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COUNTEREXAMPLES TO THE ZASSENHAUS CONJECTURE ON SIMPLE MODULAR LIE ALGEBRAS

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Abstract. We provide an infinite family of counterexamples to the conjecture of Zassenhaus on the solvability of the outer derivation algebra of a simple modular Lie algebra. In fact, we show that the simple modular Lie algebras $H(2;(1,n))^{(2)}$ of dimension $3^{n+1}-2$ in characteristic $p=3$ do not have a solvable outer derivation algebra for all $n\ge 1$. For $n=1$ this recovers the known counterexample of $\mathfrak{psl}_3(\mathbb{F})$. We show that the outer derivation algebra of $H(2;(1,n))^{(2)}$ is isomorphic to $(\mathfrak{sl}_2(\mathbb{F})\ltimes V(2))\oplus\mathbb{F}^{n-1}$, where $V(2)$ is the natural representation of $\mathfrak{sl}_2(\mathbb{F})$. We also study other known simple Lie algebras in characteristic three, but they do not yield a new counterexample.

1. Introduction

We study a conjecture by Hans Zassenhaus, which says that the outer derivation algebra $\mathrm{Out}(\mathfrak{g})$ is *solvable* for all simple modular Lie algebras $\mathfrak{g}$, over a field $\mathbb{F}$ of characteristic $p>0$. Zassenhaus posed this conjecture in 1939 in his work [22]. We have collected several results on this conjecture from the literature, and proved some results in [8]. For simple modular Lie algebras over an algebraically closed field of characteristic $p>3$ the Zassenhaus conjecture is true. The outer derivation algebra $\mathrm{Out}(\mathfrak{g})$ is solvable of derived length at most three. In characteristic $p=2$ and $p=3$, however, there is a counterexample known in each case. For $p=3$ this is a simple constituent of the classical Lie algebra $\mathfrak{g}_2$, namely $\mathfrak{psl}_3(\mathbb{F})$. For $p=2$ it is a simple constituent of dimension 26 of the classical Lie algebra $\mathfrak{f}_4$.

One motivation for us to study the Zassenhaus conjecture comes from commutative post-Lie algebra structures, or *CPA-structures*, on finite-dimensional Lie algebras over a field $\mathbb{F}$, see [8]. Indeed, every perfect modular Lie algebra in characteristic $p>2$ having a solvable outer derivation algebra admits only the trivial CPA-structure. Here CPA-structures are a special case of post-Lie algebra structures on Lie algebras, which have been studied in the context of geometric structures on Lie groups, étale representations of algebraic groups, deformation theory, homology of partition posets, Kozul operads, Yang-Baxter equations, and many other topics. For references see [3, 4, 5, 6, 7, 21].

In this article we provide an infinite family of new counterexamples to the Zassenhaus conjecture in characteristic 3. We show that the Hamiltonian Lie algebras $H(2;(1,n))^{(2)}$, which are central simple modular Lie algebras in characteristic 3 of dimension $3^{n+1}-2$, are counterexamples for all $n\ge 1$. For $n=1$ we have the isomorphism $H(2;(1,1))^{(2)}\cong\mathfrak{psl}_3(\mathbb{F})$, which recovers the known counterexample in characteristic 3. We show that there are no other counterexamples among the graded Hamiltonian Lie algebras $H(2r;\underline{n})^{(2)}$ in characteristic $p\ge 3$. We also determine

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the structure of the outer derivation algebra of $H(2r;\underline{n})^{(2)}$ in characteristic $p = 3$. Finally, we study the Zassenhaus conjecture for known simple Lie algebras of *non-standard type* over an algebraically closed field of characteristic three, such as Brown's algebras Br_8 and Br_{29}, Kostrikin's series $L(\varepsilon, \delta, \rho)$ of dimension 10, the Ermolaev algebras $R(\underline{n})$, the Brown-Kuznetsov algebras $T(n)$ and several series of new simple Lie algebras of Skryabin. We do not find new counterexamples there.

2. Preliminaries

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field $\mathbb{F}$. Denote by $\operatorname{Der}(\mathfrak{g})$ the derivation algebra of $\mathfrak{g}$ and by $\operatorname{ad}(\mathfrak{g})$ the ideal of inner derivations of the Lie algebra $\operatorname{Der}(\mathfrak{g})$. The quotient algebra $\operatorname{Out}(\mathfrak{g}) = \operatorname{Der}(\mathfrak{g})/\operatorname{ad}(\mathfrak{g})$ is called the *algebra of outer derivations* of $\mathfrak{g}$. Hans Zassenhaus posed in 1939 in his work [22] on page 80, between "Satz 7" and "Satz 8", the following conjecture.

Conjecture 2.1 (Zassenhaus)**.** *The outer derivation algebra* $\operatorname{Out}(\mathfrak{g})$ *of a simple Lie algebra* $\mathfrak{g}$ *in prime characteristic is solvable.*

For the conjecture we sometimes assume that $\mathfrak{g}$ is defined over an algebraically closed field of characteristic $p > 0$, because then we can apply the classification results. For characteristic zero, the corresponding conjecture is true, because then $\operatorname{Out}(\mathfrak{g}) \cong H^1(\mathfrak{g}, \mathfrak{g}) = 0$ for a simple Lie algebra $\mathfrak{g}$ by the first Whitehead Lemma. Clearly, this need not be true in prime characteristic, and indeed the outer derivation algebra of a simple modular Lie algebra need not be trivial in general.

Remark 2.2. The Zassenhaus conjecture for Lie algebras can be seen as an analogue of the *Schreier conjecture* for finite groups. The Schreier conjecture asserts that the outer automorphism group of every finite simple non-abelian group is solvable. It was proposed by Otto Schreier in 1926 and is known to be true as a result of the classification of finite simple groups. Up to now no simpler proof is known for it.

What is known about the Zassenhaus conjecture? There are many different results in the literature, in particular, in the context of the classification of simple modular Lie algebras over an algebraically closed field of characteristic $p > 3$. Let us summarize the main results, which we have collected in [8]. A simple modular Lie algebra in the classification is either of classical type, Cartan type, or of Melikian type in characteristic $p = 5$. The results are as follows.

Proposition 2.3. *Let* $\mathfrak{g}$ *be a classical simple Lie algebra over a field* $\mathbb{F}$ *of characteristic* $p > 3$. *Then* $\operatorname{Out}(\mathfrak{g}) = 0$ *unless* $\mathfrak{g} = \mathfrak{psl}_{n+1}(\mathbb{F})$ *with* $p \mid n+1$ *in which case* $\operatorname{Der}(\mathfrak{g}) \cong \mathfrak{pgl}_{n+1}(\mathbb{F})$ *and* $\operatorname{Out}(\mathfrak{g}) \cong \mathbb{F}$.

Proposition 2.4. *Let* $\mathfrak{g}$ *be a simple Lie algebra of Cartan type over a field of characteristic* $p > 3$. *Then* $\operatorname{Out}(\mathfrak{g})$ *is solvable. More precisely,* $\operatorname{Out}(\mathfrak{g})$ *is solvable of derived length* $d \leq 1$ *for type* W *and type* K, *of derived length* $d \leq 2$ *for type* S *and of derived length* $d \leq 3$ *for type* H.

Proposition 2.5. *Let* $\mathcal{M} = \mathcal{M}(n_1, n_2)$ *be a Melikian algebra of dimension* $5^{n_1+n_2+1}$ *over a field of characteristic* 5. *Then* $\operatorname{Out}(\mathcal{M})$ *is abelian.*

So the Zassenhaus conjecture has a positive answer for algebraically closed fields of characteristic $p > 3$:

Theorem 2.6. *Let* $\mathfrak{g}$ *be a simple modular Lie algebra over an algebraically closed field of characteristic* $p > 3$. *Then* $\mathrm{Out}(\mathfrak{g})$ *is solvable of derived length at most three.*

Recall that a Lie algebra $\mathfrak{g}$ over K is called *central simple* if its centroid coincides with K. Here the centroid is the space of all K-linear maps $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ commuting with all inner derivations. If $\mathfrak{g}$ is a central simple Lie algebra over an arbitrary field $\mathbb{F}$ of characteristic $p > 3$, then $\mathfrak{g} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is simple over $\overline{\mathbb{F}}$. Hence the Zassenhaus conjecture also holds for central simple Lie algebras over an arbitrary field of characteristic $p > 3$.

However, in characteristic $p = 3$ there is one known counterexample to the Zassenhaus conjecture. The same is true for $p = 2$. We will show in the next section that there exists a whole family of counterexamples for $p = 3$ of dimension $3^{n+1} - 2$ for all $n \geq 1$.

3. Simple modular Lie algebras in characteristic three

We want to study the Zassenhaus conjecture for simple modular Lie algebras of characteristic $p = 3$. For the theory of modular Lie algebras, see for example [18]. First we recall that there is a counterexample, see [8], Proposition 3.5.

Proposition 3.1. *Let* $\mathbb{F}$ *be a field of characteristic* $p = 3$. *Then the derivation algebra of* $\mathfrak{g} = \mathfrak{psl}_3(\mathbb{F})$ *is isomorphic to the exceptional Lie algebra* $\mathfrak{g}_2$, *and the quotient by* $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$ *is given by* $\mathrm{Out}(\mathfrak{g}) \cong \mathfrak{g}$. *In particular the outer derivation algebra of* $\mathfrak{g}$ *is simple and non-solvable.*

The next question then is, whether or not there are more counterexamples in characteristic $p = 3$. Here we distinguish *Lie algebras of standard type* (i.e., Lie algebras of classical or Cartan type) and *Lie algebras of non-standard type.*

3.1. **Classical type.** For $p > 3$ the list of classical simple modular Lie algebras is given by

$$\mathfrak{sl}_n(\mathbb{F}), p \nmid n,\ \mathfrak{psl}_n(\mathbb{F}), p \mid n,\ \mathfrak{so}_n(\mathbb{F}), \mathfrak{sp}_{2n}(\mathbb{F}), \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8.$$

For $p = 3$ these Lie algebras are still simple, except for $\mathfrak{g}_2$ and $\mathfrak{e}_6$. In fact, $\mathfrak{g}_2$ has a simple ideal $I \cong \mathfrak{psl}_3(\mathbb{F})$, generated by the short roots, with $\mathfrak{g}_2/I \cong I$. This leads to the counterexample mentioned above. The algebra $\mathfrak{e}_6$ has a 1-dimensional center so that $\mathfrak{e}_6/\mathfrak{z}$ is a simple modular Lie algebra of dimension 77 in characteristic 3. Its derivation algebra is abelian, so that we do not obtain another counterexample. It turns out that for classical simple Lie algebras in characteristic 3 there are no counterexamples, except for $\mathfrak{g}_2/I \cong \mathfrak{psl}_3(\mathbb{F})$ discussed above. Indeed, we have the following results, see [8]:

Proposition 3.2. *Let* $\mathbb{F}$ *be a field of characteristic* 3 *and* $\mathfrak{g}$ *be a simple Lie algebra of classical type different from* $\mathfrak{psl}_{3m}(\mathbb{F})$ *and* $\mathfrak{e}_6/\mathfrak{z}$. *Then* $\mathrm{Out}(\mathfrak{g}) = 0$.

Proposition 3.3. *Let* $\mathbb{F}$ *be a field of characteristic* 3. *Then we have* $\mathrm{Der}(\mathfrak{psl}_{3m}(\mathbb{F})) \cong \mathfrak{pgl}_{3m}(\mathbb{F})$ *for all* $m \geq 2$. *Hence* $\mathrm{Out}(\mathfrak{psl}_{3m}(\mathbb{F})) \cong \mathbb{F}$ *is abelian for all* $m \geq 2$. *Also* $\mathrm{Out}(\mathfrak{e}_6/\mathfrak{z})$ *is abelian.*

3.2. **Cartan type.** The list of simple modular Lie algebras of *Cartan type* for $p > 3$ is given by the *graded* simple Lie algebras of Cartan type

$$W(m;\underline{n}),\ S(m;\underline{n})^{(1)},\ H(2r;\underline{n})^{(2)},\ K(2r+1;\underline{n})^{(1)},$$

and their filtered deformations. Here $m \in \mathbb{N}$, $\underline{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$ and $|\underline{n}| := n_1 + \cdots + n_m$.

These algebras are called *Witt algebras, special algebras, Hamiltonian algebras and contact algebras.* They are the finite-dimensional versions defined over a field $\mathbb{F}$ of characteristic $p > 0$

of the infinite-dimensional Lie algebras of characteristic zero occurring in E. Cartan's work of 1909 on pseudogroups in differential geometry. For the precise definition of these algebras see H. Strade's book [20]. All these algebras are still simple for characteristic $p = 3$, where we need $m \geq 3$ for the special algebras. The dimensions of these algebras are given by

$$\begin{aligned} \dim W(m;\underline{n}) &= m \cdot p^{|\underline{n}|}, \\ \dim S(m;\underline{n})^{(1)} &= (m-1)(p^{|\underline{n}|}-1) \\ \dim H(2r;\underline{n})^{(2)} &= p^{|\underline{n}|}-2, \\ \dim K(2r+1;\underline{n})^{(1)} &= \begin{cases} p^{|\underline{n}|}, & \text{if } 2r+1 \not\equiv -3 \bmod p, \\ p^{|\underline{n}|}-1, & \text{if } 2r+1 \equiv -3 \bmod p. \end{cases} \end{aligned}$$

The derivation algebras have been computed for a field $\mathbb{F}$ of characteristic $p \geq 3$, see Theorem 7.1.2 in [20]. In particular, the result for $p > 3$ still holds for $p = 3$, except for the Hamiltonian algebras. So it follows from the work of Celousov [9], that the Zassenhaus conjecture is true for Witt algebras, special algebras and contact algebras for an algebraically closed field of characteristic $p \geq 3$. However, there are new counterexamples in the Hamiltonian case for $p = 3$. The following table gives a survey.

$\mathfrak{g}$	conditions	$\dim \operatorname{Der}(\mathfrak{g})$	$\dim \operatorname{Out}(\mathfrak{g})$	conjecture
$W(m;\underline{n})$	$p \geq 2$	$m(p^{\|\underline{n}\|}-1)+\|\underline{n}\|$	$\|\underline{n}\|-m$	✓
$S(m;\underline{n})^{(1)}$	$p > 0, m \geq 3$	$(m-1)(p^{\|\underline{n}\|}-1)+\|\underline{n}\|+1$	$\|\underline{n}\|+1$	✓
$H(2;(1,1))^{(2)}$	$p = 3$	14	7	–
$H(2;(1,n_2))^{(2)}$	$p = 3, n_2 > 1$	$3^{n_2+1}+n_2+2$	n_2+4	–
$H(2r;\underline{n})^{(2)}$	$p > 3$, or $p = 3, r > 1$, or $p = 3, r = 1, 1 < n_1 \leq n_2$	$p^{\|\underline{n}\|}+\|\underline{n}\|$	$\|\underline{n}\|+2$	✓
$K(2r+1;\underline{n})^{(1)}$	$p > 2, p \nmid 2r+4$	$p^{\|\underline{n}\|}+\|\underline{n}\|-2r-1$	$\|\underline{n}\|-(2r+1)$	✓
$K(2r+1;\underline{n})^{(1)}$	$p > 2, p \mid 2r+4$	$p^{\|\underline{n}\|}+\|\underline{n}\|-2r-1$	$\|\underline{n}\|-2r$	✓

Note that we also have

$$H(2;(1,n_2))^{(2)} \cong H(2;(n_1,1))^{(2)}$$

for $p \geq 3$, see [20], (3) on page 199.

We have first guessed these results for $p = 3$ in low dimensions by doing a computation with GAP. In fact, we computed the dimensions of the derived series of the outer derivation algebras for the Hamiltonian algebras $H(2r;\underline{n})^{(2)}$ in a few cases. The following table shows the results. The last computation was only possible on the CoCalc server of Anton Mellit, with 192 GB RAM.

$\mathfrak{g}$	$\dim(\mathfrak{g})$	$\dim \mathrm{Der}(\mathfrak{g})$	$\dim \mathrm{Out}(\mathfrak{g})^{(i)}$	$\mathrm{Out}(\mathfrak{g})$
$H(2;(1,1))^{(2)}$	7	14	$(7,7,\ldots)$	simple
$H(2;(1,2))^{(2)}$	25	31	$(6,5,5,\ldots)$	non-solvable
$H(2;(1,3))^{(2)}$	79	86	$(7,5,5,\ldots)$	non-solvable
$H(2;(2,2))^{(2)}$	79	85	$(6,3,1,0)$	solvable
$H(4;(1,1,1,1))^{(2)}$	79	85	$(6,4,0)$	solvable
$H(2;(2,3))^{(2)}$	241	248	$(7,3,1,0)$	solvable

In order to prove our results, let us introduce further notations. Let $\mathbb{F}$ be a field of characteristic $p > 2$. Denote by $\mathcal{O}(m)$ the associative and commutative algebra with unit element over $\mathbb{F}$ defined by generators $x_i^{(r)}$ for $r \geq 0$ and $1 \leq i \leq m$, and relations

$$x_i^{(0)} = 1, \quad x_i^{(r)} x_i^{(s)} = \binom{r+s}{r} x_i^{(r+s)}$$

for $r, s \geq 0$. Put $x_i := x_i^{(1)}$ and $x^{(a)} := x_1^{(a_1)} \cdots x_m^{(a_m)}$ for a tuple $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$. Then the *divided power algebra* of dimension $p^{|\underline{n}|}$ is defined by

$$\mathcal{O}(m;\underline{n}) := \mathrm{span}\{x^{(a)} \mid 0 \leq a_i < p^{n_i}\}.$$

The product is given by

$$x^{(a)} x^{(b)} := \binom{a+b}{b} x^{(a+b)},$$

where $\binom{a}{b} = \prod_{i=1}^m \binom{a_i}{b_i}$ and $x^{(c)} = 0$ if $c_i \geq p^{n_i}$ for some c_i. For each i denote by ∂_i the derivation of the algebra $\mathcal{O}(m)$ given by

$$\partial_i(x_j^{(r)}) = \delta_{i,j} x_j^{(r-1)}.$$

The *generalized Jacobson-Witt algebra* is defined by

$$W(m,\underline{n}) := \sum_{i=1}^m \mathcal{O}(m;\underline{n}) \partial_i,$$

together with the Lie bracket

$$[x^{(a)}\partial_i, x^{(b)}\partial_j] = \binom{a+b-\varepsilon_i}{a} x^{(a+b-\varepsilon_i)} \partial_j - \binom{a+b-\varepsilon_j}{b} x^{(a+b-\varepsilon_j)} \partial_i$$

where $\varepsilon_i = (\delta_{i,1}, \ldots, \delta_{i,m}) \in \mathbb{N}^m$.

Consider the linear operator $D_H \colon \mathcal{O}(2r;\underline{n}) \to W(2r;\underline{n})$ defined by

$$D_H(x^{(a)}) = \sum_{i=1}^{2r} \sigma(i) \partial_i(x^{(a)}) \partial_{i'},$$

where

$$\sigma(i) := \begin{cases} 1, & \text{if } 1 \leq i \leq r, \\ -1, & \text{if } r+1 \leq i \leq 2r, \end{cases}$$

and

$$i' := \begin{cases} i+r, & \text{if } 1 \leq i \leq r, \\ i-r, & \text{if } r+1 \leq i \leq 2r. \end{cases}$$

The *Hamiltonian algebra* is defined by

$$H(2r;\underline{n})^{(2)} = \operatorname{span}\{D_H(x^{(a)}) \mid 0 < a < \tau(\underline{n})\},$$

where $\tau(\underline{n}) = (p^{n_1}-1,\dots,p^{n_m}-1) \in \mathbb{N}^m$. The Lie bracket is given by

$$[D_H(x^{(a)}), D_H(x^{(b)})] = D_H(D_H(x^{(a)})(x^{(b)})).$$

The main result of this paper is that we obtain an infinite family of counterexamples to the Zassenhaus conjecture, which contains the known counterexample $\mathfrak{psl}_3(\mathbb{F})$ as the smallest case $n = 1$:

Theorem 3.4. *For all $n \geq 1$ the simple modular Lie algebra $H(2;(1,n))^{(2)}$ of dimension $3^{n+1}-2$ in characteristic 3 does not have a solvable outer derivation algebra.*

Proof. We will use the basis of $\mathfrak{g} = H(2;(1,n))^{(2)}$ given above, for the special case of $p = 3$, $m = 2$, and $\underline{n} = (n_1,n_2) = (1,n)$. For $x^{(\alpha)}$ we will write $x_1^a x_2^b$. Then the explicit Lie brackets are given by

$$[D_H(x_1^a x_2^b), D_H(x_1^c x_2^d)] = f_{a,b,c,d} \cdot D_H(x_1^{a+c-1} x_2^{b+d-1}),$$

where

$$f_{a,b,c,d} := e_a e_d \cdot \binom{a+c-1}{a-1}\binom{b+d-1}{d-1} - e_b e_c \cdot \binom{a+c-1}{c-1}\binom{b+d-1}{b-1},$$

with $e_k := 1 - \delta_{k,0}$.

Let us order the basis elements $D_H(x_1^a x_2^b)$ of $\mathfrak{g}$ with respect to the formal exponents as follows:

$$D_H(x_1) \prec D_H(x_1^2) \prec D_H(x_2) \prec D_H(x_1x_2) \prec D_H(x_1^2x_2) \prec \cdots,$$

so that we can write a general inner derivation $D \in \operatorname{Der}(\mathfrak{g})$ as

$$\alpha \cdot \operatorname{ad}(D_H(x_1)) + \beta \cdot \operatorname{ad}(D_H(x_1^2)) + \gamma \cdot \operatorname{ad}(D_H(x_2)) + \delta \cdot \operatorname{ad}(D_H(x_1x_2)) + \varepsilon \cdot \operatorname{ad}(D_H(x_1^2x_2)) + \cdots$$

Using the Lie brackets, the matrix of D with respect to this ordered basis is of the form

$$D = \left(\begin{array}{ccc|ccc|ccc}
-\delta & -\gamma & & & & & & & \\
-\varepsilon & \delta & 0 & & & & & & \\
 & 0 & \delta & -\gamma & & & & & \\
\hline
 & & \varepsilon & 0 & -\gamma & & & & \\
 & & & \varepsilon & -\delta & 0 & & & \\
 & & & & 0 & -\delta & & & \\
\hline
\vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\
\hline
0 & & & & & & & & \\
0 & & & & & & & & \\
0 & 0 & 0 & & & & & &
\end{array}\right).$$

For $n = 1$ we have $\mathfrak{g} = H^2(2;(1,1))^{(2)} \cong \mathfrak{psl}_3(\mathbb{F})$, where we already know that $\operatorname{Out}(\mathfrak{g}) \cong \mathfrak{g}$ is not solvable, see Proposition 3.1. So we may assume that $n > 1$. Consider the linear maps

$E, F, H \in \mathrm{End}(\mathfrak{g})$ defined by

$$\begin{aligned} E\colon \mathfrak{g} \to \mathfrak{g}, &\quad D_H(x_1^a x_2^b) \mapsto \delta_{a,2} \cdot D_H(x_2^{b+1}), \\ F\colon \mathfrak{g} \to \mathfrak{g}, &\quad D_H(x_1^a x_2^b) \mapsto \delta_{a,0} \cdot D_H(x_1^2 x_2^{b-1}), \\ H\colon \mathfrak{g} \to \mathfrak{g}, &\quad D_H(x_1^a x_2^b) \mapsto (1-a) \cdot D_H(x_1^a x_2^b). \end{aligned}$$

We claim that $E, F, H \in \mathrm{Der}(\mathfrak{g})$ are derivations of $\mathfrak{g}$. This follows easily from a direct computation. Indeed, we have

$$\begin{aligned} E([D_H(x_1^a x_2^b), D_H(x_1^c x_2^d)]) &= f_{a,b,c,d} \cdot E(D_H(x_1^{a+c-1} x_2^{b+d-1})) \\ &= f_{a,b,c,d} \cdot \delta_{a+c-1,2} \cdot D_H(x_2^{b+d}) \\ &= \binom{b+d}{b} \cdot \delta_{(a,c),(1,2)} \cdot D_H(x_2^{b+d}) - \binom{b+d}{b} \cdot \delta_{(a,c),(2,1)} \cdot D_H(x_2^{b+d}) \\ &= -\delta_{a,2} e_c \cdot \binom{b+d}{b} D_H(x_1^{c-1} x_2^{b+d}) + \delta_{c,2} e_a \cdot \binom{b+d}{b} D_H(x_1^{a-1} x_2^{b+d}) \\ &= \delta_{a,2} \cdot f_{0,b+1,c,d} \cdot D_H(x_1^{c-1} x_2^{b+d}) + \delta_{c,2} \cdot f_{a,b,0,d+1} \cdot D_H(x_1^{a-1} x_2^{b+d}) \\ &= [\delta_{a,2} \cdot D_H(x_2^{b+1}), D_H(x_1^c x_2^d)] + [D_H(x_1^a x_2^b), \delta_{c,2} \cdot D_H(x_2^{d+1})] \\ &= [E(D_H(x_1^a x_2^b)), D_H(x_1^c x_2^d)] + [D_H(x_1^a x_2^b), E(D_H(x_1^c x_2^d))]. \end{aligned}$$

Here we have used that $2 = -1$ in $\mathbb{F}$ and Pascal's identity

$$\binom{b+d-1}{d-1} + \binom{b+d-1}{d} = \binom{b+d}{d}.$$

A similar computation shows that also F and H are derivations. On the other hand, this follows anyway, because F coincides with the restriction of the inner derivation $\mathrm{ad}(D_H(x_1^3))$ of the larger Lie algebra $H(2;(1,n))$, and H coincides with the commutator $[E, F]$, and hence is a derivation. It is easy to see that we have

$$[E, H] = E = -2E, \ [F, H] = -F = 2F, [E, F] = H.$$

Thus (E, F, H) forms an $\mathfrak{sl}_2(\mathbb{F})$-triple in $\mathrm{Der}(\mathfrak{g})$, i.e., the subalgebra $\mathfrak{s}$ of $\mathrm{Der}(\mathfrak{g})$ generated by E, F, H is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Now the matrix of $\lambda E + \mu F + \nu H$ with respect to the ordered basis of $\mathfrak{g}$ has the form

$$D = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & -\nu & \mu & & & & & & \\ & \lambda & \nu & & & & & & \\ \hline & & & 0 & & & & & \\ & & & & -\nu & \mu & & & \\ & & & & \lambda & \nu & & & \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & \end{array}\right).$$

Comparing this with the form for the general inner derivation D we conclude that the subalgebra $\mathfrak{s}$ satisfies $\mathfrak{s} \cap \mathrm{ad}(\mathfrak{g}) = 0$. Hence $\mathrm{Out}(\mathfrak{g})$ contains the subalgebra

$$(\mathfrak{s} + \mathrm{ad}(\mathfrak{g}))/\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{s}/\mathfrak{s} \cap \mathrm{ad}(\mathfrak{g}) \cong \mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{F}).$$

Thus $\operatorname{Out}(\mathfrak{g})$ is not solvable. $\square$

So we have obtained an infinite family of counterexamples. In addition, we can be more precise about the structure of the outer derivation algebra of $H(2;(1,n))^{(2)}$. Denote by $V(2)$ the natural representation of $\mathfrak{sl}_2(\mathbb{F})$. Then the Lie algebra $\mathfrak{sl}_2(\mathbb{F}) \ltimes V(2)$ in characteristic 3 has a basis $(e_1,\dots,e_5)$ with Lie brackets

$$\begin{aligned}
&[e_1,e_2]=e_3, && [e_2,e_3]=2e_2, && [e_3,e_4]=e_4,\\
&[e_1,e_3]=e_1, && [e_2,e_4]=e_5, && [e_3,e_5]=2e_5.\\
&[e_1,e_5]=e_4,
\end{aligned}$$

Theorem 3.5. *Let $n>1$. Then the outer derivation algebra of $H(2;(1,n))^{(2)}$ in characteristic 3 is isomorphic to $(\mathfrak{sl}_2(\mathbb{F}) \ltimes V(2)) \oplus \mathbb{F}^{n-1}$.*

Proof. Let $\mathfrak{g} = H(2;(1,n))^{(2)}$. According to [20], Theorem 7.1.2, (3) part (b) on page 358 we have

$$\operatorname{Der}(\mathfrak{g}) \cong CH(2;(1,n)) + \sum_{i=1}^{n-1} \mathbb{F}\cdot \partial_2^{3^i} + \mathbb{F}\cdot d,$$

where d is the derivation which we called F in the proof of Theorem 3.4, and

$$CH(2;(1,n)) = H(2;(1,n)) \oplus \mathbb{F}\cdot (x_1\partial_1 + x_2\partial_2).$$

We have $\dim CH(2;(1,n)) = 3^{n+1}+2$, see [14, page 273], so that we obtain $\dim \operatorname{Der}(\mathfrak{g}) = 3^{n+1}+n+2$ and $\dim \operatorname{Out}(\mathfrak{g}) = n+4$. Consider the linear maps given by

$$\begin{aligned}
V &: L\to L, \quad D_H(x_1^a x_2^b) \mapsto \delta_{b,0}\cdot D_H(x_1^{a-1}x_2^{3^n-1}),\\
W &: L\to L, \quad D_H(x_1^a x_2^b) \mapsto \delta_{a+b,1}\cdot (-1)^a \cdot D_H(x_1^{a+1}x_2^{b+3^n-2}).
\end{aligned}$$

They are derivations of $\mathfrak{g}$, because each of them is a restriction of inner derivations of the larger Lie algebra $H(2;(1,n))$ to $\mathfrak{g}$, namely of $\operatorname{ad}(D_H(x_2^{3^n}))$, respectively of $\operatorname{ad}(D_H(x_1^2 x_2^{3^n-1}))$. By a computation we see that

$$[E,W]=V,\ [F,V]=W,\ [H,V]=V,\ [H,W]=2W,$$

where E,F,H are the derivations of $\mathfrak{g}$ given in the proof of Theorem 3.4. Hence the subalgebra $\mathfrak{t}$ of $\operatorname{Der}(\mathfrak{g})$ generated by E,F,H,V,W is isomorphic to $\mathfrak{sl}_2(\mathbb{F}) \ltimes V(2)$.
The matrix of $\lambda E+\mu F+\nu H+\eta V+\xi W$ with respect to the ordered basis of $\mathfrak{g}$ is of the form

$$D = \left(\begin{array}{ccc|ccc|ccc}
0 & & & & & & & & \\
 & -\nu & \mu & & & & & & \\
 & \lambda & \nu & & & & & & \\
\hline
 & & & 0 & & & & & \\
 & & & & -\nu & \mu & & & \\
 & & & & \lambda & \nu & & & \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
-\xi & 0 & 0 & & & & \ddots & & \\
\eta & 0 & 0 & & & & & \ddots & \\
0 & \eta & \xi & & & & & & \ddots
\end{array}\right)$$

Comparing with the matrix D of inner derivations (see the proof of Theorem 3.4) we obtain $\mathfrak{t} \cap \mathrm{ad}(\mathfrak{g}) = 0$, so that $\mathrm{Out}(\mathfrak{g})$ has a subalgebra isomorphic to $\mathfrak{sl}_n(\mathbb{F}) \ltimes V(2)$. We claim that the derivations $\partial_2^{3^i}$ belong to the center of $\mathrm{Out}(\mathfrak{g})$. Indeed, they commute pairwise, and they commute with E, F, H. Furthermore we have, using also [20, Lemma 2.1.2(1), page 61],

$$[\partial_2^{3^i}, V] = \mathrm{ad}(D(x_2^{3^n-3^i})),$$
$$[\partial_2^{3^i}, W] = \mathrm{ad}(D(x_1^2 x_2^{3^n-3^i-1})),$$

for $i = 1, \ldots, n-1$. This implies that $\mathrm{Out}(\mathfrak{g}) \cong \mathfrak{t} \oplus \mathbb{F}^{n-1}$, where $\mathfrak{t} \cong \mathfrak{sl}_2(\mathbb{F}) \ltimes V(2)$. □

We will show now that the remaining cases for the Hamiltonian Lie algebras $H(2r;\underline{n})^{(2)}$ do not provide new counterexamples to the Zassenhaus conjecture for $p = 3$. We have two cases, namely first $r > 1$, and secondly $r = 1$ and $1 < n_1 \leq n_2$, where $\underline{n} = (n_1, n_2) \in \mathbb{N}^2$. Let $\mathfrak{h}_3(\mathbb{F})$ be the Heisenberg Lie algebra over $\mathbb{F}$ with basis $\{e_1, e_2, e_3\}$ and Lie bracket $[e_1, e_2] = e_3$. Recall that a Lie algebra over a field $\mathbb{F}$ is called *almost abelian* if it is nonabelian and has an ideal of codimension 1. Hence every almost abelian Lie algebra can be written as $\mathbb{F}^r \rtimes \mathbb{F}$, and is 2-step solvable.

Theorem 3.6. *Let $\mathfrak{g}$ be the Hamiltonian Lie algebra $H(2r;\underline{n})^{(2)}$ over a field $\mathbb{F}$ of characteristic $p = 3$. Then, for $r > 1$ the outer derivation algebra $\mathrm{Out}(\mathfrak{g})$ is 2-step solvable, and for $r = 1$, $1 < n_1 \leq n_2$, it is 3-step solvable. More precisely, we have*

$$\mathrm{Out}(\mathfrak{g}) \cong \begin{cases} (\mathfrak{h}_3(\mathbb{F}) \rtimes \mathbb{F}) \oplus \mathbb{F}^{|\underline{n}|-2}, & \text{if } r = 1,\ 1 < n_1 \leq n_2, \\ (\mathbb{F}^{2r+1} \rtimes \mathbb{F}) \oplus \mathbb{F}^{|\underline{n}|-2r}, & \text{if } r > 1, r \equiv 0 \bmod 3, \\ (\mathbb{F}^{2r+1} \rtimes \mathbb{F}) \oplus \mathbb{F}^{|\underline{n}|-2r}, & \text{if } r > 1, r \equiv 1 \bmod 3, \\ (\mathbb{F}^{2r} \rtimes \mathbb{F}) \oplus \mathbb{F}^{|\underline{n}|-2r+1}, & \text{if } r > 1, r \equiv 2 \bmod 3. \end{cases}$$

Here in the first case $\mathbb{F}$ acts on $\mathfrak{h}_3(\mathbb{F})$ by the derivation $D = \mathrm{diag}(1,1,-1)$, in the second case $\mathbb{F}$ acts on $\mathbb{F}^{2r+1}$ by the derivation $D = \mathrm{id}$, in the third case $\mathbb{F}$ acts on $\mathbb{F}^{2r+1}$ by the derivation $D = \mathrm{diag}(1,\ldots,1,-1)$, and in the last case $\mathbb{F}$ acts on $\mathbb{F}^{2r}$ by the derivation $D = \mathrm{id}$.

Proof. Let us write x^a for $x^{(a)} = x_1^{a_1} \cdots x_m^{a_m}$ and

$$\tau = (3^{n_1} - 1, \ldots, 3^{n_m} - 1) \in \mathbb{N}^m.$$

By [20, Theorem 7.1.2(3)(b), page 358], the structure of $\mathrm{Der} H(2r;\underline{n})^{(2)}$ is given by

$$\mathrm{Der} H(2r;\underline{n})^{(2)} \cong CH(2r;\underline{n})^{(2)} \oplus \sum_{i=1}^{2r} \sum_{0<j_i<n_i} \mathbb{F} \cdot \partial_i^{j_i},$$

where

$$CH(2r;\underline{n})^{(2)} = H(2r;\underline{n}) \oplus \mathbb{F} \cdot \left(\sum_{i=1}^{2r} x_i \partial_i \right).$$

So we obtain the following dimensions:

$$\begin{aligned} \dim \mathrm{Der} H(2r;\underline{n})^{(2)} &= \dim H(2r;\underline{n}) + 1 + |\underline{n}| - 2r \\ &= (3^{|\underline{n}|} - 2 + 2r + 1) + 1 + |\underline{n}| - 2r \\ &= 3^{|\underline{n}|} + |\underline{n}|, \end{aligned}$$

see also [14, page 273]. So we have

$$\begin{aligned}\dim \mathrm{Out} H(2r;\underline{n})^{(2)} &= \dim \mathrm{Der} H(2r;\underline{n})^{(2)} - \dim H(2r;\underline{n})^{(2)}\\ &= 3^{|\underline{n}|} + |\underline{n}| - (3^{|\underline{n}|} - 2)\\ &= |\underline{n}| + 2.\end{aligned}$$

Consider the restrictions to $H(2r;\underline{n})^{(2)}$ of the derivations $\mathrm{ad}(D_H(x_i^{p^{n_i}}))$ and $\mathrm{ad}(D_H(x^\tau))$ of the larger Lie algebra $H(2r;\underline{n})$. They are given explicitly as the linear maps

$$\begin{aligned} A_i\colon H(2r;\underline{n})^{(2)} \to H(2r;\underline{n})^{(2)}, &\quad D_H(x^a) \mapsto \delta_{a_i,0}\cdot\sigma(i)\cdot D_H(x^{a+(\tau_i-a_i)\varepsilon_i-\varepsilon_{i'}})\\ B\colon H(2r;\underline{n})^{(2)} \to H(2r;\underline{n})^{(2)}, &\quad D_H(x^a) \mapsto \delta_{|a|,1}\cdot\sigma(k)\cdot D_H(x^{\tau-\varepsilon_k})\end{aligned}$$

for $i = 1,\dots,2r$, and where $k \in \{1,\dots,2r\}$ is the only index such that $a_{k'} \neq 0$. Recall the definition of k' before Theorem 3.4. It is clear that $A_i, B \in \mathrm{Der} H(2r;\underline{n})^{(2)}$. Moreover the derivations $C := \sum_{i=1}^{2r} x_i\partial_i$ and $D_{i,j_i} := \partial_i^{j_i}$ for $i = 1,\cdots,2r$ and $0 < j_i < n_i$ for each i are explicitly given by

$$\begin{aligned} C\colon H(2r;\underline{n})^{(2)} \to H(2r;\underline{n})^{(2)}, &\quad D_H(x^a) \mapsto (|a|-2)\cdot D_H(x^a)\\ D_{i,j_i}\colon H(2r;\underline{n})^{(2)} \to H(2r;\underline{n})^{(2)}, &\quad D_H(x^a) \mapsto D_H(x^{a-p^{j_i}\varepsilon_i}).\end{aligned}$$

We claim that

$$\{A_1,\dots,A_{2r},B,C,D_{1,1},\dots D_{1,n_1-1},\dots,D_{2r,1},\dots,D_{2r,n_{2r}-1}\}$$

are representatives of a basis of $\mathrm{Out} H(2r;\underline{n})^{(2)}$. Its cardinality is given by $2r+2+\sum_{i=1}^{2r} n_i - 2r = |\underline{n}|+2$. The arguments are the same as used in the proofs of Theorem 3.4 and Theorem 3.5, i.e., one can easily check that the intersection of the linear span of these derivations and $\mathrm{ad} H(2r;\underline{n})^{(2)}$ is zero. Indeed, this follows just from comparing the images of $D_H(x_i)$ for $i = 1,\dots,2r$, under a general inner derivation and $\sum_{i=1}^{2r}\alpha_iA_i + \beta B + \gamma C + \sum_{i=1}^{2r}\sum_{0<j_i<n_i}\delta_{i,j_i}D_{i,j_i}$. The projections onto $\mathrm{Out} H(2r;\underline{n})^{(2)}$ of A_i, B, C and D_{i,j_i} are then $|n|+2$ linearly independent derivations which therefore constitute a basis of $\mathrm{Out} H(2r;\underline{n})^{(2)}$.

It is straightforward to compute the Lie brackets between the representatives in $\mathrm{Der} H(2r;\underline{n})^{(2)}$ of the basis vectors of $\mathrm{Out} H(2r;\underline{n})^{(2)}$. The nonzero brackets are given as follows, with $1 \le i < i' \le 2r$,

$$\begin{aligned}[A_i, A_{i'}] &= \begin{cases} B & \text{if } r = 1\\ \mathrm{ad} D_H(x_i^{\tau_i} x_{i'}^{\tau_{i'}}) & \text{if } r > 1\end{cases}\\ [A_i, C] &= -A_i,\\ [A_i, D_{i,j_i}] &= -\mathrm{ad} D_H(x_i^{\tau_i - p^{j_i}+1}),\\ [B, C] &= (2r-1)B,\\ [B, D_{i,j_i}] &= -\mathrm{ad} D_H(x^{\tau - p^{j_i}\varepsilon_i}).\end{aligned}$$

Note that $[B, C] = 0$ for the case $r \equiv 2 \bmod 3$. For $r > 1$, the Lie brackets yield a direct sum of an almost abelian Lie algebra $\mathbb{F}^{2r+1} \rtimes \mathbb{F}$ (or $\mathbb{F}^{2r} \rtimes \mathbb{F}$ for $r \equiv 2 \bmod 3$), and an abelian

Lie algebra. Hence $\mathrm{Out}(\mathfrak{g})$ is 2-step solvable in this case. For $r = 1$ we have $[C, B] = -B$, $[C, A_i] = A_i$ for $i = 1, 2$, and $[A_1, A_2] = B$, so that

$$\mathrm{Out}H(2r; \underline{n})^{(2)} \cong \mathrm{span}(A_1, A_2, B, C) \oplus \mathrm{span}(D_{i,j_i}) \cong (\mathfrak{h}_3(\mathbb{F}) \rtimes \mathbb{F}) \oplus \mathbb{F}^{|\underline{n}|-2}.$$

The ideal $\mathfrak{a} = \mathrm{span}(A_1, A_2, B, C)$ satisfies $\mathfrak{a}^{(1)} = \mathrm{span}(A_1, A_2, B)$, $\mathfrak{a}^{(2)} = \mathrm{span}(B)$ and $\mathfrak{a}^{(3)} = 0$. Thus $\mathrm{Out}(\mathfrak{g})$ is 3-step solvable for $r = 1$. □

3.3. **Non-standard type.** There are several simple modular Lie algebras over a field of characteristic 3 that are neither of classical nor Cartan type. For example, the 1-parameter family of 10-dimensional Kostrikin algebras $L(\varepsilon)$, the Ermolaev algebras $R(\underline{n})$, the Brown-Kuznetsov algebras $T(n)$, and the Skryabin algebras $X(\underline{n})$ and $Y(\underline{n})$. Chan Nam Zung studied their properties in [10], published in 1993. He computed the outer derivation algebras of these algebras. It turns out that we do not obtain any new counterexample to the Zassenhaus conjecture. The following table gives a survey.

$\mathfrak{g}$	conditions	$\dim(\mathfrak{g})$	$\dim \mathrm{Out}(\mathfrak{g})$	$\mathrm{Out}(\mathfrak{g})$
$L(\varepsilon)$	$\varepsilon \in \mathbb{F}$	10	0	abelian
$R(\underline{n})$	$\underline{n} = (n_1, n_2) \in \mathbb{N}^2$	$3^{|\underline{n}|+1} - 1$	$|\underline{n}| + 1$	abelian
$T(n)$	$n \in \mathbb{N}$	$2 \cdot 3^{n+1}$	$n - 1$	abelian
$X(\underline{n})$	$\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$	$3^{|\underline{n}|+1} - 4$	$|\underline{n}| + 1$	solvable
$Y(\underline{n})$	$\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$	$2 \cdot 3^{|\underline{n}|+1}$	$|\underline{n}| - 3$	abelian

However, there are three further infinite families of simple Skryabin algebras in characteristic three, denoted by $Z'(\underline{n})$, and $X_i(\underline{n}, \omega)$, for $i = 1, 2$ of type 1 and type 2, see [19]. Zung does not determine the outer derivation algebras of these families in [10]. He mentions that the determination for $Z'(\underline{n})$ is still an open problem. However, this was solved 2001 in [15]. The outer derivation algebra is abelian. Unfortunately we could not find a result for the algebras $X_i(\underline{n}, \omega)$. But we believe that the outer derivation algebra will be solvable, too. Let us explain the result of [15] on the derivation algebra of $Z'(\underline{n})$. In the construction of the Lie algebra $Z'(\underline{n})$, Skryabin introduces a Lie algebra $Z(\underline{n})$ of dimension $3^{|\underline{n}|+2} + 1$ with

$$Z'(\underline{n}) = [Z(\underline{n}), Z(\underline{n})].$$

Using this notation, the result of [15] is as follows, see Corollary 1 on page 3925.

Proposition 3.7. *Let $\mathfrak{g} = Z'(\underline{n})$, with $\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}$. Then we have*

$$\mathrm{Der}(\mathfrak{g}) \cong \overline{\mathfrak{g}_{\overline{0}}} + Z(\underline{n}).$$

Here $\mathfrak{g}_{\overline{0}} \cong W(3, \underline{n})$ and $\overline{\mathfrak{g}_{\overline{0}}}$ denotes the p-closure of $\mathrm{ad}(\mathfrak{g}_{\overline{0}})$ in $\mathrm{Der}(\mathfrak{g})$. This implies that $\mathrm{Out}(\mathfrak{g})$ is abelian, since

$$[\mathrm{Der}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g})] \subseteq [Z(\underline{n}), Z(\underline{n})] = Z'(\underline{n}) \cong \mathrm{ad}(\mathfrak{g}).$$

Furthermore we have the 10-dimensional simple Lie algebras $L(\varepsilon, \delta, \rho)$ in characteristic three of Kostrikin [13], which are deformations of the algebras $L(\varepsilon)$. Here it is known that all derivations are inner. All known simple Lie algebras of dimension 10 for $p = 3$ can be realized within the family $L(\varepsilon, \delta, \rho)$, see [13], but a classification up to isomorphism is still not known.

Finally we have the 8-dimensional and the 29-dimensional simple Lie algebras Br_8 and Br_{29} of

Brown [2, 1]. Both Lie algebras are central simple. A direct computation shows that the outer derivation algebra is abelian in each case. Surprisingly, Br_8 is not mentioned in later works on simple Lie algebras of characteristic three. Thus, for the convenience of the reader, let us give all Lie brackets of Br_8 explicitly, with respect to the basis

$$(x_1, \dots, x_8) = (K_{12}, K_{21}, K_{13}, K_{31}, K_{23}, K_{32}, H, K)$$

introduced in [2] on page 440:

$$\begin{aligned}
&[x_1, x_2] = x_7, && [x_2, x_6] = 2x_4, && [x_4, x_5] = 2x_2,\\
&[x_1, x_4] = 2x_6, && [x_2, x_7] = 2x_2, && [x_4, x_7] = x_4,\\
&[x_1, x_5] = x_3, && [x_2, x_8] = 2x_6, && [x_5, x_6] = x_7,\\
&[x_1, x_7] = x_1, && [x_3, x_4] = 2x_7, && [x_5, x_7] = x_5,\\
&[x_2, x_3] = x_5, && [x_3, x_6] = x_1, && [x_5, x_8] = x_1,\\
&[x_2, x_5] = x_8, && [x_3, x_7] = 2x_3, && [x_6, x_7] = 2x_6.
\end{aligned}$$

This algebra is central simple and non-restricted. Its outer derivation algebra is 2-dimensional and abelian. Note that Br_8 is isomorphic to a deformed Hamiltonian algebra $H(2; (1, 1), \omega)$, where $\omega = (1 + x_1^{(2)} x_2^{(2)})(dx_1 \wedge dx_2)$. For the family of simple deformed Hamiltonian algebras $H(2r; \underline{n}, \omega)$ of dimension $p^{|\underline{n}|} - 1$ see [20], pp. $340 - 341$. The following table gives a survey of the preceding discussion.

$\mathfrak{g}$	conditions	$\dim(\mathfrak{g})$	$\dim \mathrm{Out}(\mathfrak{g})$	$\mathrm{Out}(\mathfrak{g})$
Br_8	$-$	8	2	abelian
$L(\varepsilon, \delta, \rho)$	$\varepsilon, \delta, \rho \in \mathbb{F}$	10	0	abelian
Br_{29}	$-$	29	0	abelian
$Z'(\underline{n})$	$\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$	$3^{\lvert\underline{n}\rvert+2} - 2$	$\lvert\underline{n}\rvert$	abelian
$X_1(\underline{n}, \omega)$	$\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$	$3^{\lvert\underline{n}\rvert+1} - 3$	?	?
$X_2(\underline{n}, \omega)$	$\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$	$3^{\lvert\underline{n}\rvert+1} - 1$	?	?

There are other simple Lie algebras for $p = 3$, which we have not studied here, e.g., deformed Hamiltonian and special Lie algebras of Cartan type for $p = 3$, or other families, where no explicit realization is known.

Remark 3.8. We also studied the Zassenhaus conjecture for simple Lie algebras over a field of characteristic $p = 2$. Here it was already known since 1955 that a simple constituent J of dimension 26 of the Lie algebra $\mathfrak{f}_4$ provides a counterexample, see [17], and [8] for references. Note that J is given as the simple ideal in $\mathfrak{f}_4$ generated by the short roots. We tried to find an infinite family of simple Lie algebras such that the algebra J is the lowest-dimensional member. One possibility is the family of simple Lie algebras $\mathfrak{si}(\mathfrak{sle}(n))$ of dimension $2^{2n-1} - 2^{n-1} - 2$ for $n \geq 3$, see [12], Lemma 2.2.2. This algebra is denoted by $\mathfrak{sh}(2n; \underline{m})$ in Purslow's thesis [16], Theorem 5.4.3. We used Purslow's construction for $n = 4$, see [16] pp. $138 - 141$, to compute the outer derivation algebra of this 118-dimensional algebra. It is a solvable Lie algebra of derived length 5. So it is not a counterexample, but the derived length is higher than in all other known cases. For $n = 5$ the algebra has dimension 494, but we could not compute the derivation algebra so far.

We also tested the table of B. Eick in [11] with known simple Lie algebras up to dimension 20, but found no counterexample there. There are various families of simple Lie algebras of non-standard type, and it seems to be very complicated to obtain an overview on the Zassenhaus conjecture here. So far, all families we have been able to study did not yield a new counterexample.

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