \mathbb{C}^3$ is bijective in the first case, and the column projection map $c_3: \mathfrak{b} \rightarrow \mathbb{C}^3$ is bijective in the second case. However, by Lemma 3.6, this is impossible. \square

We can now apply these lemmas to PA-structures on pairs $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$, where \mathfrak{g} has a Levi subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Proposition 3.9. *Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$ and $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. Then there is no PA-structure on the pair $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Assume that there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. Let us write $\mathfrak{g} = \mathfrak{s}_1 \oplus (\mathfrak{s}_2 \ltimes V(2))$. By Proposition 2.6 there exists an injective Lie algebra homomorphism $j: \mathfrak{g} \rightarrow \mathfrak{n} \oplus \mathfrak{n}$, $x \mapsto (j_1(x), j_2(x))$ such that $j_1 - j_2$ is a bijective linear map. We will examine the possible kernels of j_1 . Since $\ker(j_1)$ is an ideal in \mathfrak{g} , it must be one of the six possibilities given in Lemma 3.7.

Case 1: $\ker(j_1) = 0$. Then $j_1: \mathfrak{g} \rightarrow \mathfrak{n}$ is an isomorphism. Since \mathfrak{g} is not semisimple, this is a contradiction.

Case 2: $\ker(j_1) = \mathfrak{g}$. Then j_1 is the zero map. Since $j_1 - j_2$ has to be bijective, $j_2: \mathfrak{g} \rightarrow \mathfrak{n}$ is an isomorphism. This is impossible.

Case 3: $\ker(j_1) = V(2)$. Then the representation

$$j_1|_{\mathfrak{s}_1 \oplus \mathfrak{s}_2}: \mathfrak{s}_1 \oplus \mathfrak{s}_2 \rightarrow \mathfrak{sl}_3(\mathbb{C})$$

is faithful. However, the smallest dimension of a faithful representation of $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ is equal to 4, see [3], Proposition 2.5. This is a contradiction.

Case 4: $\ker(j_1) = \mathfrak{s}_1 \oplus V(2)$. Then

$$j_2|_{\mathfrak{s}_1 \oplus V(2)} = (j_2 - j_1)|_{\mathfrak{s}_1 \oplus V(2)}: \mathfrak{s}_1 \oplus V(2) \rightarrow \mathfrak{sl}_3(\mathbb{C})$$

has to be injective. However, $\mathfrak{s}_1 \oplus V(2)$ contains a 3-dimensional abelian subalgebra, whereas $\mathfrak{sl}_3(\mathbb{C})$ does not. This is a contradiction.

Case 5: $\ker(j_1) = \mathfrak{s}_2 \ltimes V(2)$. Then $(\mathfrak{s}_2 \ltimes V(2)) \cap \ker(j_2) = 0$. Since $\ker(j_2)$ is an ideal, which is nonzero as in case 1, we have $\ker(j_2) = \mathfrak{s}_1$. So for every $x \in \mathfrak{g}$ we can write $x = x_1 + x_2$ with $x_1 \in \mathfrak{s}_1$ and $x_2 \in \mathfrak{s}_2 \ltimes V(2)$. Then we have $(j_1 - j_2)(x) = j_1(x_1) - j_2(x_2)$, and $j_1 - j_2$ is injective if and only if $\text{im}(j_1) \cap \text{im}(j_2) = 0$. This is equivalent to

$$\mathfrak{sl}_3(\mathbb{C}) = \text{im}(j_1) \dot{+} \text{im}(j_2)$$

with $\text{im}(j_1) \cong \mathfrak{s}_1$ and $\text{im}(j_2) \cong \mathfrak{s}_2 \ltimes V(2)$, which is a contradiction to Lemma 3.8.

Case 6: $\ker(j_1) = \mathfrak{s}_1$. Then \mathfrak{s}_1 is not contained in $\ker(j_2)$. So from the six possibilities for the ideal $\ker(j_2)$, there are left 0, $V(2)$ and $\mathfrak{s}_2 \ltimes V(2)$. We already know that $\ker(j_2)$ must be nonzero. Also, $\ker(j_2) = V(2)$ leads to a contradiction as in case 3. Finally $\ker(j_2) = \mathfrak{s}_2 \ltimes V(2)$ and $\ker(j_1) = \mathfrak{s}_1$ is exactly the symmetric situation to case 5, and so also leads to a contradiction. \square

Now we can prove the following result.

Theorem 3.10. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is perfect non-semisimple and $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. Then there is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Denote by \mathfrak{s} a Levi subalgebra of \mathfrak{g} . Then either $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C})$. In the first case we have $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$ by Proposition 2.3. Then the claim follows by Proposition 3.9. In the second case, by Proposition 2.3, $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{r}$, where \mathfrak{r} is isomorphic to one of the following five Lie algebras

$$V(5), V(2) \oplus V(3), V(2) \oplus \mathfrak{n}_3(\mathbb{C}), \mathfrak{n}_5(\mathbb{C}), \mathfrak{f}_{2,3}(\mathbb{C}).$$

Again we are using the maps $j_i: \mathfrak{g} \rightarrow \mathfrak{n}$, and assume that $j_1 - j_2$ is bijective. In the first two cases, either j_1 or j_2 must be injective on the factor $V(3)$, respectively $V(5)$. This contradicts the fact that $\mathfrak{sl}_3(\mathbb{C})$ does not contain an abelian subalgebra of dimension $n \geq 3$.

Assume that $\mathfrak{r} \cong V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. We will look again at the possibilities for $\ker(j_1)$. If it has a non-trivial Levi factor, then $\ker(j_1) = \mathfrak{g}$, so that $j_2: \mathfrak{g} \rightarrow \mathfrak{n}$ is bijective. This is a contradiction. Hence we may assume that $\ker(j_1)$ is solvable, so that it is contained in $V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. Here we can view $\ker(j_1)$ both as a subalgebra and as a submodule of $V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. As an $\mathfrak{sl}_2(\mathbb{C})$ -submodule, $V(2) \oplus \mathfrak{n}_3(\mathbb{C})$ is isomorphic to $V(2) \oplus V(2) \oplus V(1)$, because $\mathfrak{n}_3(\mathbb{C}) \cong V(2) \oplus V(1)$, see Example 2.2. The submodule $V(2) \oplus V(2)$ cannot occur as an ideal, since there is no subalgebra corresponding to it. So we have the following possibilities for $\ker(j_1)$ as a submodule:

Case 1: $\ker(j_1) = 0$. Then j_1 is an isomorphism. This is a contradiction.

Case 2: $\ker(j_1) \cong V(2)$. Then $j_1|_{\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})}$ is injective. This is impossible, because $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$ has a 3-dimensional abelian subalgebra, see Example 2.2, but $\mathfrak{sl}_3(\mathbb{C})$ does not have one.

Case 3: $\ker(j_1) \cong \mathfrak{n}_3(\mathbb{C})$. Since $\ker(j_1) \cap \ker(j_2) = 0$, it follows that j_2 induces an injective homomorphism $\mathfrak{sl}_2(\mathbb{C}) \ltimes \ker(j_1) \hookrightarrow \mathfrak{sl}_3(\mathbb{C})$, which is impossible as in case 2.

Case 4: $\ker(j_1) \cong V(2) \oplus V(1)$, with $V(1) \cong Z(\mathfrak{n}_3(\mathbb{C}))$. Then j_2 is injective on $\ker(j_1)$, which is impossible, because $V(2) \oplus V(1)$ is an abelian subalgebra of dimension 3.

Case 5: $\ker(j_1) \cong V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. Then $j_2|_{V(2) \oplus \mathfrak{n}_3(\mathbb{C})}$ is injective. This is impossible, because $V(2) \oplus \mathfrak{n}_3(\mathbb{C})$ has a 3-dimensional abelian subalgebra, but $\mathfrak{sl}_3(\mathbb{C})$ does not have one.

Assume that $\mathfrak{r} \cong \mathfrak{n}_5(\mathbb{C})$. We claim that at least one of the maps j_1, j_2 must be injective on $\mathfrak{n}_5(\mathbb{C})$. Otherwise $j_1(Z(\mathfrak{n}_5(\mathbb{C}))) = j_2(Z(\mathfrak{n}_5(\mathbb{C}))) = 0$, so that $(j_1 - j_2)|_{Z(\mathfrak{n}_5(\mathbb{C}))} = 0$, which is a contradiction to the fact that $j_1 - j_2$ is bijective. So we may assume that j_1 or j_2 is injective. This is impossible since $\mathfrak{n}_5(\mathbb{C})$ contains a 3-dimensional abelian subalgebra, but $\mathfrak{sl}_3(\mathbb{C})$ does not have one.

Finally assume that $\mathfrak{r} \cong \mathfrak{f}_{2,3}(\mathbb{C})$. Then again at least one of the maps j_1, j_2 must be injective on \mathfrak{r} . If j_1 is not injective on \mathfrak{r} , then $Z(\mathfrak{r}) \cap \ker(j_1) \neq 0$. We claim that $Z(\mathfrak{r}) \subseteq \ker(j_1)$. In fact, every ideal \mathfrak{a} of \mathfrak{g} satisfying $\mathfrak{a} \cap Z(\mathfrak{r}) \neq 0$ also satisfies $Z(\mathfrak{r}) \subseteq \mathfrak{a}$. To see this, note that $Z(\mathfrak{r}) = [\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]]$, and that the action of $\mathfrak{sl}_2(\mathbb{C})$ on $[\mathfrak{r}, \mathfrak{r}]/[\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]}$ is trivial, since the quotient is 1-dimensional. It follows that the action of $\mathfrak{sl}_2(\mathbb{C})$ on $Z(\mathfrak{r})$ coincides with the action on $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$. By Proposition 2.1 it has no trivial 1-dimensional submodule, since \mathfrak{g} is perfect. This also implies that $Z(\mathfrak{r})$ has no trivial 1-dimensional submodule, so that $\dim(\mathfrak{a} \cap Z(\mathfrak{r})) \geq 2$. Since $\dim Z(\mathfrak{r}) = 2$ it follows that $Z(\mathfrak{r}) \subseteq \mathfrak{a}$. Hence if both j_1 and j_2 are not injective on \mathfrak{r} , the center $Z(\mathfrak{r})$ is contained in $\ker(j_1)$ and $\ker(j_2)$, so that $(j_1 - j_2)|_{Z(\mathfrak{r})} = 0$. This is a contradiction. Consequently, $j_i: \mathfrak{f}_{2,3}(\mathbb{C}) \rightarrow \mathfrak{sl}_3(\mathbb{C})$ is an injection for some i , contradicting the fact that $\mathfrak{f}_{2,3}(\mathbb{C})$ has a 3-dimensional abelian subalgebra, but $\mathfrak{sl}_3(\mathbb{C})$ does not have one. \square

It is not clear how to generalize this proof for other simple Lie algebras \mathfrak{n} .

For case (e) we can prove the following general result.

Proposition 3.11. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is perfect non-semisimple and \mathfrak{n} is semisimple. Assume that we have a Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes V$, where \mathfrak{s} is a simple subalgebra and V is an irreducible \mathfrak{s} -module, considered as abelian Lie algebra. Then there is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Suppose that there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then by Proposition 2.7 it is of the form $x \cdot y = \{R(x), y\}$ for a Rota-Baxter operator $R: \mathfrak{g} \rightarrow \mathfrak{n}$ of weight 1 on \mathfrak{n} . Moreover, since \mathfrak{g} and \mathfrak{n} are not isomorphic, both $\ker(R)$ and $\ker(R + \text{id})$ are nonzero ideals of \mathfrak{g} with

$$\ker(R) \cap \ker(R + \text{id}) = 0.$$

We will show that the only ideals of $\mathfrak{g} = \mathfrak{s} \ltimes V$ are $0, V, \mathfrak{g}$. Then it is clear that V is contained in the above intersection, so that $V = 0$. This is a contradiction. So let \mathfrak{a} be an ideal of \mathfrak{g} . Then we obtain a Levi decomposition for \mathfrak{a} by

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes (\mathfrak{a} \cap V).$$

Since $\mathfrak{a} \cap \mathfrak{s}$ is an ideal in \mathfrak{s} , and \mathfrak{s} is simple, we have either $\mathfrak{a} \cap \mathfrak{s} = 0$ or $\mathfrak{a} \cap \mathfrak{s} = \mathfrak{s}$. Also, since $\mathfrak{a} \cap V$ is an \mathfrak{s} -submodule of V and V is irreducible, we have either $\mathfrak{a} \cap V = 0$ or $\mathfrak{a} \cap V = V$.

Case 1: $\mathfrak{a} \cap V = 0$. Then $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{s}$ is either zero or \mathfrak{s} . It follows that either $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{s}$. But \mathfrak{s} is not an ideal in \mathfrak{g} , so that we obtain $\mathfrak{a} = 0$.

Case 2: $\mathfrak{a} \cap V = V$. Then $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes V$, which means either $\mathfrak{a} = V$ or $\mathfrak{a} = \mathfrak{g}$.

So we obtain $V = 0$ and hence a contradiction. □

For $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ we obtain a further result for case (e). In Theorem 4.1 of [9] we have classified all Lie algebras \mathfrak{g} , such that the pair $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ admits a PA-structure. Here we have used the theory of Rota-Baxter operators. It is easy to see that none of the eight cases for \mathfrak{g} in this classification yields a perfect, non-semisimple Lie algebra. Hence we obtain the following result.

Proposition 3.12. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is perfect non-semisimple and $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Then there is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

For case (f) consider the perfect non-semisimple Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$. Let (e_1, \dots, e_5) be a basis of \mathfrak{g} with $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e_1, e_2, e_3\}$, $V(2) = \text{span}\{e_4, e_5\}$, and Lie brackets given as follows:

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_2, e_3] &= 2e_2, & [e_3, e_4] &= e_4, \\ [e_1, e_3] &= -2e_1, & [e_2, e_4] &= e_5, & [e_3, e_5] &= -e_5, \\ [e_1, e_5] &= e_4. \end{aligned}$$

Example 3.13. *The pair of Lie algebras $(\mathfrak{g}, \mathfrak{n}) = (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2), \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2)$ admits a PA-structure given by*

$$\begin{aligned} e_1 \cdot e_5 &= e_4, & e_2 \cdot e_4 &= e_5, \\ e_3 \cdot e_4 &= e_4, & e_3 \cdot e_5 &= -e_5. \end{aligned}$$

Here \mathfrak{n} is reductive with a 2-dimensional center. Such examples are impossible when \mathfrak{n} is reductive with a 1-dimensional center, as the following result shows.

Proposition 3.14. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is perfect and \mathfrak{n} is reductive with a 1-dimensional center. Then there is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Assume that there exists a PA-structure $x \cdot y = L(x)(y)$ on $(\mathfrak{g}, \mathfrak{n})$. Then by Proposition 2.11 in [5] we have an injective Lie algebra homomorphism

$$\varphi: \mathfrak{g} \hookrightarrow \mathfrak{n} \rtimes \text{Der}(\mathfrak{n}), \quad x \mapsto (x, L(x)).$$

Writing $\mathfrak{h} = L(\mathfrak{g})$ we obtain a direct vector space decomposition $\mathfrak{n} \rtimes \mathfrak{h} = \varphi(\mathfrak{g}) \dot{+} \mathfrak{h}$. Note that \mathfrak{h} is nonzero. Since \mathfrak{g} is perfect and φ and L are homomorphisms, $\varphi(\mathfrak{g})$ and \mathfrak{h} are perfect subalgebras of $\mathfrak{n} \rtimes \mathfrak{h}$. Hence also $\varphi(\mathfrak{g}) \dot{+} \mathfrak{h}$ is perfect, see the proof of Lemma 2.3 in [13], so that $\mathfrak{n} \rtimes \mathfrak{h}$ is perfect. By assumption we have $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus Z(\mathfrak{n})$, where $\mathfrak{s} = [\mathfrak{n}, \mathfrak{n}]$ is semisimple and $Z(\mathfrak{n})$ is 1-dimensional. Since the commutator and the center of \mathfrak{n} are characteristic ideals in \mathfrak{n} , and \mathfrak{n} is an ideal in $\mathfrak{n} \rtimes \mathfrak{h}$, both $[\mathfrak{n}, \mathfrak{n}]$ and $Z(\mathfrak{n})$ are ideals in $\mathfrak{n} \rtimes \mathfrak{h}$. We claim that $[Z(\mathfrak{n}), \mathfrak{h}] = 0$ for the Lie bracket in $\mathfrak{n} \rtimes \mathfrak{h}$. Since $Z(\mathfrak{n})$ is an ideal in $\mathfrak{n} \rtimes \mathfrak{h}$, we have $[\mathfrak{h}, Z(\mathfrak{n})] \subseteq Z(\mathfrak{n})$, so that $Z(\mathfrak{n})$ is a 1-dimensional \mathfrak{h} -module. However, for a perfect Lie algebra, every 1-dimensional module is trivial. The proof is the same as for a semisimple Lie algebra. Hence we obtain $[Z(\mathfrak{n}), \mathfrak{h}] = 0$. It follows that

$$[\mathfrak{n}, \mathfrak{h}] = [\mathfrak{s} + Z(\mathfrak{n}), \mathfrak{h}] = [\mathfrak{s}, \mathfrak{h}] \subseteq \mathfrak{s},$$

since \mathfrak{s} is an ideal in $\mathfrak{n} \rtimes \mathfrak{h}$. Because $\mathfrak{n} \rtimes \mathfrak{h}$ is perfect, we have

$$\begin{aligned} \mathfrak{n} + \mathfrak{h} &= [\mathfrak{n} + \mathfrak{h}, \mathfrak{n} + \mathfrak{h}] \\ &= [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{n}, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \\ &= \mathfrak{s} + \mathfrak{h}. \end{aligned}$$

However, because of $\mathfrak{n} = \mathfrak{s} \oplus Z(\mathfrak{n})$ we have $\dim(\mathfrak{n}) = \dim(\mathfrak{s}) + 1$. Since $\mathfrak{n} \cap \mathfrak{h} = \mathfrak{s} \cap \mathfrak{h} = 0$, this implies $\dim(\mathfrak{n} + \mathfrak{h}) = \dim(\mathfrak{s} + \mathfrak{h}) + 1$. This is a contradiction to $\mathfrak{n} + \mathfrak{h} = \mathfrak{s} + \mathfrak{h}$. \square

For case (g) we have the following result.

Proposition 3.15. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is perfect and \mathfrak{n} is complete non-perfect. Then there is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Assume that there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Since \mathfrak{n} is complete, this PA-structure is given by $x \cdot y = \{R(x), y\}$ for a Rota-Baxter operator R of weight 1. Because \mathfrak{g} is perfect, it follows by Corollary 2.20 in [9] that \mathfrak{n} is also perfect. This is a contradiction. \square

4. PA-STRUCTURES WITH \mathfrak{n} PERFECT

In this section we study the existence question of PA-structures on pairs of complex Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} is perfect non-semisimple. We consider 7 different cases for \mathfrak{g} , namely (a) \mathfrak{g} is abelian, (b) \mathfrak{g} is nilpotent non-abelian, (c) \mathfrak{g} is solvable non-nilpotent, (d) \mathfrak{g} is simple, (e) \mathfrak{g} is semisimple non-simple, (f) \mathfrak{g} is reductive non-semisimple, and (g) \mathfrak{g} is complete non-perfect.

For case (a) we have the following result.

Proposition 4.1. *There is no PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is abelian and \mathfrak{n} is perfect.*

Proof. Any PA-structure on $(\mathfrak{g}, \mathfrak{n})$ with \mathfrak{g} abelian corresponds to an LR-structure on \mathfrak{n} . However, every Lie algebra admitting an LR-structure is 2-step solvable by Proposition 2.1 in [4]. Hence there exists no PA-structure on $(\mathfrak{g}, \mathfrak{n})$. \square

For case (b) we have the following result.

Proposition 4.2. *There is no PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is nilpotent non-abelian and \mathfrak{n} is perfect.*

Proof. Assume that there is a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Since \mathfrak{g} is nilpotent, \mathfrak{n} must be solvable by Proposition 4.3 in [5]. This is a contradiction. \square

For case (c), consider the perfect non-semisimple Lie algebra $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ with the Lie brackets given before Example 3.13. We have a decomposition $\mathfrak{n} = \mathfrak{n}_1 \dot{+} \mathfrak{n}_2$ into subalgebras $\mathfrak{n}_1 = \text{span}\{e_1, e_4, e_5\}$ and $\mathfrak{n}_2 = \text{span}\{e_2, e_3\}$. Then by Propositions 2.7 and 2.13 in [9], the Rota-Baxter operator R given by $R(x + y) = -y$ for all $x \in \mathfrak{n}_1, y \in \mathfrak{n}_2$ defines a PA-structure on the pair $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g} \cong \mathfrak{n}_1 \oplus \mathfrak{n}_2 \cong \mathfrak{n}_3(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ is solvable non-nilpotent. The matrix of R is given by

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the Lie brackets of \mathfrak{g} are given by $[e_1, e_5] = e_4$ and $[e_2, e_3] = -2e_2$.

Example 4.3. *The pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g} \cong \mathfrak{n}_3(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ and $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ admits a PA-structure given by*

$$\begin{aligned} e_2 \cdot e_1 &= e_3, & e_3 \cdot e_1 &= -2e_1, & e_3 \cdot e_4 &= -e_4, \\ e_2 \cdot e_3 &= -2e_2, & e_3 \cdot e_2 &= 2e_2, & e_3 \cdot e_5 &= e_5, \\ e_2 \cdot e_4 &= -e_5. \end{aligned}$$

For the cases (d) and (e) we have the following result.

Proposition 4.4. *There is no PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is semisimple and \mathfrak{n} is perfect non-semisimple.*

Proof. Assume that there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is semisimple. Then by Theorem 3.3 in [12], \mathfrak{g} is isomorphic to \mathfrak{n} . This is a contradiction. \square

For case (f) we have the following example.

Example 4.5. *The pair of Lie algebras $(\mathfrak{g}, \mathfrak{n}) = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2, \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$ admits a PA-structure given by*

$$\begin{aligned} e_4 \cdot e_2 &= e_5, & e_5 \cdot e_1 &= e_4, \\ e_4 \cdot e_3 &= e_4, & e_5 \cdot e_3 &= -e_5. \end{aligned}$$

Here we use the Lie brackets for $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ as in Example 3.13, and the standard Lie brackets of $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e_1, e_2, e_3\}$ for \mathfrak{g} . This PA-structure can also be realized by the Rota-Baxter operator

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

for the decomposition $\mathfrak{n} = \mathfrak{n}_1 \dot{+} \mathfrak{n}_2$, where $\mathfrak{n}_1 = \text{span}\{e_1, e_2, e_3\}$, $\mathfrak{n}_2 = \text{span}\{e_4, e_5\}$, $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ and $x \cdot y = \{R(x), y\}$.

Finally, for case (g) we have the following example.

Example 4.6. *The pair of Lie algebras $(\mathfrak{g}, \mathfrak{n}) = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$ admits a PA-structure given by*

$$\begin{aligned} e_4 \cdot e_2 &= e_5, & e_5 \cdot e_1 &= e_4, & e_5 \cdot e_4 &= -e_4, \\ e_4 \cdot e_3 &= e_4, & e_5 \cdot e_3 &= -e_5, & e_5 \cdot e_5 &= -e_5. \end{aligned}$$

Here \mathfrak{g} is complete non-perfect, and the Lie brackets of \mathfrak{g} are given by the standard brackets for $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e_1, e_2, e_3\}$, and by $[e_4, e_5] = e_4$ for $\mathfrak{r}_2(\mathbb{C}) = \text{span}\{e_4, e_5\}$.

5. THE EXISTENCE QUESTION

We summarize the existence results for post-Lie algebra structures from the previous sections and from the papers [5, 6, 7, 8, 9, 10, 12] as follows.

Theorem 5.1. *The existence table for post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$ is given as follows:*

$(\mathfrak{g}, \mathfrak{n})$	\mathfrak{n} <i>abe</i>	\mathfrak{n} <i>nil</i>	\mathfrak{n} <i>sol</i>	\mathfrak{n} <i>sim</i>	\mathfrak{n} <i>sem</i>	\mathfrak{n} <i>red</i>	\mathfrak{n} <i>com</i>	\mathfrak{n} <i>per</i>
\mathfrak{g} <i>abelian</i>	✓	✓	✓	—	—	—	✓	—
\mathfrak{g} <i>nilpotent</i>	✓	✓	✓	—	—	—	✓	—
\mathfrak{g} <i>solvable</i>	✓	✓	✓	✓	✓	✓	✓	✓
\mathfrak{g} <i>simple</i>	—	—	—	✓	—	—	—	—
\mathfrak{g} <i>semisimple</i>	—	—	—	—	✓	—	—	—
\mathfrak{g} <i>reductive</i>	✓	✓	✓	—	?	✓	✓	✓
\mathfrak{g} <i>complete</i>	✓	✓	✓	✓	✓	✓	✓	✓
\mathfrak{g} <i>perfect</i>	—	?	—	?	?	✓	—	✓

A checkmark only means that there is *some* non-trivial pair $(\mathfrak{g}, \mathfrak{n})$ of Lie algebras with the given algebraic properties admitting a PA-structure. A dash means that there does not exist any PA-structure on such a pair. Recall that the classes are (to avoid unnecessary overlap) abelian, nilpotent non-abelian, solvable non-nilpotent, simple, semisimple non-simple, reductive non-semisimple and complete non-perfect.

ACKNOWLEDGMENTS

Dietrich Burde and Mina Monadjem are supported by the Austrian Science Foundation FWF, grant P 33811. Karel Dekimpe is supported by Methusalem grant Meth/21/03 - long term structural funding of the Flemish Government.

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