SYMPATHETIC LIE ALGEBRAS AND ADJOINT COHOMOLOGY FOR LIE ALGEBRAS

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Abstract. We study sympathetic (i.e., perfect and complete) Lie algebras. Among other topics they arise in the study of adjoint Lie algebra cohomology. Here a motivation comes from a conjecture of Pirashvili, which says that a finite-dimensional complex perfect Lie algebra is semisimple if and only if its adjoint cohomology vanishes. We prove several general results for sympathetic Lie algebras and for the adjoint Lie algebra cohomology of arbitrary finite-dimensional Lie algebras in characteristic zero using a result of Hochschild and Serre. Moreover, for certain semidirect products we obtain explicit results for the adjoint cohomology.

1. Introduction

It is well known that one can characterize finite-dimensional semisimple Lie algebras $\mathfrak{g}$ over a field of characteristic zero by the vanishing of certain Lie algebra cohomology groups. For example, by Whitehead's first lemma, we have $H^1(\mathfrak{g}, M) = 0$ for every finite-dimensional $\mathfrak{g}$-module M. The converse statement is also true - any Lie algebra whose first cohomology with coefficients in any finite-dimensional module vanishes is semisimple, see [9], Theorem 25.1. By Whitehead's second lemma, for a semisimple Lie algebra $\mathfrak{g}$ we also have $H^2(\mathfrak{g}, M) = 0$ for every finite-dimensional $\mathfrak{g}$-module M. However, the converse is no longer true, see [20].

It has also been asked, whether or not the vanishing of the adjoint cohomology groups for $\mathfrak{g}$ implies that $\mathfrak{g}$ is semisimple. This is not true in general. In fact, it is well known that the adjoint cohomology of any parabolic subalgebra $\mathfrak{p}$ of a semisimple Lie algebra vanishes, i.e., $H^n(\mathfrak{p}, \mathfrak{p}) = 0$ for all $n \geq 0$, see [19]. It is natural, however, to add the condition $H^1(\mathfrak{g}, \mathbb{C}) = 0$ for the trivial module. Note that this is a strong condition on $\mathfrak{g}$, which is equivalent to $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, i.e., $\mathfrak{g}$ is perfect.

The study of perfect Lie algebras with vanishing adjoint cohomology groups has already a long history. In 1988 Angelopoulos stated in [1] that the question goes back to M. Flato some decades ago, who asked whether semisimple Lie algebras $\mathfrak{g}$ are characterized by the vanishing conditions $H^1(\mathfrak{g}, K) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$. Afterwards several authors constructed complex non-semisimple Lie algebras $\mathfrak{g}$ satisfying

$$H^1(\mathfrak{g}, \mathbb{C}) = H^2(\mathfrak{g}, \mathbb{C}) = H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = H^2(\mathfrak{g}, \mathfrak{g}) = 0,$$

see [2, 3, 4, 5]. Angelopoulos also introduced in [1] the notion of a *sympathetic* Lie algebra, i.e., a Lie algebra which is perfect and complete, so that it satisfies $H^1(\mathfrak{g}, \mathbb{C}) = H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$. In 1996, Benayadi [4] constructed a non-semisimple sympathetic Lie algebra $\mathfrak{g}$ of dimension 25 over the complex numbers. This Lie algebra has the lowest dimension among

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the known examples of complex non-semisimple sympathetic Lie algebras. The Lie algebra of Angelopoulos has dimension 35, is sympathetic and satisfies $H^2(\mathfrak{g},\mathfrak{g}) = 0$. In this article we will provide a basis and explicit Lie brackets for Benayadi's Lie algebra in dimension 25, and show that it satisfies $\dim H^2(\mathfrak{g},\mathfrak{g}) = 1$.

In 2013 T. Pirashvili [18] studied Lie algebra and Leibniz algebra cohomology and posed the conjecture, that a non-trivial finite-dimensional complex Lie algebra $\mathfrak{g}$ is semisimple if and only if it is perfect and satisfies $H^n(\mathfrak{g},\mathfrak{g}) = 0$ for all $n \geq 0$. He called this conjecture the "Weak Conjecture" (see page 1624 in [18]) and also formulated a "Strong Conjecture". He proved one direction of the weak conjecture, namely, that a semisimple Lie algebra has vanishing adjoint cohomology and satisfies $H^1(\mathfrak{g},\mathbb{C}) = 0$.

The outline of this paper is as follows. In the second section we recall the definition and basic properties of sympathetic Lie algebras and provide results on the adjoint cohomology of Lie algebras. Then we discuss the conjecture by Pirashvili as stated above. Moreover we consider a few special cases and obtain some partial results.

In the third section we study the adjoint cohomology of Lie algebras also for Lie algebras that are not necessarily sympathetic. Here we use the Hochschild-Serre formula and other tools from homological algebra. For semidirect products $\mathfrak{g} = \mathfrak{s} \ltimes V$, where $\mathfrak{s}$ is semisimple and V is an $\mathfrak{s}$-module, we obtain non-vanishing results for $H^k(\mathfrak{g},\mathfrak{g})$. In particular, if $\mathfrak{s} = \mathfrak{sl}_n(\mathbb{C})$ and V is the natural representation of $\mathfrak{s}$, we obtain an explicit result for all cohomology groups $H^k(\mathfrak{g},\mathfrak{g})$.

In the last section we show that Benayadi's non-semisimple sympathetic Lie algebra $\mathfrak{g}$ of dimension 25 satisfies $\dim H^2(\mathfrak{g},\mathfrak{g}) = 1$. The crucial step here is to provide explicit Lie brackets for $\mathfrak{g}$ from the implicit construction in [5]. Then it is possible to compute the cohomology by using a computer algebra system. It follows that this Lie algebra cannot be a counterexample to the Pirashvili conjecture.

2. Sympathetic Lie algebras and a conjecture by Pirashvili

We always assume that all Lie algebras are finite-dimensional and defined over the complex numbers. Many results also hold for arbitrary fields of characteristic zero, but it is enough to consider complex numbers for our main results. For a given Lie algebra $\mathfrak{g}$ we denote by $\mathrm{Der}(\mathfrak{g})$ the Lie algebra of derivations of $\mathfrak{g}$, and by $\mathrm{ad}(\mathfrak{g})$ the ideal of inner derivations in $\mathrm{Der}(\mathfrak{g})$. Furthermore $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$. Let us recall the notion of a sympathetic Lie algebra, see [5].

Definition 2.1. A Lie algebra $\mathfrak{g}$ is called *sympathetic*, if it is perfect and complete, i.e., if it satisfies $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$, $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Note that every sympathetic Lie algebra $\mathfrak{g}$ is *unimodular*, i.e., it satisfies $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for all $x \in \mathfrak{g}$.

One may characterize sympathetic Lie algebras in terms of their cohomology. The result is as follows, see Proposition 1 in [4].

Proposition 2.2. *A Lie algebra $\mathfrak{g}$ is sympathetic if and only if $H^1(\mathfrak{g},\mathbb{C}) = H^0(\mathfrak{g},\mathfrak{g}) = H^1(\mathfrak{g},\mathfrak{g}) = 0$.*

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra. Denote by $\mathrm{rad}(\mathfrak{g})$ the solvable radical of $\mathfrak{g}$ and by $\mathrm{nil}(\mathfrak{g})$ the nilradical of $\mathfrak{g}$.

We begin by proving that the solvable radical of every sympathetic Lie algebra is nilpotent.

Lemma 2.4. *Let $\mathfrak{g}$ be a sympathetic Lie algebra. Then we have* $\mathrm{rad}(\mathfrak{g}) = \mathrm{nil}(\mathfrak{g})$.

Proof. It is enough to assume that $\mathfrak{g}$ is perfect. Let us write $\mathfrak{r}$ for $\mathrm{rad}(\mathfrak{g})$ and $\mathfrak{n}$ for $\mathrm{nil}(\mathfrak{g})$, and let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition of $\mathfrak{g}$. Then we have a direct vector space sum $\mathfrak{g} = \mathfrak{s} \dotplus \mathfrak{r}$. Since $\mathfrak{g}$ is perfect, we have

$$\mathfrak{s} \dotplus \mathfrak{r} = \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{s} \dotplus \mathfrak{r}, \mathfrak{s} \dotplus \mathfrak{r}] = \mathfrak{s} \dotplus ([\mathfrak{s}, \mathfrak{r}] + [\mathfrak{r}, \mathfrak{r}]).$$

Because the sum is direct it follows that

$$\mathfrak{r} = [\mathfrak{s}, \mathfrak{r}] + [\mathfrak{r}, \mathfrak{r}] \subseteq \mathrm{ad}(\mathfrak{g})(\mathfrak{r}) \subseteq \mathfrak{n}.$$

The last inclusion follows from the fact that $D(\mathfrak{r}) \subseteq \mathfrak{n}$ holds for all derivations $D \in \mathrm{Der}(\mathfrak{g})$, hence in particular for inner derivations $D = \mathrm{ad}(x)$. For a reference see Theorem 7 in chapter III in [14]. It follows that $\mathfrak{r} = \mathfrak{n}$ is nilpotent. □

Sympathetic Lie algebras have been studied by many authors, see for example [1, 2, 3, 4, 5]. Several examples of sympathetic non-semisimple Lie algebras were constructed. This is of particular interest in connection with a conjecture by Pirashvili, the so-called *Weak Conjecture* from [18].

Pirashvili Conjecture 2.5. *A finite-dimensional complex Lie algebra is semisimple if and only if it satisfies $H^1(\mathfrak{g}, \mathbb{C}) = 0$ and $H^n(\mathfrak{g}, \mathfrak{g}) = 0$ for all $n \geq 0$.*

Remark 2.6. The conjecture can also be formulated in terms of vanishing Leibniz homology with trivial coefficients, i.e., that $HL_n(\mathfrak{g}, \mathbb{C}) = 0$ for all $n \geq 1$. For the equivalence of these cohomological conditions see Lemma 4.2 of [18]. See [11] for results on the cohomology of Leibniz algebras and Lie algebras.

Let us call these cohomological vanishing conditions the *Pirashvili conditions* for $\mathfrak{g}$. It is known that every semisimple Lie algebra satisfies these conditions, see [17]. In fact, this follows from the first Whitehead Lemma and the following result of Carles, see Lemma 2.2 in [8]:

Proposition 2.7. *Let $\mathfrak{g}$ be a complete Lie algebra whose solvable radical is abelian. Then we have $H^n(\mathfrak{g}, \mathfrak{g}) = 0$ for all $n \geq 0$.*

The converse direction of Pirashvili's conjecture is still open. A Lie algebra satisfying the Pirashvili conditions is sympathetic, but we don't know, whether or not it is necessarily semisimple.

It is also interesting to note that the conjecture need not be true if we omit the condition $H^1(\mathfrak{g}, \mathbb{C}) = 0$ from the Pirashvili conditions. Let us give another explicit example (we have already mentioned that the adjoint cohomology of any parabolic subalgebra of a semisimple Lie algebra vanishes).

Example 2.8. *Let $\mathfrak{g} = \mathfrak{aff}(\mathbb{C}^m) \cong \mathfrak{gl}_m(\mathbb{C}) \ltimes \mathbb{C}^m$ be the affine Lie algebra with $m \geq 1$. Then we have $H^n(\mathfrak{g}, \mathfrak{g}) = 0$ for all $n \geq 0$, but $\mathfrak{g}$ is not semisimple.*

Indeed, it is easy to see that $\mathfrak{g}$ is complete, i.e., that $H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$, see Theorem 4.2 in [15]. Therefore it follows from Proposition 2.7 that $H^n(\mathfrak{g}, \mathfrak{g}) = 0$ for all $n \geq 0$.

The following result shows that the Pirashvili conjecture is true for sympathetic Lie algebras, whose solvable radical is abelian.

Lemma 2.9. *Let $\mathfrak{g}$ be a sympathetic Lie algebra whose solvable radical $\mathfrak{r}$ is abelian. Then $\mathfrak{g}$ is semisimple.*

Proof. Assume that $\mathfrak{r}$ is abelian. Then $V=\mathfrak{r}$ is an m-dimensional vector space. We have the Levi decomposition $\mathfrak{g}=\mathfrak{s}\ltimes V$ with a Levi subalgebra $\mathfrak{s}$ and an $\mathfrak{s}$-module V. Denote by D the linear map on $\mathfrak{g}$ which is zero on $\mathfrak{s}$ and the identity on V. It can be represented by a block-diagonal matrix with a zero matrix in the left-upper block and an identity matrix in the right-lower block which shows that $\operatorname{tr}(D)=m$. We claim that D is a derivation of $\mathfrak{g}$. The Lie bracket on $\mathfrak{s}\dotplus V$ is given by

$$[(x,v),(y,u)]=([x,y],x\cdot u-y\cdot v)$$

for all $x,y\in\mathfrak{s}$ and $u,v\in V$. Then $D([(x,v),(y,u)])=(0,x\cdot u-y\cdot v)$ and

$$\begin{aligned}[D((x,v)),(y,u)]&=[(0,v),(y,u)]=(0,-y\cdot v),\\ [(x,v),D((y,u))]&=[(x,v),(0,u)]=(0,x\cdot u).\end{aligned}$$

Since $\mathfrak{g}$ is complete, D is an inner derivation. However, since $\mathfrak{g}$ is perfect, all adjoint operators $\operatorname{ad}(x)$ have zero trace. Hence $\operatorname{tr}(D)=m=0$, so that $V=0$ and therefore $\mathfrak{g}$ is semisimple. ☐

It follows that a potential counterexample to Pirashvili's conjecture must have a nilpotent, non-abelian radical.

Corollary 2.10. *Let $\mathfrak{g}$ be a non-semisimple Lie algebra that satisfies the Pirashvili conditions. Then the solvable radical of $\mathfrak{g}$ is nilpotent and non-abelian.*

Proof. The solvable radical is nilpotent by Lemma 2.4 and non-abelian by Lemma 2.9. ☐

Similarly, we can also obtain the following result.

Proposition 2.11. *Let $\mathfrak{g}$ be a sympathetic Lie algebra with solvable radical $\mathfrak{n}$. Suppose that we have a split short exact sequence*

$$0\to Z(\mathfrak{n})\to\mathfrak{n}\to\mathfrak{n}/Z(\mathfrak{n})\to 0.$$

Then $\mathfrak{g}$ is semisimple.

Proof. Let $\mathfrak{g}\cong\mathfrak{s}\ltimes\mathfrak{n}$ be a Levi decomposition. The Lie bracket on the vector space $\mathfrak{g}=\mathfrak{s}\dotplus\mathfrak{n}$ is given by

$$[(s,n),(t,m)]=([s,t],[n,m]+s\cdot m-t\cdot n)$$

for $s,t\in\mathfrak{s}$ and $n,m\in\mathfrak{n}$. Since $\mathfrak{n}$ is a central extension of $\mathfrak{n}/Z(\mathfrak{n})$ by $Z(\mathfrak{n})$, the Lie bracket on the vector space $\mathfrak{n}=(\mathfrak{n}/Z(\mathfrak{n}))\dotplus Z(\mathfrak{n})$ is given by

$$[n,m]=[(x,a),(y,b)]=([x,y],\omega(x,y))$$

for $x,y\in\mathfrak{n}/Z(\mathfrak{n})$, $a,b\in Z(\mathfrak{n})$ and $[\omega]\in H^2(\mathfrak{n}/Z(\mathfrak{n}),Z(\mathfrak{n}))$. Since the extension is central and split we may assume that $\omega=0$. Writing $s\cdot m=s\cdot(y,b)=(s\cdot y,s\cdot b)$ and $t\cdot n=t\cdot(x,a)=(t\cdot x,t\cdot a)$, the Lie bracket on $\mathfrak{g}=\mathfrak{s}\dotplus(\mathfrak{n}/Z(\mathfrak{n}))\dotplus Z(\mathfrak{n})$ becomes

$$[(s,x,a),(t,y,b)]=([s,t],[x,y]+s\cdot y-t\cdot x,s\cdot b-t\cdot a)$$

for $s,t\in\mathfrak{s}$, $x,y\in\mathfrak{n}/Z(\mathfrak{n})$ and $a,b\in Z(\mathfrak{n})$.

Now define a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ by $(s,x,a) \mapsto (0,0,a)$. It is a derivation of $\mathfrak{g}$, because we have

$$\begin{aligned} D([(s,x,a),(t,y,b)]) &= (0,0,s\cdot b - t\cdot a),\\ [D(s,x,a),(t,y,b)] &= [(0,0,a),(t,y,b)] = (0,0,-t\cdot a),\\ [(s,x,a),D(t,y,b)] &= [(s,x,a),(0,0,b)] = (0,0,s\cdot b). \end{aligned}$$

Since $\mathfrak{g}$ is complete, D is an inner derivation. Let $D = \operatorname{ad}(z)$ and $m = \dim Z(\mathfrak{n})$. Now $\mathfrak{g}$ is also perfect so that the adjoint operators have trace zero. Hence $\operatorname{tr}(D) = 0$ and thus $m = 0$ and $Z(\mathfrak{n}) = 0$. By Lemma 2.4, $\mathfrak{n}$ is nilpotent. Hence $Z(\mathfrak{n}) = 0$ implies that $\mathfrak{n} = 0$. It follows that $\mathfrak{g}$ is semisimple. ☐

Corollary 2.12. *Let $\mathfrak{g}$ be a non-semisimple sympathetic Lie algebra with solvable radical $\mathfrak{n}$. Then $\mathfrak{n}$ is nilpotent, non-abelian, and the extension $0 \to Z(\mathfrak{n}) \to \mathfrak{n} \to \mathfrak{n}/Z(\mathfrak{n}) \to 0$ does not split.*

3. Adjoint Lie algebra cohomology

A well-known construction for perfect but non-semisimple Lie algebras is the semidirect product $\mathfrak{g} = \mathfrak{s} \ltimes V$ of a semisimple Lie algebra with a non-trivial simple $\mathfrak{s}$-module V, where the latter is considered as an abelian Lie algebra, i.e., $\operatorname{rad}(\mathfrak{g})$ is abelian. Suppose that $\mathfrak{g}$ is complete. Then $\mathfrak{g}$ is sympathetic and hence semisimple by Lemma 2.9. This is a contradiction. Thus $\mathfrak{g}$ cannot be complete. In fact, this is true more generally, even if $\mathfrak{g}$ is not perfect.

Proposition 3.1. *Let $\mathfrak{g} = \mathfrak{s} \ltimes V$, where $\mathfrak{s}$ is semisimple and V is an $\mathfrak{s}$-module. Then we have*

$$H^1(\mathfrak{g},\mathfrak{g}) \cong H^1(V,V)^{\mathfrak{s}} \cong \operatorname{Hom}_{\mathfrak{s}}(V,V).$$

In particular, $\dim H^1(\mathfrak{g},\mathfrak{g}) \geq 1$ *and $\mathfrak{g}$ is not complete.*

Proof. By Proposition 5.11 in [6] we have $H^1(\mathfrak{g},\mathfrak{g}) \cong \operatorname{Hom}_{\mathfrak{s}}(V,V)$. Since V is an abelian Lie algebra and a trivial V-module, $H^1(V,V) \cong \operatorname{Hom}(V,V)$, so that $H^1(V,V)^{\mathfrak{s}} \cong \operatorname{Hom}_{\mathfrak{s}}(V,V)$. Now we always have the identity in $\operatorname{Hom}_{\mathfrak{s}}(V,V)$. Hence this space is at least 1-dimensional. ☐

Corollary 3.2. *Let $\mathfrak{g} = \mathfrak{s} \ltimes V$, where $\mathfrak{s}$ is semisimple and V is a simple $\mathfrak{s}$-module. Then we have* $\dim H^1(\mathfrak{g},\mathfrak{g}) = 1$.

Proof. By Schur's Lemma we have $\operatorname{Hom}_{\mathfrak{s}}(V,V) \cong \mathbb{C}\cdot \operatorname{id}$. Hence the space is 1-dimensional. ☐

Proposition 3.3. *Let $\mathfrak{g} = \mathfrak{s} \ltimes V$, where $\mathfrak{s}$ is semisimple and V is an $\mathfrak{s}$-module. Then we have an exact sequence*

$$0 \to H^1(V,V)^{\mathfrak{s}} \to H^1(V,\mathfrak{g})^{\mathfrak{s}} \to H^1(V,\mathfrak{g}/V)^{\mathfrak{s}}.$$

Proof. Consider the short exact sequence of $\mathfrak{g}$-modules

$$0 \to V \to \mathfrak{g} \to \mathfrak{g}/V \to 0,$$

which is also a short exact sequence of V-modules by restriction to $V \subset \mathfrak{g}$. Here V and $\mathfrak{g}/V$ are trivial V-modules. Applying the long exact sequence in cohomology we obtain

$$\cdots \to H^0(V,\mathfrak{g}/V) \to H^1(V,V) \to H^1(V,\mathfrak{g}) \to H^1(V,\mathfrak{g}/V) \to \cdots$$

Applying the functor of $\mathfrak{s}$-invariants, which is exact on the subcategory of finite-dimensional $\mathfrak{s}$-modules we obtain

$$\cdots \to H^0(V,\mathfrak{g}/V)^{\mathfrak{s}} \to H^1(V,V)^{\mathfrak{s}} \to H^1(V,\mathfrak{g})^{\mathfrak{s}} \to H^1(V,\mathfrak{g}/V)^{\mathfrak{s}} \to \cdots$$

Since $H^0(V,\mathfrak{g}/V)$ is the space of V-invariants of the trivial module $\mathfrak{g}/V$, we obtain $H^0(V,\mathfrak{g}/V)\cong \mathfrak{g}/V$. But we have $(\mathfrak{g}/V)^{\mathfrak{s}}=0$, because the quotient module $\mathfrak{g}/V\cong\mathfrak{s}$ does not contain non-zero $\mathfrak{s}$-invariants. Hence we have $H^0(V,\mathfrak{g}/V)^{\mathfrak{s}}=0$. This yields the claimed exact sequence. $\square$

By the Hochschild-Serre formula, see Theorem 13 of [13] and the paragraph before on page 603, we obtain

$$H^1(\mathfrak{g},\mathfrak{g})\cong\bigoplus_{i+j=1}H^i(\mathfrak{s},\mathbb{C})\otimes H^j(V,\mathfrak{g})^{\mathfrak{s}}\cong H^1(V,\mathfrak{g})^{\mathfrak{s}}.$$

Thus the morphism $H^1(V,V)^{\mathfrak{s}}\to H^1(V,\mathfrak{g})^{\mathfrak{s}}$ in the exact sequence of Proposition 3.3 is in fact an isomorphism. The Hochschild-Serre formula also implies $H^2(\mathfrak{g},\mathfrak{g})\cong H^2(V,\mathfrak{g})^{\mathfrak{s}}$.

Proposition 3.4. *Let $\mathfrak{g}=\mathfrak{s}\ltimes V$, where $\mathfrak{s}$ is semisimple and V is an $\mathfrak{s}$-module. Assume that V does not contain any factor isomorphic to an ideal of $\mathfrak{s}$ in its decomposition as an $\mathfrak{s}$-module. Then we have an exact sequence*

$$0\to H^2(V,V)^{\mathfrak{s}}\to H^2(V,\mathfrak{g})^{\mathfrak{s}}.$$

If we assume in addition that the $\mathfrak{s}$-module $\Lambda^2(V)$ contains a factor isomorphic to V, then we obtain

$$H^2(V,\mathfrak{g})^{\mathfrak{s}}\cong H^2(\mathfrak{g},\mathfrak{g})\neq 0.$$

Proof. We have $H^1(V,\mathfrak{g}/V)^{\mathfrak{s}}\cong \operatorname{Hom}_{\mathfrak{s}}(V,\mathfrak{g}/V)$, because the Lie algebra V is abelian and the V-module $\mathfrak{g}/V$ is trivial. Both $\mathfrak{g}/V\cong\mathfrak{s}$ and V decompose into direct factors, and by assumption they do not share an isomorphic factor. Hence we have $\operatorname{Hom}_{\mathfrak{s}}(V,\mathfrak{g}/V)=0$ and $H^1(V,\mathfrak{g}/V)^{\mathfrak{s}}=0$. So the continuation to degree two of the exact sequence in Proposition 3.3 yields

$$0=H^1(V,\mathfrak{g}/V)^{\mathfrak{s}}\to H^2(V,V)^{\mathfrak{s}}\to H^2(V,\mathfrak{g})^{\mathfrak{s}}.$$

Here we have noted above that $H^2(V,\mathfrak{g})^{\mathfrak{s}}\cong H^2(\mathfrak{g},\mathfrak{g})$. By the additional assumption we have $H^2(V,V)^{\mathfrak{s}}\cong\operatorname{Hom}_{\mathfrak{s}}(\Lambda^2(V),V)\neq 0$. $\square$

The results on $H^1(\mathfrak{g},\mathfrak{g})$ and $H^2(\mathfrak{g},\mathfrak{g})$ can be generalized to higher cohomology groups as follows. Note that we have for all k,

$$\begin{aligned} H^k(V,\mathfrak{g}/V)^{\mathfrak{s}}&\cong\operatorname{Hom}_{\mathfrak{s}}(\Lambda^k(V),\mathfrak{s}),\\ H^k(V,V)^{\mathfrak{s}}&\cong\operatorname{Hom}_{\mathfrak{s}}(\Lambda^k(V),V).\end{aligned}$$

Proposition 3.5. *Let $\mathfrak{g}=\mathfrak{s}\ltimes V$, where $\mathfrak{s}$ is semisimple and V is an $\mathfrak{s}$-module. Let $k\geq 1$ and suppose that the $\mathfrak{s}$-module $\Lambda^{k-1}(V)$ does not contain a submodule isomorphic to $\mathfrak{s}$. Then we have an exact sequence*

$$0\to H^k(V,V)^{\mathfrak{s}}\to H^k(V,\mathfrak{g})^{\mathfrak{s}}\to H^k(V,\mathfrak{g}/V)^{\mathfrak{s}}.$$

Suppose that in addition the $\mathfrak{s}$-module $\Lambda^k(V)$ does contain a submodule isomorphic to V. Then we have $\dim H^k(\mathfrak{g},\mathfrak{g})\geq 1$.

Proof. As in the proof of Proposition 3.3 we have a long exact sequence

$$\cdots\to H^{k-1}(V,\mathfrak{g}/V)^{\mathfrak{s}}\to H^k(V,V)^{\mathfrak{s}}\to H^k(V,\mathfrak{g})^{\mathfrak{s}}\to H^k(V,\mathfrak{g}/V)^{\mathfrak{s}}\to\cdots$$

Here we have by the first assumption that

$$H^{k-1}(V,\mathfrak{g}/V)^{\mathfrak{s}}\cong\operatorname{Hom}_{\mathfrak{s}}(\Lambda^{k-1}(V),\mathfrak{s})=0.$$

So the first assertion follows.

By the second assumption we have that $H^k(V,V)^{\mathfrak{s}} \cong \operatorname{Hom}_{\mathfrak{s}}(\Lambda^k(V),V)$ is nonzero. Thus the exact sequence implies that also $H^k(V,\mathfrak{g})^{\mathfrak{s}}$ is nonzero. Using again the Hochschild-Serre formula for $M=\mathfrak{g}$, $L=V$ and $K=\mathfrak{s}$, we have

$$H^k(\mathfrak{g},\mathfrak{g}) \cong \bigoplus_{i+j=k} H^i(\mathfrak{s},\mathbb{C}) \otimes H^j(V,\mathfrak{g})^{\mathfrak{s}}$$

For $i=0$ and $j=k$ this direct sum contains the summand

$$H^0(\mathfrak{s},\mathbb{C}) \otimes H^k(V,\mathfrak{g})^{\mathfrak{s}} \cong \mathbb{C} \otimes H^k(V,\mathfrak{g})^{\mathfrak{s}} \cong H^k(V,\mathfrak{g})^{\mathfrak{s}},$$

which is nonzero. Hence $H^k(\mathfrak{g},\mathfrak{g})$ is nonzero. □

In some cases we can explicitly compute all cohomology groups $H^k(\mathfrak{g},\mathfrak{g})$ by using the above arguments.

Proposition 3.6. *Let $\mathfrak{g}=\mathfrak{s}\ltimes V$ with $\mathfrak{s}=\mathfrak{sl}_n(\mathbb{C})$, $n\geq 2$, and $V=L(\omega_1)$ be the natural $\mathfrak{s}$-module of dimension n. Then we have for all $k\geq 1$*

$$H^k(\mathfrak{g},\mathfrak{g}) \cong H^{k-1}(\mathfrak{sl}_n(\mathbb{C}),\mathbb{C})$$

Here $H^*(\mathfrak{sl}_n(\mathbb{C}),\mathbb{C})$ is isomorphic to the exterior algebra $\Lambda^*(c_3,c_5,\dots,c_{2n-1})$ generated by cocycles c_{2i+1} for $i=1,\dots,n-1$.

Proof. The $\mathfrak{s}$-module $\Lambda^k(V)$ has dimension $\binom{n}{k}$. It is irreducible for every $1\leq k\leq n$, and it is not isomorphic to the adjoint $\mathfrak{s}$-module $\mathfrak{s}$. Indeed, their dimensions are always different. Namely, suppose that $\binom{n}{k}=n^2-1$. It is well known that

$$\frac{n}{(n,k)} \mid \binom{n}{k},\ 1\leq k\leq n,$$

where $(n,k)=\gcd(n,k)$. If $k<n$, then $d=\frac{n}{(n,k)}$ is a divisor $d>1$ of n and hence does not divide n^2-1. Hence $\binom{n}{k}=n^2-1$ is impossible. For $k=n$ this is also impossible. Hence by Schur's Lemma we have

$$H^k(V,\mathfrak{g}/V)^{\mathfrak{s}} \cong \operatorname{Hom}_{\mathfrak{s}}(\Lambda^k(V),\mathfrak{s})=0$$

for all $k\geq 1$. Then the long exact sequence from the proof of Proposition 3.5 yields

$$H^k(V,\mathfrak{g})^{\mathfrak{s}} \cong H^k(V,V)^{\mathfrak{s}} \cong \operatorname{Hom}_{\mathfrak{s}}(\Lambda^k(V),V) \cong \begin{cases} \mathbb{C} & \text{for } k=1 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for $n=2$, the modules V and $\Lambda^k(V)$ have the same dimension only for $k=1$. For $n>2$, also the $\mathfrak{sl}_n(\mathbb{C})$-module $\Lambda^{n-1}(V)$ has the same dimension as V, but it is not isomorphic to V. It is isomorphic as an $\mathfrak{sl}_n(\mathbb{C})$-module to V^*, because $\mathfrak{sl}_n(\mathbb{C})$ is unimodular, and V is self-dual if and only if $n=2$. Thus the Hochschild-Serre formula yields,

$$\begin{aligned} H^k(\mathfrak{g},\mathfrak{g}) &\cong \bigoplus_{i+j=k} H^i(\mathfrak{s},\mathbb{C}) \otimes H^j(V,\mathfrak{g})^{\mathfrak{s}} \\ &\cong H^{k-1}(\mathfrak{s},\mathbb{C}) \otimes \mathbb{C}. \end{aligned}$$

It is well known that the cohomology $H^\ell(\mathfrak{sl}_n(\mathbb{C}),\mathbb{C})$ is isomorphic to the ℓ-th component of

$$\Lambda^*(c_3,c_5,\dots,c_{2n-1})$$

with generators c_{2i+1}, see table 4 in [16]. □

This yields, for example, with the n-dimensional natural $\mathfrak{sl}_n(\mathbb{C})$-module $V = L(\omega_1)$,

$$H^k(\mathfrak{sl}_2(\mathbb{C}) \ltimes V, \mathfrak{sl}_2(\mathbb{C}) \ltimes V) \cong H^{k-1}(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } k = 1, 4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$H^k(\mathfrak{sl}_3(\mathbb{C}) \ltimes V, \mathfrak{sl}_3(\mathbb{C}) \ltimes V) \cong H^{k-1}(\mathfrak{sl}_3(\mathbb{C}), \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } k = 1, 4, 6, 9 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.7. One can also consider Proposition 3.6 for $\mathfrak{g} = \mathfrak{s} \ltimes V$ with other classical Lie algebras $\mathfrak{s}$ and their natural representation V. We have used in the proof two facts that must be satisfied for $\mathfrak{s}$, namely that the exterior powers $\Lambda^k(V)$ are again irreducible, and that $\Lambda^k(V)$ and $\mathfrak{s}$ are non-isomorphic as $\mathfrak{s}$-modules for all $k \geq 1$. Unfortunately, this need not be true in general for simple Lie algebras $\mathfrak{s}$ of type B_n, C_n, D_n. For example, for type C_n, the exterior powers $\Lambda^k(V)$ of dimension $\binom{2n}{k}$ are no longer irreducible for $2 \leq k \leq 2n-1$, see [12], §17.2. And for types B_n and D_n, the $\mathfrak{s}$-modules $\Lambda^2(V)$ and $\mathfrak{s}$ are isomorphic. So one would need additional arguments for the computation of the adjoint cohomology.

Remark 3.8. Using the Hochschild-Serre formula as in the proof of Proposition 3.6, one may also compute the cohomology with trivial coefficients of a semi-direct product $\mathfrak{g} = \mathfrak{s} \ltimes V$ for a general complex semisimple Lie algebra $\mathfrak{s}$ and a general irreducible $\mathfrak{s}$-module V. In particular, if the $\mathfrak{s}$-module $\Lambda^k(V)$ is non-trivial and irreducible for all $0 < k < m$, where $m := \dim(V)$, then $H^k(V, \mathbb{C})^{\mathfrak{s}} = \Lambda^k(V^*)^{\mathfrak{s}} = 0$ for all $0 < k < m$, and we obtain

$$H^n(\mathfrak{g}, \mathbb{C}) = H^n(\mathfrak{s}, \mathbb{C}) \oplus H^{n-m}(\mathfrak{s}, \mathbb{C}).$$

This can also be used to compute the low degree *Leibniz cohomology* $HL^n(\mathfrak{g}, \mathfrak{g})$ with adjoint coefficients in some cases. Consider the 5-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V$ from above. It satisfies $Z(\mathfrak{g}) = 0$. Hence by Proposition 2.2 and Theorem 2.6 in [11] in conjunction with Proposition 3.6 and the formula right after its proof, we obtain that

$$\dim HL^n(\mathfrak{g}, \mathfrak{g}) \cong \begin{cases} 0, & \text{for } n = 0, 2, \\ 1, & \text{for } n = 1. \end{cases}$$

Thus the Lie algebra $\mathfrak{g}$ is rigid as a Leibniz algebra. Note that we used $H^\bullet(\mathfrak{g}, \mathfrak{g}) \cong H^\bullet(\mathfrak{g}, \mathfrak{g}^*)$ for this computation. The latter follows from the existence of an invariant, non-degenerate, symmetric bilinear form on $\mathfrak{g}$.

Finally, we can also use the Hochschild-Serre formula to compute the adjoint cohomology $H^n(\mathfrak{g}, \mathfrak{g})$ of certain semidirect products $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ for the top degree n, i.e., for $n = \dim(\mathfrak{g})$.

Proposition 3.9. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra with Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$, where $\mathfrak{n} = \operatorname{rad}(\mathfrak{g})$ is nilpotent. Assume that the $\mathfrak{s}$-module $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ does not contain the trivial $\mathfrak{s}$-module $\mathbb{C}$. Then we have*

$$H^n(\mathfrak{g}, \mathfrak{g}) = 0.$$

Proof. Let $\dim(\mathfrak{n}) = m$ and $\dim(\mathfrak{s}) = n - m$. By the Hochschild-Serre formula we have

$$\begin{aligned} H^n(\mathfrak{g}, \mathfrak{g}) &= \bigoplus_{p+q=n} H^p(\mathfrak{s}, \mathbb{C}) \otimes H^q(\mathfrak{n}, \mathfrak{g})^{\mathfrak{s}} \\ &= H^{n-m}(\mathfrak{s}, \mathbb{C}) \otimes H^m(\mathfrak{n}, \mathfrak{g})^{\mathfrak{s}}, \end{aligned}$$

because $\dim H^k(\mathfrak{n},\mathfrak{g})=0$ for all $k>m$, and $\dim H^k(\mathfrak{s},\mathbb{C})=0$ for all $k>m-n$. To compute $H^m(\mathfrak{n},\mathfrak{g})^{\mathfrak{s}}$ we use the long exact sequence as in the proof of Proposition 3.5 to obtain

$$\cdots \to H^m(\mathfrak{n},\mathfrak{n})^{\mathfrak{s}} \to H^m(\mathfrak{n},\mathfrak{g})^{\mathfrak{s}} \to H^m(\mathfrak{n},\mathfrak{g}/\mathfrak{n})^{\mathfrak{s}} \to 0.$$

Here $\mathfrak{g}/\mathfrak{n}$ is a trivial $\mathfrak{n}$-module. Since $\mathfrak{n}$ is nilpotent, we have $\dim H^m(\mathfrak{n},\mathbb{C})\ge 1$ by Théorème 2 of [10]. On the other hand, $\dim \operatorname{Hom}(\Lambda^m(\mathfrak{n}),\mathbb{C})=1$, so that $\dim H^m(\mathfrak{n},\mathbb{C})=1$. Therefore we have

$$H^m(\mathfrak{n},\mathfrak{g}/\mathfrak{n})\cong H^m(\mathfrak{n},\mathbb{C})\otimes \mathfrak{g}/\mathfrak{n}\cong \mathfrak{g}/\mathfrak{n}\cong\mathfrak{s}$$

as $\mathfrak{s}$-modules. Hence $H^m(\mathfrak{n},\mathfrak{g}/\mathfrak{n})^{\mathfrak{s}}\cong\mathfrak{s}^{\mathfrak{s}}=0$.

To compute $H^m(\mathfrak{n},\mathfrak{n})^{\mathfrak{s}}$, we use Poincaré duality. Since $\mathfrak{n}$ is unimodular, the duality is $\mathfrak{s}$-equivariant, and we obtain

$$H^m(\mathfrak{n},\mathfrak{n})\cong H_0(\mathfrak{n},\mathfrak{n})\cong \mathfrak{n}/[\mathfrak{n},\mathfrak{n}].$$

By assumption, the $\mathfrak{s}$-module $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ does not contain the trivial module. Hence we have $H^m(\mathfrak{n},\mathfrak{n})^{\mathfrak{s}}=0$. Hence $H^m(\mathfrak{n},\mathfrak{g})^{\mathfrak{s}}=0$, so that $H^n(\mathfrak{g},\mathfrak{g})=0$ by the Hochschild-Serre formula. $\square$

4. Cohomology of Benayadi's Lie algebra

Benayadi constructed in [5] non-semisimple sympathetic Lie algebras of dimension 25 by taking the vector space

$$\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})\ltimes(V(7)\oplus V(5)\oplus V(7)\oplus V(3)),$$

where $V(n)$ denotes the n-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. He equipped $\mathfrak{g}$ with a Lie bracket such that the Lie brackets of $\mathfrak{sl}_2(\mathbb{C})$ with $\mathfrak{g}_1=V(7)$, $\mathfrak{g}_2=V(5)$, $\mathfrak{g}_3=V(7)$ and $\mathfrak{g}_4=V(3)$ are given by the action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathfrak{g}_i$, and such that

$$[\mathfrak{g}_1,\mathfrak{g}_1]=\mathfrak{g}_3,\ [\mathfrak{g}_1,\mathfrak{g}_2]=\mathfrak{g}_4,\ [\mathfrak{g}_2,\mathfrak{g}_2]=\mathfrak{g}_4,\ [\mathfrak{g}_1,\mathfrak{g}_3]=\mathfrak{g}_4.$$

In order to obtain explicit Lie brackets, we want to introduce a basis $\{e_1,\dots,e_{25}\}$ of $\mathfrak{g}$. Then the cohomology can be computed by a computer algebra system, e.g., GAP. So fix a basis of $\mathfrak{g}$, such that $\{e_1,e_2,e_3\}$ is a basis of $\mathfrak{sl}_2(\mathbb{C})$, $\{e_4,\dots,e_{10}\}$ is a basis of $\mathfrak{g}_1$, $\{e_{11},\dots,e_{15}\}$ is a basis of $\mathfrak{g}_2$, $\{e_{16},\dots,e_{22}\}$ is a basis of $\mathfrak{g}_3$ and $\{e_{23},e_{24},e_{25}\}$ is a basis of $\mathfrak{g}_4$.

The action of $\mathfrak{sl}_2(\mathbb{C})$ on $V(n)$ may be given by

$$\rho(e_1)=\begin{pmatrix}0&1&0&\cdots&0\\0&0&2&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&0\\\vdots&\ddots&\ddots&\ddots&n-1\\0&0&\cdots&0&0\end{pmatrix},\quad \rho(e_2)=\begin{pmatrix}0&0&\cdots&0&0\\n-1&0&\ddots&0&0\\0&\ddots&\ddots&\ddots&\vdots\\\vdots&\ddots&2&0&0\\0&\cdots&0&1&0\end{pmatrix}$$

and

$$\rho(e_3)=\begin{pmatrix}n-1&0&\cdots&0&0\\0&n-3&\cdots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\ddots&3-n&0\\0&0&\cdots&0&1-n\end{pmatrix}$$

So the nonzero brackets are determined as follows:

1. The brackets for $\mathfrak{sl}_2(\mathbb{C})$:

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -2e_1, \qquad [e_2, e_3] = 2e_2.$$

2. The brackets between $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g}_4$:

$$\begin{aligned}
[e_1, e_{24}] &= e_{23}, & [e_2, e_{23}] &= 2e_{24}, & [e_3, e_{23}] &= 2e_{23},\\
[e_1, e_{25}] &= 2e_{24}, & [e_2, e_{24}] &= e_{25}, & [e_3, e_{25}] &= -2e_{25},
\end{aligned}$$

3. The brackets between $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g}_2$:

$$\begin{aligned}
[e_1, e_{12}] &= e_{11}, & [e_2, e_{11}] &= 4e_{12}, & [e_3, e_{11}] &= 4e_{11},\\
[e_1, e_{13}] &= 2e_{12}, & [e_2, e_{12}] &= 3e_{13}, & [e_3, e_{12}] &= 2e_{12},\\
[e_1, e_{14}] &= 3e_{13}, & [e_2, e_{13}] &= 2e_{14}, & [e_3, e_{14}] &= -2e_{14},\\
[e_1, e_{15}] &= 4e_{14}, & [e_2, e_{14}] &= e_{15}, & [e_3, e_{15}] &= -4e_{15}.
\end{aligned}$$

4. The brackets between $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g}_3$:

$$\begin{aligned}
[e_1, e_{17}] &= e_{16}, & [e_2, e_{16}] &= 6e_{17}, & [e_3, e_{16}] &= 6e_{16},\\
[e_1, e_{18}] &= 2e_{17}, & [e_2, e_{17}] &= 5e_{18}, & [e_3, e_{17}] &= 4e_{17},\\
[e_1, e_{19}] &= 3e_{18}, & [e_2, e_{18}] &= 4e_{19}, & [e_3, e_{18}] &= 2e_{18},\\
[e_1, e_{20}] &= 4e_{19}, & [e_2, e_{19}] &= 3e_{20}, & [e_3, e_{20}] &= -2e_{20},\\
[e_1, e_{21}] &= 5e_{20}, & [e_2, e_{20}] &= 2e_{21}, & [e_3, e_{21}] &= -4e_{21},\\
[e_1, e_{22}] &= 6e_{21}, & [e_2, e_{21}] &= e_{22}, & [e_3, e_{22}] &= -6e_{22}.
\end{aligned}$$

5. The brackets between $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g}_1$:

$$\begin{aligned}
[e_1, e_5] &= e_4, & [e_2, e_4] &= 6e_5, & [e_3, e_4] &= 6e_4,\\
[e_1, e_6] &= 2e_5, & [e_2, e_5] &= 5e_6, & [e_3, e_5] &= 4e_5,\\
[e_1, e_7] &= 3e_6, & [e_2, e_6] &= 4e_7, & [e_3, e_6] &= 2e_6,\\
[e_1, e_8] &= 4e_7, & [e_2, e_7] &= 3e_8, & [e_3, e_8] &= -2e_8,\\
[e_1, e_9] &= 5e_8, & [e_2, e_8] &= 2e_9, & [e_3, e_9] &= -4e_9,\\
[e_1, e_{10}] &= 6e_9, & [e_2, e_9] &= e_{10}, & [e_3, e_{10}] &= -6e_{10}.
\end{aligned}$$

6. The brackets $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_3$:

$$\begin{aligned}
[e_4, e_5] &= a_1 e_{16} + \cdots + a_7 e_{22},\\
[e_4, e_6] &= a_8 e_{16} + \cdots + a_{14} e_{22},\\
\cdots &= \qquad \cdots\\
[e_9, e_{10}] &= a_{141} e_{16} + \cdots + a_{147} e_{22}.
\end{aligned}$$

7. The brackets $[\mathfrak{g}_1, \mathfrak{g}_2] = \mathfrak{g}_4$:

$$\begin{aligned}
[e_4, e_{11}] &= b_1 e_{23} + b_2 e_{24} + b_3 e_{25},\\
[e_4, e_{12}] &= b_4 e_{23} + b_5 e_{24} + b_6 e_{25},\\
\cdots &= \cdots\\
[e_{10}, e_{15}] &= b_{103} e_{23} + b_{104} e_{24} + b_{105} e_{25}.
\end{aligned}$$

8. The brackets $[\mathfrak{g}_2, \mathfrak{g}_2] = \mathfrak{g}_4$:

$$\begin{aligned}
[e_{11}, e_{12}] &= c_1 e_{23} + c_2 e_{24} + c_3 e_{25},\\
[e_{11}, e_{13}] &= c_4 e_{23} + c_5 e_{24} + c_6 e_{25},\\
\cdots &= \cdots\\
[e_{14}, e_{15}] &= c_{28} e_{23} + c_{29} e_{24} + c_{30} e_{25}.
\end{aligned}$$

9. The brackets $[\mathfrak{g}_1, \mathfrak{g}_3] = \mathfrak{g}_4$:

$$\begin{aligned}
[e_4, e_{16}] &= d_1 e_{23} + d_2 e_{24} + d_3 e_{25},\\
[e_4, e_{17}] &= d_4 e_{23} + d_5 e_{24} + d_6 e_{25},\\
\cdots &= \cdots\\
[e_{10}, e_{22}] &= d_{145} e_{23} + d_{146} e_{24} + d_{147} e_{25}.
\end{aligned}$$

The Jacobi identity is equivalent to some polynomial equations in the variables a_i, b_i, c_i, d_i. Here these equations reduce to some easy linear equations. It turns out that there are solutions. The solution space only depends on the four nonzero parameters $a_{15}, b_{13}, c_7, d_{16}$. We obtain a family of *Lie algebras* $L(a, b, c, d)$ with

$$(a_{15}, b_{13}, c_7, d_{16}) = (3a, 60b, 2c, 15d).$$

The rewriting in terms of nonzero complex parameters a, b, c, d is only for our convenience, to avoid fractions.

Proposition 4.1. *The family of Lie algebras $L(a, b, c, d)$ has the following explicit Lie brackets with respect to the basis $(e_1, \ldots, e_{25})$.*

$$\begin{aligned}
[e_1, e_2] &= e_3, & [e_2, e_{23}] &= 2e_{24}, & [e_6, e_8] &= -4ae_{19},\\
[e_1, e_3] &= -2e_1, & [e_2, e_{24}] &= e_{25}, & [e_6, e_{10}] &= 12ae_{21},\\
[e_1, e_5] &= e_4, & [e_3, e_4] &= 6e_4, & [e_6, e_{13}] &= 4be_{25},\\
[e_1, e_6] &= 2e_5, & [e_3, e_5] &= 4e_5, & [e_6, e_{14}] &= -8be_{24},\\
[e_1, e_7] &= 3e_6, & [e_3, e_6] &= 2e_6, & [e_6, e_{15}] &= 4be_{25},\\
[e_1, e_8] &= 4e_7, & [e_3, e_8] &= -2e_8, & [e_6, e_{19}] &= 3de_{23},\\
[e_1, e_9] &= 5e_8, & [e_3, e_9] &= -4e_9, & [e_6, e_{20}] &= 2de_{24},\\
[e_1, e_{10}] &= 6e_9, & [e_3, e_{10}] &= -6e_{10}, & [e_6, e_{21}] &= -5de_{25},
\end{aligned}$$

$$
\begin{array}{lll}
[e_1,e_{12}] = e_{11}, & [e_3,e_{11}] = 4e_{11}, & [e_7,e_8] = -3ae_{20}, \\
[e_1,e_{13}] = 2e_{12}, & [e_3,e_{12}] = 2e_{12}, & [e_7,e_9] = -3ae_{21}, \\
[e_1,e_{14}] = 3e_{13}, & [e_3,e_{14}] = -2e_{14}, & [e_7,e_{10}] = 3ae_{22}, \\
[e_1,e_{15}] = 4e_{14}, & [e_3,e_{15}] = -4e_{15}, & [e_7,e_{12}] = -3be_{23}, \\
[e_1,e_{17}] = e_{16}, & [e_3,e_{16}] = 6e_{16}, & [e_7,e_{13}] = 6be_{24}, \\
[e_1,e_{18}] = 2e_{17}, & [e_3,e_{17}] = 4e_{17}, & [e_7,e_{14}] = -3be_{25}, \\
[e_1,e_{19}] = 3e_{18}, & [e_3,e_{18}] = 2e_{18}, & [e_7,e_{18}] = -3de_{23}, \\
[e_1,e_{20}] = 4e_{19}, & [e_3,e_{20}] = -2e_{20}, & [e_7,e_{20}] = 3de_{25},
\end{array}
$$

$$
\begin{array}{lll}
[e_1,e_{21}] = 5e_{20}, & [e_3,e_{21}] = -4e_{21}, & [e_8,e_9] = -2ae_{22}, \\
[e_1,e_{22}] = 6e_{21}, & [e_3,e_{22}] = -6e_{22}, & [e_8,e_{11}] = 4be_{23}, \\
[e_1,e_{24}] = e_{23}, & [e_3,e_{23}] = 2e_{23}, & [e_8,e_{12}] = -8be_{24}, \\
[e_1,e_{25}] = 2e_{24}, & [e_3,e_{25}] = -2e_{25}, & [e_8,e_{13}] = 4be_{25}, \\
[e_2,e_3] = 2e_2, & [e_4,e_7] = 3ae_{16}, & [e_8,e_{17}] = 5de_{23}, \\
[e_2,e_4] = 6e_5, & [e_4,e_8] = 12ae_{17}, & [e_8,e_{18}] = -2de_{24}, \\
[e_2,e_5] = 5e_6, & [e_4,e_9] = 30ae_{18}, & [e_8,e_{19}] = -3de_{25}, \\
[e_2,e_6] = 4e_7, & [e_4,e_{10}] = 60ae_{19}, & [e_9,e_{11}] = 20be_{24},
\end{array}
$$

$$
\begin{array}{lll}
[e_2,e_7] = 3e_8, & [e_4,e_{15}] = 60be_{23}, & [e_9,e_{12}] = -10be_{25}, \\
[e_2,e_8] = 2e_9, & [e_4,e_{21}] = 15de_{23}, & [e_9,e_{16}] = -15de_{23}, \\
[e_2,e_9] = e_{10}, & [e_4,e_{22}] = 90de_{24}, & [e_9,e_{17}] = 10de_{24}, \\
[e_2,e_{11}] = 4e_{12}, & [e_5,e_6] = -2ae_{16}, & [e_9,e_{18}] = 5de_{25}, \\
[e_2,e_{12}] = 3e_{13}, & [e_5,e_7] = -3ae_{17}, & [e_{10},e_{11}] = 60be_{25}, \\
[e_2,e_{13}] = 2e_{14}, & [e_5,e_9] = 10ae_{19}, & [e_{10},e_{16}] = -90de_{24}, \\
[e_2,e_{14}] = e_{15}, & [e_5,e_{10}] = 30ae_{20}, & [e_{10},e_{17}] = -15de_{25}, \\
[e_2,e_{16}] = 6e_{17}, & [e_5,e_{14}] = -10be_{23}, & [e_{11},e_{14}] = 2ce_{23},
\end{array}
$$

$$
\begin{array}{lll}
[e_2,e_{17}] = 5e_{18}, & [e_5,e_{15}] = 20be_{24}, & [e_{11},e_{15}] = 8ce_{24}, \\
[e_2,e_{18}] = 4e_{19}, & [e_5,e_{20}] = -5de_{23}, & [e_{12},e_{13}] = -ce_{23}, \\
[e_2,e_{19}] = 3e_{20}, & [e_5,e_{21}] = -10de_{24}, & [e_{12},e_{14}] = -ce_{24}, \\
[e_2,e_{20}] = 2e_{21}, & [e_5,e_{22}] = 15de_{25}, & [e_{12},e_{15}] = 2ce_{25}, \\
[e_2,e_{21}] = e_{22}, & [e_6,e_7] = -3ae_{18}, & [e_{13},e_{14}] = -ce_{25}.
\end{array}
$$

It turns out that all Lie algebras $L(a,b,c,d)$ are isomorphic.

Proposition 4.2. *We have an isomorphism $L(a_1,b_1,c_1,d_1)\cong L(a_2,b_2,c_2,d_2)$ for all nonzero complex numbers $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2$.*

Proof. Let $\varphi\colon L(a_1,b_1,c_1,d_1)\to L(a_2,b_2,c_2,d_2)$ be the map given by $\varphi(e_i)=\xi_i e_i$ for all i with $1\le i\le 25$. A direct computation shows that φ is a Lie algebra homomorphism if and only if

$$\xi_1=\xi_2=\xi_3=1$$

$$\begin{aligned}\xi_4=\cdots=\xi_{10}&=\frac{a_1b_2^2c_1d_1}{a_2b_1^2c_2d_2},\\ \xi_{11}=\cdots=\xi_{15}&=\frac{a_1b_2^3c_1^2d_1}{a_2b_1^3c_2^2d_2},\\ \xi_{16}=\cdots=\xi_{22}&=\frac{a_1b_2^4c_1^2d_1^2}{a_2b_1^4c_2^2d_2^2},\\ \xi_{23}=\xi_{24}=\xi_{25}&=\frac{a_1^2b_2^6c_1^3d_1^2}{a_2^2b_1^6c_2^3d_2^2}.\end{aligned}$$

Obviously the determinant of the diagonal matrix associated to φ is nonzero. So the map is a Lie algebra isomorphism. $\square$

Hence we may choose the parameters as $(a_{15},b_{13},c_7,d_{16})=(3,60,2,15)$, namely by taking

$$(a,b,c,d)=(1,1,1,1).$$

Note that then all structure constants are integers. We call the Lie algebra

$$L_{25}:=L(1,1,1,1)$$

Benayadi's Lie algebra. It is uniquely determined up to isomorphism. Since we have explicit Lie brackets, the cohomology can be easily computed by using a computer algebra system like GAP. We already know that

$$H^1(L_{25},\mathbb{C})=H^0(L_{25},L_{25})=H^1(L_{25},L_{25})=0.$$

Our result is the following.

Theorem 4.3. *Benayadi's Lie algebra L_{25} satisfies* $\dim H^2(L_{25},L_{25})=1$.

We also can determine the highest degree adjoint cohomology of $\mathfrak{g}$ without a computation.

Proposition 4.4. *Benayadi's Lie algebra L_{25} satisfies $H^{25}(L_{25},L_{25})=0$.*

Proof. Since L_{25} is perfect, it is unimodular. Hence we obtain

$$H^{25}(L_{25},L_{25})\cong H_0(L_{25},L_{25})\cong L_{25}/[L_{25},L_{25}]=0$$

by Poincaré duality. $\square$

Note that the result also follows from Proposition 3.9.

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References

[1] E. Angelopoulos: *Algèbres de Lie $\mathfrak{g}$ satisfaisant $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$, $\mathrm{Der}(\mathfrak{g})=\mathrm{ad}(\mathfrak{g})$*. C. R. Acad. Sci. Paris, t. **306**, Série I (1988), 523–525.

[2] E. Angelopoulos, S. Benayadi: *Construction d'algèbres de Lie sympathiques non semi-simples munies de produits scalaires invariants*. C. R. Acad. Sci. Paris, t. **317**, Série I (1993), 741–744.

[3] D. Arnal, H. Benamor, S. Benayadi, G. Pinczon: *Une algèbre de Lie non semi-simple rigide et sympathique: $H^1(\mathfrak{g})=H^2(\mathfrak{g})=H^0(\mathfrak{g},\mathfrak{g})=H^1(\mathfrak{g},\mathfrak{g})=H^2(\mathfrak{g},\mathfrak{g})=0$*. C. R. Acad. Sci. Paris, t. **315**, Série I (1992), 261–263.

[4] S. Benayadi: *Certaines propriétés d'une classe d'algèbres de Lie qui généralisent les algèbres de Lie semi-simples*. Ann. Fac. Sci. Toulouse, 5^e série, t. **12**, 1 (1991), 29–35.

[5] S. Benayadi: *Structure of perfect Lie algebras without center and outer derivations*. Ann. Fac. Sci. Toulouse, 6^e série, t. **5**, 2 (1996), 203–231.

[6] D. Burde, K. Dekimpe: *Post-Lie algebra structures on pairs of Lie algebras*. Journal of Algebra, Vol. **464** (2016), 226–245.

[7] R. Carles: *Sur la structure des algèbres de Lie rigides*. Ann. Inst. Fourier Grenoble, t. **34**, 3 (1984), 65–82.

[8] R. Carles: *Sur certaines classes d'algèbres de Lie rigides*. Math. Ann. **272** (1985), 477–488.

[9] C. Chevalley, S. Eilenberg: *Cohomology theory of Lie groups and Lie algebras*. Trans. AMS **63** (1948), 85–124.

[10] J. Dixmier: *Cohomologie des algèbres de Lie nilpotentes*. Acta Sci. Math. Szeged **16** (1955), 246–250.

[11] J. Feldvoss, F. Wagemann: *On Leibniz cohomology*. J. Algebra **569** (2021), 276–317.

[12] W. Fulton, J. Harris: *Representation theory. A first course*. Graduate Texts in Mathematics, **129**. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp.

[13] G. Hochschild, J-P. Serre: *Cohomology of Lie algebras*. Ann. Math. (2) **57** (1953), no. 3, 591–603.

[14] N. Jacobson: *Lie algebras*. Dover Publications, Inc., New York, 1979 (unabridged and corrected republication of the original edition from 1962).

[15] D. Meng: *Some results on complete Lie algebras*. Communications in Algebra **22** (1994), no. 13, 5457–5507.

[16] A. L. Onishchik, E. B. Vinberg: *Lie groups and algebraic groups*. Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics, 1990, xx+328 pp.

[17] T. Pirashvili: *On Leibniz homology*. Ann. Inst. Fourier Grenoble **44** (1994), no. 2, 401–411.

[18] T. Pirashvili: *On strongly perfect Lie algebras*. Communications in Algebra **41** (2013), no. 5, 1619–1625.

[19] A. K. Tolpygo: *Cohomologies of parabolic Lie algebras*, Math. Notes **12** (1972), 585–587.

[20] P. Zusmanovich: *A converse to the second Whitehead lemma*. J. Lie Theory **18** (2008), no. 2, 295-299.

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