Lie algebras and representation theory

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Contents

Introduction

Lie algebras arise in many areas of mathematics and physics, such as differential geometry, Lie theory, number theory, combinatorics and quantum field theory, just to name a few of them. Originally Lie algebras were called "infinitesimal groups" of Lie groups. Herman Weyl introduced the name *Lie algebra* only in the nineteen twenties, at the suggestion of Nathan Jacobson. One can associate Lie algebras to Lie groups and algebraic groups. It is a vector space, the tangent space at the identity, together with a Lie bracket. The relationship between Lie groups and Lie algebras is very close and Sophus Lie proved three important theorems on this relationship. Many problems about Lie groups can be "linearized", i.e., formulated on the level of Lie algebras, which often makes them more accessible.

We also give an introduction in this lecture to representation theory of Lie algebras. We study highest weight modules for complex semisimple Lie algebras. The basic case here is the classification of all finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Let us give another basic example of a Lie algebra, namely $\mathfrak{so}_3(\mathbb{R})$, which can be viewed as the vector space $\mathbb{R}^3$ together with the cross product as Lie bracket. We have the following two identities

$$\begin{aligned} 0 &= v \times v, \\ 0 &= (u \times v) \times w + (v \times w) \times u + (w \times u) \times v. \end{aligned}$$

The Lie brackets for a basis (e_1, e_2, e_3) of $\mathfrak{so}_3(\mathbb{R})$, consisting of skew-symmetric 3×3-matrices are given by

$$[e_1, e_2] = e_3, \ [e_3, e_1] = e_2, \ [e_2, e_3] = e_1.$$

The representation theory of this algebra is also important in quantum mechanics, as the theory of angular momentum.

I want to present some historical information on the work of mathematicians which have worked on Lie algebras. A short list of these mathematicians is as follows:

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Elie Joseph Cartan (1869-1951)
Claude Chevalley (1909-1984)
Friedrich Engel (1861-1941)
Carl Gustav Jacob Jacobi (1804-1851)
Wilhelm Karl Joseph Killing (1847-1923)
Marius Sophus Lie (1842-1899)
Anatoly Ivanovich Malcev (1909-1967)
Hermann Klaus Hugo Weyl (1885-1955)
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• *Elie Cartan* was a French mathematician in Montpellier, Lyon, Nancy and Paris. He was a student of Marius Sophus Lie. In his dissertation of 1894 he completed Wilhelm Killings classification of finite-dimensional semisimple complex Lie algebras. He initiated among other things the theory of Riemannian symmetric spaces.

• *Claude Chevalley* was a French mathematician in Princeton, Columbia University and Paris. He proved fundamental results in the theory of algebraic groups and in algebraic geometry. He wrote a three-volume work on Lie groups.

• *Friedrich Engel* was a German mathematician in Leipzig, Greifswald and Gießen. He was a student of Marius Sophus Lie. Lie and Engel wrote in the eighteen nineties a three-volume work on transformation groups.

• *Carl Jacobi* was a German mathematician in Berlin and Königsberg. He made important contributions to number theory, determinants and partial differential equations. He invented the "Jacobi-identity" around 1830 in the context of Poisson brackets arsing in Hamiltonian mechanics.

• *Wilhelm Killing* was a German mathematician in Münster. he was a student of Weierstraß. He introduced Lie groups in the context of non-Euclidean geometry. He gave the first classification of finite-dimensional complex semisimple Lie algebras, although it was incomplete. After the death of his four sons he joined at the age of 39 with his spouse the Third Order of Franciscans in 1886.

• *Sophus Lie* was a Norwegian mathematician in Kristiania (Oslo) and Leipzig. He created the theory of continuous transformation groups, which later on became the concept of a Lie group. He applied this to the study of geometry and differential equations.

• *Anatoly Malcev* was a Russian mathematician in Moscow and Ivanovo. He studied among other things solvable groups, Lie groups, topological algebras and decision problems in algebra.

• *Hermann Weyl* was a German mathematician in Zürich, Göttingen and Princeton. His research has had major significance for theoretical physics as well as purely mathematical disciplines including number theory and Lie theory. He was one of the most influential mathematicians of the twentieth century, and an important member of the Institute for Advanced Study during its early years.

CHAPTER 1

Basic notions of Lie algebra theory

1.1. Definitions and examples

Let k be an arbitrary field and V be a k-vector space equipped with a k-bilinear product $x \cdot y$. For three elements $x, y, z \in V$ define the *associator* by

$$(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z).$$

DEFINITION 1.1.1. A *k-algebra* A is a k-vector space together with a k-bilinear map

$$A \times A \to A, \quad (x, y) \mapsto x \cdot y.$$

The algebra A is called *left-symmetric*, or a *pre-Lie algebra*, if falls

$$(x, y, z) = (y, x, z)$$

for all $x, y, z \in A$. It is called *associative* if

$$(x, y, z) = 0$$

for all $x, y, z \in A$.

A Lie algebra is a special type of a k-algebra, named after the mathematician Sophus Lie.

DEFINITION 1.1.2. A *Lie algebra* $\mathfrak{g}$ over k is a k-vector space together with a k-bilinear map, the so-called *Lie bracket*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y],$$

such that we have for all $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} 0 &= [x, x], \\ 0 &= [x, [y, z]] + [y, [z, x]] + [z, [x, y]]. \end{aligned}$$

The first identity is called *skew-symmetry*, because it implies the identity $[y, x] = -[x, y]$:

$$\begin{aligned} 0 &= [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] \\ &= [x, y] + [y, x]. \end{aligned}$$

However, conversely this identity implies $[x, x] = 0$ only for characteristic of k different from two. For $y = x$ one obtains $[x, x] = -[x, x]$, hence $2[x, x] = 0$ and $[x, x] = 0$. To include the case of characteristic $p = 2$ one defines skew-symmetry by $[x, x] = 0$.

The second identity is called *Jacobi identity*. It arises by "derivation" of the group structure of

Lie groups. Note that the Jacobi identity does not imply associativity in general (only if the Lie algebra is 2-step nilpotent).

For two subspaces $\mathfrak{h}, \mathfrak{k}$ of $\mathfrak{g}$ the space $[\mathfrak{h}, \mathfrak{k}]$ is defined as the subspace generated by all products $[h,k]$ with $h \in \mathfrak{h}$ and $k \in \mathfrak{k}$. Each element of $[\mathfrak{h}, \mathfrak{k}]$ is a sum

$$[h_1, k_1] + \cdots + [h_r, k_r]$$

with $h_i \in \mathfrak{h}, k_i \in \mathfrak{k}$. The multiplication of subspaces in a Lie algebra is commutative.

LEMMA 1.1.3. *For subspaces $\mathfrak{h}, \mathfrak{k}$ of $\mathfrak{g}$ we have $[\mathfrak{h}, \mathfrak{k}] = [\mathfrak{k}, \mathfrak{h}]$.*

PROOF. Let $h \in \mathfrak{h}$ and $k \in \mathfrak{k}$. Then $[h,k] = -[k,h] \in [\mathfrak{k}, \mathfrak{h}]$, hence $[\mathfrak{h}, \mathfrak{k}] \subseteq [\mathfrak{k}, \mathfrak{h}]$. Similarly we obtain $[\mathfrak{k}, \mathfrak{h}] \subseteq [\mathfrak{h}, \mathfrak{k}]$. □

A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is called *subalgebra* respectively *ideal*, if $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$ respectively if $[\mathfrak{g}, \mathfrak{a}] = [\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$. The commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ because of the Jacobi identity. Indeed, we have $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subseteq [\mathfrak{g}, \mathfrak{g}]$.

For every LSA A, hence also for every associative k-algebra the commutator defines a Lie bracket on the underlying vector space.

LEMMA 1.1.4. *Let A be an LSA with product $(x,y) \mapsto x \cdot y$. Then $(A, [\,,])$ is a Lie algebra with bracket $[x,y] = x \cdot y - y \cdot x$.*

PROOF. The claim follows directly from the following identity

$$[[a,b],c] + [[b,c],a] + [[c,a],b] =$$
$$(a,b,c) + (b,c,a) + (c,a,b) - (b,a,c) - (a,c,b) - (c,b,a),$$

which holds in any k-algebra. □

EXAMPLE 1.1.5. *Let $A = M_n(k)$ be the associative matrix algebra of $n \times n$-matrices over k. Then taking commutators we obtain the general linear Lie algebra $\mathfrak{gl}_n(k)$ of dimension n^2.*

The Lie bracket for elements $A, B \in M_n(k)$ is given by $[A,B] = AB - BA$. The commutator of $\mathfrak{gl}_n(k)$ has a "special" name.

LEMMA 1.1.6. *The commutator subalgebra of $\mathfrak{gl}_n(k)$ is the special linear Lie algebra*

$$\mathfrak{sl}_n(k) = \{X \in \mathfrak{gl}_n(k) \mid \operatorname{tr}(X) = 0\}$$

of dimension $n^2 - 1$.

PROOF. We have to show that $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] = \mathfrak{sl}_n(k)$. Let $A, B \in \mathfrak{gl}_n(k)$. Then $[A,B] \in \mathfrak{sl}_n(k)$ because of

$$\operatorname{tr}([A,B]) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.$$

Thus we have $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] \subseteq \mathfrak{sl}_n(k)$. Conversely consider the matrices E_{ij}, having the entry 1 at position (i,j), and zero entries otherwise. We have

$$[E_{jk}, E_{\ell m}] = \delta_{k\ell} E_{jm} - \delta_{jm} E_{\ell k}. \tag{1.1}$$

In particular, we have for $j \neq k, m$

$$\begin{aligned}[E_{jk}, E_{kj}] &= E_{jj} - E_{kk}, \\ [E_{jm}, E_{mk}] &= E_{jk}.\end{aligned}$$

These matrices generate the Lie algebra $\mathfrak{sl}_n(k)$ for $j \neq k$. Obviously they are in the subspace of commutators. Hence we have $\mathfrak{sl}_n(k) \subseteq [\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)]$. Finally, we easily see that the elements E_{ij} for $i \neq j$ and the elements $E_{ii} - E_{i+1,i+1}$ for $1 \leq i \leq n-1$ form a basis of $\mathfrak{sl}_n(k)$. It has $(n^2 - n) + (n-1) = n^2 - 1$ elements. $\square$

REMARK 1.1.7. The algebra $\mathfrak{sl}_n(k)$ is a Lie subalgebra of $\mathfrak{gl}_n(k)$, but not a subalgebra of $M_n(k)$ with the matrix product.

Indeed, for $A = B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(k)$ we have $AB = \mathrm{id} \notin \mathfrak{sl}_2(k)$.

EXAMPLE 1.1.8. *For a given matrix $J \in \mathfrak{gl}_m(k)$ the subspace*

$$\mathfrak{g}(J) = \{X \in \mathfrak{gl}_m(k) \mid JX + X^t J = 0\}$$

is a Lie subalgebra of $\mathfrak{gl}_m(k)$.

Indeed, for $X, Y \in \mathfrak{g}(J)$ we have

$$\begin{aligned}[X,Y]^t J + J[X,Y] &= Y^t X^t J - X^t Y^t J + JXY - JYX \\ &= -Y^t JX + X^t JY + JXY - JYX \\ &= JYX - JXY + JXY - JYX \\ &= 0.\end{aligned}$$

For $m = 2n$ and

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

we obtain the Lie algebra $\mathfrak{g}(J) = \mathfrak{sp}_{2n}(k)$, the *symplectic Lie algebra* of order n. So a block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ lies in $\mathfrak{sp}_{2n}(k)$ if and only if

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} = - \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

hence if

$$\begin{pmatrix} -C^t & A^t \\ -D^t & B^t \end{pmatrix} = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix},$$

hence if

$$C^t = C, \quad B^t = B, \quad D = -A^t.$$

Thus the dimension of the symplectic Lie algebra is

$$\dim(\mathfrak{sp}_{2n}(k)) = n(n+1) + n^2 = 2n^2 + n.$$

For $J = E_m$ we obtain the Lie algebra $\mathfrak{g}(J) = \mathfrak{so}_m(k)$, the *orthogonal Lie algebra* of order m. It consists of all skew-symmetric matrices

$$\mathfrak{so}_m(k) = \{X \in \mathfrak{gl}_m(k) \mid X + X^t = 0\}.$$

Clearly we have $\dim(\mathfrak{so}_m(k)) = \frac{m(m-1)}{2}$. Sometimes we also need another representation of the orthogonal Lie algebra, distinguishing the case $m = 2n$ even and the case $m = 2n+1$ odd. One chooses for J the first matrix for even m and the second matrix for odd m,

$$J = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & E_n \\ 0 & E_n & 0 \end{pmatrix}.$$

Then we obtain two series of orthogonal Lie algebra, namely $\mathfrak{so}_{2n}(k)$ and $\mathfrak{so}_{2n+1}(k)$. Their dimensions are $n(2n-1)$ respectively $n(2n+1)$. Of course, these two different representations give isomorphic Lie algebras. There is only "one" orthogonal Lie algebra for any given order m.

Example 1.1.9. *The space of upper-triangular matrices of size n over k forms a Lie subalgebra of $\mathfrak{gl}_n(k)$, which we denote by $\mathfrak{t}_n(k)$.*

Clearly the strictly upper-triangular matrices form a Lie subalgebra of $\mathfrak{t}_n(k)$, which we denote by $\mathfrak{n}_n(k)$. The Lie subalgebra of diagonal matrices in $\mathfrak{t}_n(k)$ is denoted by $\mathfrak{d}_n(k)$. We have the following equalities.

$$\begin{aligned} \mathfrak{d}_n(k) + \mathfrak{n}_n(k) &= \mathfrak{t}_n(k), \\ [\mathfrak{d}_n(k), \mathfrak{n}_n(k)] &= \mathfrak{n}_n(k), \\ [\mathfrak{t}_n(k), \mathfrak{t}_n(k)] &= \mathfrak{n}_n(k). \end{aligned}$$

The last equality follows from the first two equalities.

Let us explicitly consider the Lie algebra $\mathfrak{n}_3(k)$, the so-called 3-dimensional *Heisenberg* Lie algebra. A basis (X, Y, Z) is given by the strictly upper-triangular matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with Lie brackets determined by $[X, Y] = Z$.

1.2. Derivations and representations of Lie algebras

Definition 1.2.1. Let A be a k-algebra with product $(x, y) \mapsto x \cdot y$ and $\mathrm{End}(A)$ be the space of all vector space endomorphisms of A. A linear map $D \in \mathrm{End}(A)$ is called a *derivation* of A if

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$$

for all $x, y \in A$. The space of derivations of A is denoted by $\mathrm{Der}(A)$.

It is clear that $\mathrm{Der}(A)$ is a subspace of $\mathrm{End}(A)$ ist.

Example 1.2.2. *For the $\mathbb{R}$-algebra $A = C^\infty(\mathbb{R})$ of smooth functions the map $D \colon A \to A$ with $D(f) = f'$ is a derivation.*

This follows from the product rule (Leibniz rule).

EXAMPLE 1.2.3. *The derivation algebra of the pre-Lie algebra* $A = \mathbb{C}x \oplus \mathbb{C}y$ *with product*

$$\begin{aligned} x \cdot x &= 2x, \ x \cdot y = y \\ y \cdot x &= 0, \quad y \cdot y = x \end{aligned}$$

is trivial. We have $\operatorname{Der}(A) = 0$.

Note that A is a simple algebra, i.e., having no two-sided ideals except for 0 and A. The Lie algebra of A is given by $[x, y] = y$.

Let D be a derivation of A. Write $D(x) = \alpha_1 x + \alpha_2 y$ and $D(y) = \alpha_3 x + \alpha_4 y$. Then $D(x \cdot x) = D(x) \cdot x + x \cdot D(x)$ is equivalent to $2\alpha_1 x - \alpha_2 y = 0$, so $\alpha_1 = \alpha_2 = 0$ and $D(x) = 0$. In the same way we have

$$\begin{aligned} D(y \cdot y) - D(y) \cdot y - y \cdot D(y) &= D(x) - (\alpha_3 x + \alpha_4 y) \cdot y - y \cdot (\alpha_3 x + \alpha_4 y) \\ &= -2\alpha_4 x - \alpha_3 y \\ &= 0. \end{aligned}$$

This gives $D(y) = 0$ and D is the zero map.

Let us denote the Lie algebra of $\operatorname{End}(A)$ with commutator as Lie bracket, by $\mathfrak{gl}(A)$. Consider $A = k^n$ as vector space. Then we identify $\mathfrak{gl}(A)$ with $\mathfrak{gl}_n(k)$. The subspace $\operatorname{Der}(A)$ becomes a Lie algebra with commutator as Lie bracket, too.

LEMMA 1.2.4. *Let A be a k-algebra. Then* $\operatorname{Der}(A)$ *is a Lie subalgebra of* $\mathfrak{gl}(A)$.

PROOF. For given $D_1, D_2 \in \operatorname{Der}(A)$ we need to show that $[D_1, D_2] \in \operatorname{Der}(A)$. For $x, y \in A$ we have

$$\begin{aligned} [D_1, D_2](x \cdot y) &= (D_1 D_2)(x \cdot y) - (D_2 D_1)(x \cdot y) \\ &= D_1(D_2(x) \cdot y + x \cdot D_2(y)) - D_2(D_1(x) \cdot y + x \cdot D_1(y)) \\ &= (D_1 D_2(x)) \cdot y + x \cdot (D_1 D_2(y))) + D_2(x) \cdot D_1(y) + D_1(x) \cdot D_2(y) \\ &\quad - (D_2 D_1(x)) \cdot y - x \cdot (D_2 D_1(y)) - D_1(x) \cdot D_2(y) - D_2(x) \cdot D_1(y) \\ &= [D_1, D_2](x) \cdot y + x \cdot [D_1, D_2](y). \end{aligned}$$

In the penultimate line we have added and subtracted the terms $D_1(x) \cdot D_2(y)$ and $D_2(x) \cdot D_1(y)$. $\square$

Now we can chose A to be a Lie algebra $\mathfrak{g}$. This yields the derivation Lie algebra $\operatorname{Der}(\mathfrak{g})$.

EXAMPLE 1.2.5. *Let* $\mathfrak{r}_2(k) = kx \oplus ky$ *be the* 2*-dimensional Lie algebra given by the Lie bracket* $[x, y] = y$, *see Example* 1.2.3. *Then we have*

$$\operatorname{Der}(\mathfrak{r}_2(k)) = \left\{ \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix} \mid \alpha, \beta \in k \right\}.$$

To see this, we only need to check one non-trivial condition, namely for the basis elements x, y,

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

This yields the claim. The Lie algebra $\operatorname{Der}(\mathfrak{r}_2(k))$ is also 2-dimensional with basis D_1, D_2 corresponding to the choices $(\alpha, \beta) = (0, 1)$ and $(\alpha, \beta) = (1, 0))$, and $[D_1, D_2] = D_2$. So the Lie algebras $\operatorname{Der}(\mathfrak{r}_2(k))$ and $\mathfrak{r}_2(k)$ are isomorphic (can be rewritten).
There are several generalizations of derivations. For a particular one see [**4**]:

Definition 1.2.6. A k-linear map $P\colon \mathfrak{g} \to \mathfrak{g}$ is called a *prederivation* of $\mathfrak{g}$ if

$$P([x,[y,z]]) = [P(x),[y,z]] + [x,[P(y),z]] + [x,[y,P(z)]]$$

for all $x,y,z \in \mathfrak{g}$.

the space of prederivations of $\mathfrak{g}$ forms a Lie subalgebra $\mathrm{Pder}(\mathfrak{g})$ of $\mathfrak{gl}(\mathfrak{g})$ containing the Lie algebra $\mathrm{Der}(\mathfrak{g})$.

Lemma 1.2.7. *Let $\mathfrak{g}$ be a Lie algebra. Then* $\mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{Pder}(\mathfrak{g})$.

Proof. Let $D \in \mathrm{Der}(\mathfrak{g})$. Then we have

$$D([x,[y,z]]) = [x, D([y,z])] + [D(x),[y,z]]$$

Substituting $D([y,z]) = [D(y),z] + [y,D(z)]$ we obtain

$$D([x,[y,z]]) = [x,[D(y),z]] + [x,[y,D(z)]] + [D(x),[y,z]].$$

□

Example 1.2.8. *We have* $\mathrm{Pder}(\mathfrak{r}_2(k)) = \mathrm{Der}(\mathfrak{r}_2(k))$, *but*

$$\mathrm{Der}(\mathfrak{n}_3(k)) = \left\{ \begin{pmatrix} \alpha & \delta & 0 \\ \beta & \varepsilon & 0 \\ \gamma & \zeta & \alpha+\varepsilon \end{pmatrix} \mid \alpha,\beta,\gamma,\delta,\varepsilon,\zeta \in k \right\}.$$

which is different from $\mathrm{Pder}(\mathfrak{n}_3(k)) = \mathfrak{gl}_3(k)$.

Recall that the Lie bracket of $\mathfrak{n}_3(k)$ is given by $[x,y] = z$, so that we have $[[u,v],w] = 0$ for all $u,v,w \in \mathfrak{n}_3(k)$. Hence every term in the prederivation identity 1.2.6 is equal to zero. Hence every $P \in \mathfrak{gl}(k^3)$ is a prederivation. On the other hand, $\mathrm{Der}(\mathfrak{n}_3(k))$ consists of the linear maps $D = (a_{ij})_{1\le i,j\le 3}$ with $a_{13} = a_{23} = 0$ and $a_{33} = a_{11} + a_{22}$. We have $D(Z) \subseteq Z$, where $Z = \langle z \rangle$ is the center. Note that $\mathfrak{n}_3(k)$ has nonsingular derivations. In this respect, there are Lie algebras having only nilpotent derivations. They are called *characteristically nilpotent.* Here is an example. Consider a 7-dimensional Lie algebra $\mathfrak{g}$ with basis $(e_1,\ldots,e_7)$ and Lie brackets

$$\begin{aligned} [e_1,e_i] &= e_{i+1}, \quad 2 \le i \le 6, \\ [e_2,e_3] &= e_6 + e_7 \\ [e_2,e_4] &= e_7. \end{aligned}$$

Then all derivations in $\mathrm{Der}(\mathfrak{g})$ are nilpotent. However, there are even invertible prederivations, such as $P = \mathrm{diag}(1,3,3,5,5,7,7)$. Note that $\dim(\mathrm{Der}(\mathfrak{g})) = 11$ and $\dim(\mathrm{Pder}(\mathfrak{g})) = 16$.

Remark 1.2.9. Every prederivation of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over a field k of characteristic zero is a derivation, and hence an *inner derivation*, i.e., $\mathrm{Pder}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Let us now come to Lie algebra representations.

Definition 1.2.10. A *Lie algebra homomorphism* $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a linear map satisfying $\varphi([x,y]_{\mathfrak{g}}) = [\varphi(x),\varphi(y)]_{\mathfrak{h}}$ for all $x,y \in \mathfrak{g}$. It is called a *Lie algebra isomorphism* if φ is bijective.

Definition 1.2.11. A *representation* of a Lie algebra $\mathfrak{g}$ over k is a pair (V,ρ) consisting of a k-vector space V and a homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. If ρ is injective, the representation is called *faithful.*

REMARK 1.2.12. The term representation means that an abstract Lie algebra $\mathfrak{g}$ is expressed in terms of explicit matrices. If it is faithful, we realize $\mathfrak{g}$ this way as a Lie subalgebra of $\mathfrak{gl}_n(k)$. The famous Theorem of **Ado and Iwasawa** says that every finite-dimensional Lie algebra over a field k has a finite-dimensional *faithful* representation.

EXAMPLE 1.2.13. *The linear map* $\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ *given by* $\mathrm{ad}(x)(y) = [x, y]$ *defines a Lie algebra representation*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto \mathrm{ad}(x).$$

Indeed, we have

$$\begin{aligned}
[\mathrm{ad}(x), \mathrm{ad}(y)](z) - \mathrm{ad}([x, y])(z) &= [x, [y, z]] - [y, [x, z]] - [[x, y], z] \\
&= [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= 0.
\end{aligned}$$

The representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is called the *adjoint representation* of $\mathfrak{g}$. It also is a representation of $\mathfrak{g}$ into the subalgebra $\mathrm{Der}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$.

PROPOSITION 1.2.14. *The endomorphisms* $\mathrm{ad}(x)$ *are derivations of* $\mathfrak{g}$. *For* $D \in \mathrm{Der}(\mathfrak{g})$ *and* $x \in \mathfrak{g}$ *we have* $[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x))$. *Hence* $\mathrm{ad}(\mathfrak{g})$ *is a Lie ideal in* $\mathrm{Der}(\mathfrak{g})$*, called the ideal of inner derivations.*

PROOF. For $x, y, z \in \mathfrak{g}$ we have

$$\begin{aligned}
\mathrm{ad}(x)([y, z]) - [\mathrm{ad}(x)(y), z] - [y, \mathrm{ad}(x)(z)] &= [x, [y, z]] - [[x, y], z] - [y, [x, z]] \\
&= [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= 0.
\end{aligned}$$

Furthermore

$$\begin{aligned}
[D, \mathrm{ad}(x)](y) &= D([x, y]) - [x, D(y)] \\
&= [D(x), y] \\
&= \mathrm{ad}(D(x))(y).
\end{aligned}$$

□

The kernel of the homomorphism $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is called the *center* of $\mathfrak{g}$,

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \; \forall\, y \in \mathfrak{g}\}.$$

It coincides with $Z_{\mathfrak{g}}(\mathfrak{g})$ defined below.

DEFINITION 1.2.15. Let $A \subseteq \mathfrak{g}$ be a subset of a Lie algebra $\mathfrak{g}$. Then the *centralizer* of A in $\mathfrak{g}$ is defined by

$$Z_{\mathfrak{g}}(A) = \{x \in \mathfrak{g} \mid [x, y] = 0 \quad \forall\, y \in A\},$$

and the *normalizer* of A in $\mathfrak{g}$ is defined by

$$N_{\mathfrak{g}}(A) = \{x \in \mathfrak{g} \mid [x, y] \in A \quad \forall\, y \in A\}.$$

Both centralizer and normalizer are Lie subalgebras of $\mathfrak{g}$ and $Z_{\mathfrak{g}}(\mathfrak{g}) = Z(\mathfrak{g})$.

Let us define $\mathfrak{sl}_2(k) = kx \oplus ky \oplus kh$ abstractly (so not by explicit matrices) by the Lie brackets $[x, y] = h$, $[x, h] = -2x$ and $[y, h] = 2y$.

EXAMPLE 1.2.16. *The map*

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

defines a 2-dimensional representation of $\mathfrak{sl}_2(k)$.

Indeed, let $\varphi\colon \mathfrak{sl}_2(k) \to \mathfrak{gl}(k^2)$ be the linear map, defined on a basis as above. Then we have

$$\varphi([x,y]) = \varphi(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = [\varphi(x), \varphi(y)].$$

Similarly we see that $\varphi([x,h]) = [\varphi(x), \varphi(h)]$ and $\varphi([y,h]) = [\varphi(y), \varphi(h)]$.

EXAMPLE 1.2.17. *The adjoint representation of* $\mathfrak{sl}_2(k)$ *is the* 3*-dimensional representation given by*

$$\mathrm{ad}(x) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The identity map defines an n-dimensional representation of the Lie algebra $\mathfrak{gl}_n(k)$:

$$\mathrm{id}\colon \mathfrak{gl}_n(k) \to \mathfrak{gl}(k^n).$$

It is called the *natural representation*. This term is also applied to any subalgebra of $\mathfrak{g} \subset \mathfrak{gl}_n(k)$ see Example 1.2.16.

If (ρ, V) a representation of a Lie algebra $\mathfrak{g}$, then we obtain by $x.v = \rho(x)(v)$ a k-bilinear operation $\mathfrak{g} \times V \to V$, $(x,v) \mapsto x.v$ such that for all $x, y \in \mathfrak{g}$ and all $v \in V$,

$$[x,y].v = x.(y.v) - y.(x.v). \tag{1.2}$$

Then V together with this map is called a $\mathfrak{g}$*-module* M. We often identify the terms $\mathfrak{g}$-module M and representation ρ of $\mathfrak{g}$.

EXAMPLE 1.2.18. *The trivial action* $x.v = 0$ *for all* $x \in \mathfrak{g}$ *and* $v \in V$ *equips every* k*-vector space* V *with a* $\mathfrak{g}$*-module structure.*

The field k equipped with the trivial action is called the *trivial representation*. The zero space equipped with the trivial action is called the *zero representation* on $\mathfrak{g}$.

DEFINITION 1.2.19. A linear map $\varphi\colon V \to W$ between two representations, i.e., $\mathfrak{g}$-modules V and W of a Lie algebra $\mathfrak{g}$ is called a *homomorphism of representations* or $\mathfrak{g}$*-module homomorphism* if

$$\varphi(x.v) = x.\varphi(v) \quad \forall\, v \in V, x \in \mathfrak{g}.$$

Two representations are called *isomorphic*, if there is a homomorphism between them, which is an isomorphism of the underlying vector spaces. A subspace U of a representation V of $\mathfrak{g}$ is called a *subrepresentation* if $x.u \in U$ for all $x \in \mathfrak{g}, u \in U$.

EXAMPLE 1.2.20. *For a linear form* $\lambda \in \mathfrak{g}^*$ *on* $\mathfrak{g}$ *the map* $\rho_\lambda\colon \mathfrak{g} \to \mathfrak{gl}(k)$ *with* $x \mapsto \lambda(x)$ *is a representation if and only if* λ *vanishes on* $[\mathfrak{g}, \mathfrak{g}]$.

The linear forms of $\mathfrak{g}$ vanishing on $[\mathfrak{g}, \mathfrak{g}]$ are called *characters of* $\mathfrak{g}$. The 1-dimensional representation then is denoted by k_λ. The map $\lambda \to k_\lambda$ induces a bijection of $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ and the isomorphism classes of all 1-dimensional representations of $\mathfrak{g}$.

DEFINITION 1.2.21. A representation V of a Lie algebra $\mathfrak{g}$ is called *simple*, or *irreducible*, if it is nonzero and its only proper subrepresentation is the zero representation.

LEMMA 1.2.22. *If $\varphi\colon V \to W$ is a homomorphism of representations of a Lie algebra $\mathfrak{g}$, then $\ker(\varphi)$ is a subrepresentation of V and $\operatorname{im}(\varphi)$ is a subrepresentation of W.*

If $U \subset V$ is a subrepresentation of V, then there is exactly one representation of $\mathfrak{g}$ on the quotient space V/U such that $\pi\colon V \to V/U$, $v \mapsto v+U$ is a homomorphism of representations. We call V/U the *quotient representation*.

REMARK 1.2.23. The ideals of a Lie algebra $\mathfrak{g}$ are exactly the subrepresentations of the adjoint representation of $\mathfrak{g}$.

If U is a subrepresentation of V, then a subrepresentation $W \subset V$ is called a *complement* of U in V, if V is the direct vector space sum of U and W, i.e., if $V \cong U \oplus W$. The map $U \oplus W \to V$, $(u,w) \mapsto u+w$ then is an isomorphism of representations.

DEFINITION 1.2.24. A representation V of $\mathfrak{g}$ is called *semisimple*, if each subrepresentation has a complement.

A simple representation only has 0 and V as subrepresentations and hence is semisimple.

LEMMA 1.2.25. *Subrepresentations and quotient representations of semisimple representations are semisimple.*

PROOF. Let V be a semisimple representation and $W \subset V$ be a subrepresentation. Let us show first that W is semisimple. So let $U \subset W$ be a subrepresentation. We have to find a complement of U in W. Since V is semisimple there is a complement U' of U in V, so that $V = U \oplus U'$. But then $U' \cap W$ is a complement of U in W, because

$$U \cap (U' \cap W) \subset U \cap U' = \{0\},$$

and because of $W \subset U + U'$ and $U \subset W$ we also have $W \subset U + (U' \cap W)$. Hence we have $(U' \cap W) \oplus U = W$.
Secondly, we show that V/W is semisimple. Let $\pi\colon V \to V/W$ be the canonical projection and W' be a complement of W in V. Then $\pi\mid_{W'} : W' \to V/W$ is an isomorphism of representations. Since W' is semisimple by the first part, this is also true for V/W. $\square$

We can now state the following important result for semisimple representations (we always assume that the representations are finite-dimensional).

PROPOSITION 1.2.26. *For a representation V of a Lie algebra $\mathfrak{g}$ the following assertions are equivalent:*

(1) *V is semisimple.*
(2) *V is the sum of simple representations.*
(3) *V is the direct sum of simple representations.*

REMARK 1.2.27. The result enables us to reduce the classification of semisimple representations of a Lie algebra $\mathfrak{g}$ to the case of simple representations.

EXAMPLE 1.2.28. *Let $\mathfrak{g} = \mathbb{C}$ be the 1-dimensional Lie algebra with trivial Lie bracket and $V = \mathbb{C}^2$ be the representation of $\mathfrak{g}$ defined by $1.v = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) v$. Then V is not semisimple.*

Assume that V is semisimple. Then V is the direct sum of two 1-dimensional representations since V itself is not simple. Then there exists a basis in which the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ has block form, i.e., is of the form $\left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right)$. This is impossible since the matrix is not diagonalizable.

DEFINITION 1.2.29. For representations V and W of a Lie algebra $\mathfrak{g}$ denote by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ the space of homomorphisms $\varphi\colon V \to W$ of representations. Define $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{Hom}_{\mathfrak{g}}(V, V)$.

It is clear that $\mathrm{End}_{\mathfrak{g}}(V)$ is an associative subalgebra of $\mathrm{End}(V)$ ist. Denote by $\overline{k}$ an algebraic closure of k.

PROPOSITION 1.2.30 (Lemma of Schur). *Let V and W be simple representations of $\mathfrak{g}$. Then we have:*

(1) $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$*, if V and W are not isomorphic.*

(2) $\mathrm{End}_{\mathfrak{g}}(V)$ *is a division algebra, i.e., every nonzero element is invertible.*

(3) *If $\dim_k V < \infty$ and $k = \overline{k}$, then* $\mathrm{End}_{\mathfrak{g}}(V) = k \cdot \mathrm{id}$.

PROOF. For (1): Let $\varphi\colon V \to W$ be a homomorphism of $\mathfrak{g}$-representations. If $\varphi \neq 0$, then $\varphi(V) \subset W$ is a subrepresentation, different from the zero representation. Since W is simple, we obtain $\varphi(V) = W$. Similarly we see that $\ker(\varphi) = 0$. Hence φ is an isomorphism, i.e., $V \cong W$.

For (2): It is clear that $\mathrm{End}_{\mathfrak{g}}(V)$ is an algebra. Let $\varphi \in \mathrm{End}_{\mathfrak{g}}(V)$ be different from zero. As in (1) we see that φ is invertible.

For (3): Let $\varphi \in \mathrm{End}_{\mathfrak{g}}(V)$. Since k is algebraically closed, the characteristic polynomial $\chi_{\varphi}(t) = \det(\varphi - t\,\mathrm{id})$ has a root λ in k. Hence there is an eigenvector $v \neq 0$ to the eigenvalue λ. Since the eigenspace $V^{\lambda}(\varphi)$ of φ corresponding to λ is a subrepresentation of V, it follows that $V^{\lambda}(\varphi) = V$, since V is simple. Hence $\varphi = \lambda\,\mathrm{id}$ and $\mathrm{End}_{\mathfrak{g}}(V) = k \cdot \mathrm{id}$. □

COROLLARY 1.2.31. *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a simple representation of a Lie algebra $\mathfrak{g}$ over an algebraically closed field k. Then the only endomorphisms of V commuting with all $\rho(x)$, $x \in \mathfrak{g}$ are the scalars, i.e., the $\lambda \cdot \mathrm{id}$ with $\lambda \in k$.*

PROOF. Let $\varphi \in \mathrm{End}(V)$ and $x.v = \rho(x)(v)$. Then $\varphi \in \mathrm{End}_{\mathfrak{g}}(V)$ if $\varphi(x.v) = x.\varphi(v)$, see the definition of a $\mathfrak{g}$-module homomorphism. Using ρ this condition reads as $(\varphi \circ \rho(x))(v) = (\rho(x) \circ \varphi)(v)$. Hence φ commutes with all $\rho(x)$. By Schur's Lemma it follows that $\mathrm{End}_{\mathfrak{g}}(V) = k \cdot \mathrm{id}_V$. □

REMARK 1.2.32. Schur's Lemma need not be true if k is not algebraically closed. Indeed, consider the simple representation of $\mathfrak{g} = \mathbb{R}$ over $k = \mathbb{R}$ in the real vector space $V = \mathbb{C} \cong \mathbb{R}^2$, where $\lambda \in \mathfrak{g}$ acts on V by multiplication with λi. Then $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C} \cdot \mathrm{id} \neq \mathbb{R} \cdot \mathrm{id} = k \cdot \mathrm{id}$.

1.3. Semidirect sums of Lie algebras

Suppose that $\mathfrak{g}_j$, $j \in J$ is a family of Lie algebras. Then we can form the direct sum of the vector spaces $\mathfrak{g}_j$,

$$\mathfrak{g} = \bigoplus_{j \in J} \mathfrak{g}_j.$$

The elements of $\mathfrak{g}$ are denoted by (x_j). Then $[(x_j), (y_j)] = ([x_j, y_j])$ defines a Lie bracket for $\mathfrak{g}$. This Lie algebra is called the *direct sum* or *direct product* of the Lie algebras $\mathfrak{g}_j$. Here the term "product" refers to the underlying Cartesian product, which gives the name for the group case.

DEFINITION 1.3.1. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ be a subalgebra in $\mathfrak{g}$ and $\mathfrak{b}$ an *ideal* of $\mathfrak{g}$ such that we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ as vector spaces. Then $\mathfrak{g}$ is called the *inner semidirect sum (or product)* of $\mathfrak{a}$ and $\mathfrak{b}$. We write $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$.

Note that a semidirect sum $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ is direct if and only if both summands are ideals in $\mathfrak{g}$.

LEMMA 1.3.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras and $\varphi\colon \mathfrak{a} \to \operatorname{Der}(\mathfrak{b})$ be a Lie algebra homomorphism. Then we obtain a Lie bracket on $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ by*

$$[(x,a),(y,b)] = ([x,y],[a,b] + \varphi(x)(b) - \varphi(y)(a)). \tag{1.3}$$

This Lie algebra is denoted by $\mathfrak{g} = \mathfrak{a} \ltimes_\varphi \mathfrak{b}$.

PROOF. Clearly we have $[(x,a),(x,a)] = (0,0)$. Let

$$J(x,y,z) = [x,[y,z]] + [y,[z,x]] + [z,[x,y]]$$

for $x, y, z \in \mathfrak{a} \times \mathfrak{b}$. Note that $J(x,y,z) = J(y,z,x) = J(z,x,y)$. One may check that J is trilinear, so we may restrict ourselves for verifying $J \equiv 0$ to the following four cases:

$$\begin{aligned} x, y, z &\in \mathfrak{b}, \\ x, y \in \mathfrak{b}, z &\in \mathfrak{a}, \\ x \in \mathfrak{b}, y, z &\in \mathfrak{a}, \\ x, y, z &\in \mathfrak{a}. \end{aligned}$$

In the first and the last case we obtain $J \equiv 0$, because $\mathfrak{b}$ respectively $\mathfrak{a}$ is a Lie algebra. The other two cases follow from the facts that the images of φ are derivations of $\mathfrak{b}$ and that φ is a Lie algebra homomorphism. □

DEFINITION 1.3.3. The Lie algebra $\mathfrak{g} = \mathfrak{a} \ltimes_\varphi \mathfrak{b}$ is called the *outer semidirect sum* of $\mathfrak{a}$ and $\mathfrak{b}$.

Obviously $\mathfrak{a} \times 0 \cong \mathfrak{a}$ is a subalgebra in $\mathfrak{g}$, and $0 \times \mathfrak{b} \cong \mathfrak{b}$ is an ideal in $\mathfrak{g}$. Hence $\mathfrak{g}$ is also an *inner* direct sum of $\mathfrak{a}$ and $\mathfrak{b}$. Conversely the following holds.

PROPOSITION 1.3.4. *Let $\mathfrak{g}$ be an inner semidirect sum of $\mathfrak{a}$ and $\mathfrak{b}$, and φ as above. Then $\mathfrak{a} \ltimes_\varphi \mathfrak{b} \to \mathfrak{g}$, $(x,a) \to x + a$ is an isomorphism of Lie algebras.*

REMARK 1.3.5. A semidirect sum $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ corresponds to a split short exact sequence of Lie algebras,

$$0 \to \mathfrak{b} \to \mathfrak{g} \xrightarrow{\beta} \mathfrak{a} \to 0.$$

Here *split* means that there is a Lie algebra homomorphism $\tau\colon \mathfrak{a} \to \mathfrak{g}$ such that $\beta \circ \tau = \mathrm{id}_{|\mathfrak{a}}$.

Let $D\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$, where we consider V as an abelian Lie algebra. Then we have $\operatorname{Der}(V) = \mathfrak{gl}(V)$. Hence (1.3) shows the following.

EXAMPLE 1.3.6. *The semidirect product* $\mathfrak{g} \ltimes V$ *with abelian Lie algebra* V *becomes a Lie algebra by*

$$[(x,v),(y,w)] = ([x,y], D(x)(w) - D(y)(v)),$$

for $x, y \in \mathfrak{g}$ *and* $v, w \in V$.

For $\mathfrak{g} = \mathfrak{gl}(V)$ and $D = \mathrm{id}$ we obtain the Lie algebra

$$\mathfrak{aff}(V) := \mathfrak{gl}(V) \ltimes V$$

with Lie bracket $[(A,v),(B,w)] = ([A,B], Aw - Bv)$. Identifying V with k^n, we obtain that $\mathfrak{aff}(V)$ is isomorphic to the following subalgebra of $\mathfrak{gl}_{n+1}(k)$:

$$\mathfrak{aff}(V) \cong \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \mid A \in M_n(k),\ v \in k^n \right\}.$$

The Lie bracket here is given by the commutator of matrices,

$$\left[\begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & w \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [A,B] & Aw - Bv \\ 0 & 0 \end{pmatrix}$$

The algebra $\mathfrak{aff}(V)$ is the Lie algebra of the group $\mathrm{Aff}(V)$ of *affine transformations* $L_{A,v} \colon V \to V,\ x \mapsto Ax + v$.

1.4. Simple, semisimple and reductive Lie algebras

DEFINITION 1.4.1. A Lie algebra $\mathfrak{g}$ is called *simple*, if its adjoint representation is simple. It is called *reductive*, if its adjoint representation is semisimple.

Because of remark 1.2.23 we see that $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ has only the two ideals 0 and $\mathfrak{g}$, and the commutator $[\mathfrak{g}, \mathfrak{g}]$ is nonzero, so that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. As in group theory, Lie algebras coinciding with their commutator ideal are called *perfect*.

By definition $\mathfrak{g}$ is reductive if and only if for every ideal $\mathfrak{a}$ in $\mathfrak{g}$ there is a complementary ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}.$$

Note that a reductive Lie algebra need not be semisimple.

REMARK 1.4.2. A Lie algebra $\mathfrak{g}$ is simple if and only if it is nonabelian and every nonzero homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is injective.

In the theory of k-algebras, semisimple is equivalent to being a sum of simple algebras. We take this as a definition here.

DEFINITION 1.4.3. A Lie algebra $\mathfrak{g}$ is called *semisimple*, if it is a direct sum of simple Lie algebras.

Obviously every simple Lie algebra is semisimple. The converse is not true. Furthermore every semisimple Lie algebra is reductive.

LEMMA 1.4.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ is perfect, reductive and has a trivial center.*

PROOF. Let $\mathfrak{g}$ be semisimple. Then there exist simple Lie algebras $\mathfrak{g}_j$ with $\mathfrak{g} = \bigoplus_{j\in J} \mathfrak{g}_j$. Since all $\mathfrak{g}_j$ are perfect we obtain

$$[\mathfrak{g}, \mathfrak{g}] = \bigoplus_{j\in J} [\mathfrak{g}_j, \mathfrak{g}_j] = \bigoplus_{j\in J} \mathfrak{g}_j = \mathfrak{g}.$$

Now every $\mathfrak{g}_j$ is a simple ideal in $\mathfrak{g}$, and therefore a simple subrepresentation of the adjoint representation of $\mathfrak{g}$. Thus the adjoint representation of $\mathfrak{g}$ is semisimple by Proposition 1.2.26, and hence $\mathfrak{g}$ is reductive.

Since $Z(\mathfrak{g}_j)$ is an ideal in $\mathfrak{g}_j$, it is zero or $\mathfrak{g}_j$. The latter is impossible since $\mathfrak{g}_j$ is nonabelian by assumption. Hence we have $Z(\mathfrak{g}_j) = 0$ and

$$Z(\mathfrak{g}) = \bigoplus_{j\in J} Z(\mathfrak{g}_j) = 0.$$

$\square$

We can say more on the structure of reductive Lie algebras.

PROPOSITION 1.4.5. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then the following holds.*

1. *For any ideal $\mathfrak{a}$ in $\mathfrak{g}$, both $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are reductive.*
2. *We have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.*
3. *$\mathfrak{g}$ is semisimple if and only if $Z(\mathfrak{g}) = 0$.*

PROOF. For (1): by assumption there is an ideal $\mathfrak{b}$ in $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. In particular we have $[\mathfrak{a}, \mathfrak{b}] = 0$, so that every ideal of $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. The ideals of $\mathfrak{a}$ are hence just the subrepresentations of the adjoint representation of $\mathfrak{a}$, which is semisimple by Lemma 1.2.25. Hence there exists to every ideal in $\mathfrak{a}$ a complementary ideal in $\mathfrak{a}$, so that $\mathfrak{a}$ is reductive. Because of $\mathfrak{g}/\mathfrak{a} \cong \mathfrak{b}$ we see that $\mathfrak{g}/\mathfrak{a}$ is reductive by the above argument applied to the ideal $\mathfrak{b}$.

For (2): Since the adjoint representation of $\mathfrak{g}$ is semisimple, the subrepresentation $[\mathfrak{g}, \mathfrak{g}]$ has a complement W and we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus W$. Since $[\mathfrak{g}, W] \subset [\mathfrak{g}, \mathfrak{g}] \cap W = 0$ we have $W \subset Z(\mathfrak{g})$, so $[\mathfrak{g}, \mathfrak{g}] + Z(\mathfrak{g}) = \mathfrak{g}$. The sum is in fact direct: the subrepresentation $[\mathfrak{g}, \mathfrak{g}]$ is semisimple, so that there exists a complement U of $[\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g})$. So we have $U \oplus ([\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g})) = [\mathfrak{g}, \mathfrak{g}]$. On the other hand we have

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}] + Z(\mathfrak{g})] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, U] \subset U,$$

hence $[\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g}) = 0$. Finally, we need to show that the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. The subrepresentation $[\mathfrak{g}, \mathfrak{g}]$ of the adjoint representation of $\mathfrak{g}$ is semisimple, hence it is a direct sum of simple representations $\mathfrak{g}_j$, $j \in J$, which are also ideals in $\mathfrak{g}$. We have $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i, j \in J$. So we see that the ideals $\mathfrak{g}_j$ are simple. Hence their direct sum is semisimple.

For (3): Assume that $Z(\mathfrak{g}) = 0$. Then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple by (2). Conversely assume that $\mathfrak{g}$ is semisimple. Then $Z(\mathfrak{g}) = 0$ by Lemma 1.4.4. □

We want to show that the Lie algebra $\mathfrak{sl}_2(k)$ is simple, except for characteristic 2, where it is the Heisenberg Lie algebra, without using any structure theory.

PROPOSITION 1.4.6. *Let k be a field of characteristic different from two. Then the Lie algebra $\mathfrak{sl}_2(k)$ is simple.*

PROOF. Let $\mathfrak{a}$ be a nonzero ideal in $\mathfrak{sl}_2(k)$ and $w \in \mathfrak{a}$ with $w \neq 0$. We can write $w = \alpha x + \beta y + \gamma h$ in the basis (x, y, h) of $\mathfrak{sl}_2(k)$ with $[x, y] = h$, $[x, h] = -2x$ and $[y, h] = 2y$. We need to show that $\mathfrak{a} = \mathfrak{sl}_2(k)$. We have

$$\begin{aligned} [x, [x, w]] &= [x, \beta h - 2\gamma x] = -2\beta x \in \mathfrak{a}, \\ [y, [y, w]] &= -2\alpha y \in \mathfrak{a}. \end{aligned}$$

For α nonzero we have $y \in \mathfrak{a}$, and then $h = [x, y] \in \mathfrak{a}$, $\alpha x = w - \beta y - \gamma h \in \mathfrak{a}$, hence $\mathfrak{a} = \mathfrak{sl}_2(k)$. For β nonzero we obtain $\mathfrak{a} = \mathfrak{sl}_2(k)$ the same way. For $\alpha = \beta = 0$ we have $w = \gamma h \in \mathfrak{a}$ with $\gamma \neq 0$, hence $h \in \mathfrak{a}$. Then we have $2x = [h, x] \in \mathfrak{a}$ and $2y = [y, h] \in \mathfrak{a}$, so that again $\mathfrak{a} = \mathfrak{sl}_2(k)$. □

What about the Lie algebra $\mathfrak{gl}_2(k)$, in characteristic zero? We have $\mathfrak{gl}_2(k) = \mathfrak{sl}_2(k) \oplus k$, so that it is reductive. These result do not only hold for $n = 2$, but in general. Let us state the following result here without proof.

PROPOSITION 1.4.7. *Let k be a field of characteristic zero. Then the Lie algebra $\mathfrak{gl}_n(k)$ is reductive and its commutator subalgebra $\mathfrak{sl}_n(k)$ is simple.*

For a proof see [**10**] and Corollary 2.2.15.

1.5. Classification of simple representations of $\mathfrak{sl}_2(\mathbb{C})$

We already know simple representations of $\mathfrak{sl}_2(\mathbb{C})$ in low dimensions, namely the trivial representation in dimension 1, the natural representation in dimension 2 and the adjoint representation in dimension 3. We will prove now that there is up to isomorphism a *unique* simple representation in any dimension $n \geq 1$. Let $k[X] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables and let ∂_i be the partial derivative with respect to the variable x_i.

Lemma 1.5.1. *The linear map*

$$\begin{aligned} \rho\colon \mathfrak{gl}_n(k) &\to \mathfrak{gl}(k[X]), \\ E_{ij} &\mapsto x_i\partial_j \end{aligned}$$

is a representation of $\mathfrak{gl}_n(k)$ on the polynomial ring $k[X]$.

Proof. For all polynomials $p \in k[X]$ we have the formula

$$x_i\partial_j x_k\partial_\ell(p) = \delta_{jk}x_i\partial_\ell(p) + x_ix_k\partial_j\partial_\ell(p).$$

This yields

$$\begin{aligned} [\rho(E_{ij}), \rho(E_{k\ell})] &= [x_i\partial_j, x_k\partial_\ell] \\ &= \delta_{jk}x_i\partial_\ell - \delta_{\ell i}x_k\partial_j \\ &= \rho(\delta_{jk}E_{i\ell} - \delta_{\ell i}E_{kj}) \\ &= \rho([E_{ij}, E_{k\ell}]). \end{aligned}$$

□

Theorem 1.5.2. *In every dimension $n \geq 1$ there is up to isomorphism exactly one simple representation of $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. First we construct for every dimension n a simple representation of $\mathfrak{sl}_2(\mathbb{C})$. Then we show that every two simple representations of dimension n are isomorphic. Let (x, y, h) be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. By Lemma 1.5.1 we obtain a representation $\rho\colon \mathfrak{sl}_2(k) \to \mathfrak{gl}(k[X, Y])$ by

$$\begin{aligned} \rho(x) &= X\partial_Y, \\ \rho(y) &= Y\partial_X, \\ \rho(h) &= X\partial_X - Y\partial_Y. \end{aligned}$$

However, this representation is neither finite-dimensional nor simple. But we can consider the subrepresentation $V(m)$ formed by the polynomials of fixed total degree m, given by

$$V(m) = k[X, Y]^m \subset k[X, Y]$$

This representation has dimension $m + 1$ with basis $v_i = Y^iX^{m-i}$ for $i = 0, 1, \ldots m$. With respect to this basis the action of $\mathfrak{sl}_2(k)$ on $V(m)$ is given as follows:

$$\begin{aligned} x.v_i &= iv_{i-1}, \\ y.v_i &= (m - i)v_{i+1}, \\ h.v_i &= (m - 2i)v_i. \end{aligned}$$

Here we set $v_{-1} = v_{m+1} = 0$. The representing matrices are given by

$$\rho(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & m-1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ m & 0 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 2 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and

$$\rho(h) = \begin{pmatrix} m & 0 & \cdots & 0 & 0 \\ 0 & m-2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 2-m & 0 \\ 0 & 0 & \cdots & 0 & -m \end{pmatrix}$$

These representations are *simple*. Indeed, every nonzero subrepresentation $U \subset V(m)$ contains an eigenvector for h, hence one of the v_i, since we have $h.U \subset U$ by definition. The formulas then immediately imply that $U = V(m)$. First we have $v_0 \in U$ by repeated action of x, then $v_1, v_2, \ldots v_m$ by action of y. So we have found a simple representation in each dimension $m \geq 1$.

Now we have to show that any two simple representations of dimension m are isomorphic. Let $\rho\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be an arbitrary representation of dimension m and

$$V_\mu = \ker(\rho(h) - \mu \,\mathrm{id})$$

be the eigenspace of $\rho(h)$ corresponding to the eigenvalue $\mu \in \mathbb{C}$. We have

$$\begin{aligned} h.(x.v) - x.(h.v) &= [h,x].v = 2xv, \\ h.(x.v) &= x.(h+2).v, \end{aligned}$$

and hence $x.V_\mu \subset V_{\mu+2}$. In the same way we see that $y.V_\mu \subset V_{\mu-2}$ because of $h.(y.v) = y.(h-2).v$.
Since V is finite-dimensional and nonzero there is a $\lambda \in \mathbb{C}$ with $V_\lambda \neq 0$, but $V_{\lambda+2} = 0$. For $v \in V_\lambda$ we obtain $x.v = 0$ and $h.v = \lambda v$. By induction we verify the following identities, with $y.(y.v) = y^2.v$ and so on for all $n \geq 1$:

$$\begin{aligned} h.(y^n.v) &= (\lambda - 2n) y^n.v, \\ x.(y^n.v) &= n(\lambda - n + 1) y^{n-1}.v. \end{aligned}$$

Thus the subspace generated by all $y^n.v$ with $n \geq 0$ is a subrepresentation. Assuming now that V is simple and $v \neq 0$, we see that the $y^n.v$ span all of V. If $y^n.v \neq 0$ then all elements $v, y.v, \ldots, y^n.v$ are eigenvectors of h to pairwise distinct eigenvalues, and hence linearly independent. Because of $\dim V < \infty$ there exists a $d \geq 1$ such that $y^d.v = 0$. Choosing this d minimal, the set of vectors $(v, y.v, \ldots, y^{d-1}.v)$ is a basis of V and we have $\dim V = d$. Now $y^d.v = 0$ implies that

$$0 = x.(y^d.v) = d(\lambda - d + 1) y^{d-1}.v,$$

and therefore $\lambda = d - 1$, since we had assumed that $d \neq 0$ and $y^{d-1}.v \neq 0$. For every simple representation ρ of $\mathfrak{sl}_2(\mathbb{C})$ the matrices for $\rho(x), \rho(y)$ and $\rho(h)$ in the basis $(v, y.v, \ldots, y^{d-1}.v)$ only depend on d. Hence any two of them in the same dimension are isomorphic. $\square$

Every simple representation $V(m)$ of dimension $m+1$ of $\mathfrak{sl}_2(\mathbb{C})$ decomposes under h in 1-dimensional eigenspaces to the eigenvalues $m, m-2, \ldots, 2-m, -m$. We write

$$V = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{2-m} \oplus V_{-m}.$$

REMARK 1.5.3. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the complexified Lie algebra of the rotation group $SO_3(\mathbb{R})$ and its universal cover, the spin group S^3, i.e.,

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie}(SO_3(\mathbb{R})) \cong \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie}(S^3).$$

It follows from Theorem 1.5.2 that the dimension induces a bijection between the simple finite-dimensional continuous complex representations of S^3 up to isomorphism, and the set of positive integers. In fact, the integers corresponding to $SO_3(\mathbb{R})$ are $\{1, 3, 5, 7, \ldots\}$.

1.6. Abelian, nilpotent and solvable Lie algebras

Let $\mathfrak{g}$ be a nonzero Lie algebra if not said otherwise.

DEFINITION 1.6.1. A Lie algebra $\mathfrak{g}$ over k is called *abelian*, if $[\mathfrak{g}, \mathfrak{g}] = 0$.

An abelian Lie algebras has a trivial Lie bracket. So it is just a k-vector space like k^n. Note that an abelian Lie algebra $\mathfrak{g} \neq 0$ is not semisimple by definition, but only reductive. Let us inductively define two sequences of ideals of $\mathfrak{g}$:

- The *descending central series* $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$;
- The *derived series* $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$.

We have $\mathfrak{g}^i \subset \mathfrak{g}^{i-1}$ and $\mathfrak{g}^{(i)} \subset \mathfrak{g}^{(i-1)}$. The next lemma shows that indeed all subspaces $\mathfrak{g}^i$ and $\mathfrak{g}^{(i)}$ are Lie ideals in $\mathfrak{g}$.

LEMMA 1.6.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $\mathfrak{g}$. Then also $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]$ are ideals in $\mathfrak{g}$.*

PROOF. The first two claims are obvious, and the third one is implied by the Jacobi identity, which we have already shown earlier in the special case of $\mathfrak{g} = \mathfrak{a} = \mathfrak{b}$ for the commutator ideal. So we have

$$\begin{aligned}
[\mathfrak{g}, [\mathfrak{a}, \mathfrak{b}]] &\subset [[\mathfrak{g}, \mathfrak{a}], \mathfrak{b}] + [\mathfrak{a}, [\mathfrak{g}, \mathfrak{b}]] \\
&\subset [\mathfrak{a}, \mathfrak{b}] + [\mathfrak{a}, \mathfrak{b}] \\
&\subset [\mathfrak{a}, \mathfrak{b}].
\end{aligned}$$

Here we used that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ and $[\mathfrak{g}, \mathfrak{b}] \subset \mathfrak{b}$. □

DEFINITION 1.6.3. A Lie algebra $\mathfrak{g} \neq 0$ is called k-step *nilpotent*, if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$. It is called k-step *solvable* if $\mathfrak{g}^{(k)} = 0$ and $\mathfrak{g}^{(k-1)} \neq 0$.

We may consider the zero Lie algebra as being nilpotent and solvable, too. Abelian Lie algebras are solvable and nilpotent of step or class 1. Because of $\mathfrak{g}^{(i)} \subset \mathfrak{g}^i$ every nilpotent Lie algebra is solvable. If $\mathfrak{g}$ is solvable of class k, then we obtain an identity of iterated Lie brackets with 2^k elements. For, say, $k = 3$ the condition $\mathfrak{g}^{(3)} = 0$ reads as

$$[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]]] = 0$$

for all $x_i \in \mathfrak{g}$.

EXAMPLE 1.6.4. *The Heisenberg Lie algebra $\mathfrak{n}_3(k)$ is 2-step nilpotent.*

Note that the center of $\mathfrak{n}_3(k)$ is 1-dimensional and hence abelian. So $\mathfrak{n}_3(k)$ is 2-step solvable and 2-step nilpotent. This explains again, why $\mathfrak{sl}_2(k) = \mathfrak{n}_3(k)$ is nilpotent in characteristic two.

EXAMPLE 1.6.5. *The Lie algebra $\mathfrak{t}_n(k)$ of upper-triangular matrices is solvable and its commutator algebra $\mathfrak{n}_n(k)$ is nilpotent.*

LEMMA 1.6.6. *For $r, s \in \mathbb{N}$ we have $[\mathfrak{g}^r, \mathfrak{g}^s] \subset \mathfrak{g}^{r+s+1}$.*

PROOF. We show this by induction on $r \geq 0$. For $r = 0$ the claim is true by definition. The step $r \mapsto r+1$ goes as follows:

$$\begin{aligned}
[\mathfrak{g}^{r+1}, \mathfrak{g}^s] &= [[\mathfrak{g}, \mathfrak{g}^r], \mathfrak{g}^s] \\
&\subset [[\mathfrak{g}, \mathfrak{g}^s], \mathfrak{g}^r] + [\mathfrak{g}, [\mathfrak{g}^s, \mathfrak{g}^r]] \\
&\subset [\mathfrak{g}^r, \mathfrak{g}^{s+1}] + [\mathfrak{g}, \mathfrak{g}^{r+s+1}] \\
&\subset \mathfrak{g}^{r+s+2} + \mathfrak{g}^{r+s+2} \\
&\subset \mathfrak{g}^{r+s+2}.
\end{aligned}$$

□

REMARK 1.6.7. A Lie algebra $\mathfrak{g}$ is called *residually nilpotent*, if

$$\bigcap_{n \in \mathbb{N}} \mathfrak{g}^n = 0.$$

If $\mathfrak{g}$ is finite-dimensional, then this is equivalent to being nilpotent. In general however, a residually nilpotent Lie algebra need not be nilpotent.

PROPOSITION 1.6.8. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then we have the following statements:*

(1) *For any ideal $\mathfrak{a} \neq 0$ in $\mathfrak{g}$ we have $\mathfrak{a} \cap Z(\mathfrak{g}) \neq 0$. In particular we have $Z(\mathfrak{g}) \neq 0$.*
(2) *Every subalgebra and every homomorphic image of $\mathfrak{g}$ is nilpotent.*
(3) *Given a short exact sequence of Lie algebras*

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

where both $\mathfrak{a}$ and $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}$ are nilpotent, and with $\mathfrak{a} \subset Z(\mathfrak{h})$, then it follows that also $\mathfrak{h}$ is nilpotent.

PROOF. For (1): Since $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ we have that $\mathfrak{g}$ acts on $\mathfrak{a}$ by the adjoint representation. By Lemma 1.6.13 there is a $v \neq 0$ in $\mathfrak{a}$ such that $0 = \mathfrak{g}.v = [\mathfrak{g}, v]$, so with $v \in \mathfrak{a} \cap Z(\mathfrak{g})$. This shows the claim. On the other hand, there is also a direct proof. If $\mathfrak{g}$ is nilpotent of class k then $\mathfrak{g}^{k-1} \neq 0$ and $\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}] = 0$, hence $\mathfrak{g}^{k-1} \subset Z(\mathfrak{g})$.

For (2): If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ then we have $\mathfrak{a}^n \subset \mathfrak{g}^n$ for all $n \geq 0$. Hence also $\mathfrak{a}$ is nilpotent. If $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ is a surjective Lie algebra Homomorphism then $\varphi(\mathfrak{g}^n) = \mathfrak{h}^n$. Hence also $\mathfrak{h}$ is nilpotent.

For (3): Let $\pi \colon \mathfrak{h} \to \mathfrak{h}/\mathfrak{a}$ be the quotient map. Since $\mathfrak{h}/\mathfrak{a}$ is nilpotent, there exists a $n \geq 1$ with $(\mathfrak{h}/\mathfrak{a})^n = 0$. Because of (2) we then have $\pi(\mathfrak{h}^n) = (\mathfrak{h}/\mathfrak{a})^n = 0$. Hence $\mathfrak{h}^n \subset \mathfrak{a} \subset Z(\mathfrak{h})$, and thus $\mathfrak{h}^{n+1} \subset [\mathfrak{h}, Z(\mathfrak{h})] = 0$. □

There is an important remark for (3). If $\mathfrak{a}$ and $\mathfrak{h}/\mathfrak{a}$ are nilpotent then $\mathfrak{h}$ need not be nilpotent in general. So we cannot abandon the condition that $\mathfrak{a} \subset Z(\mathfrak{h})$. Nilpotency is not an extension property in general. We demonstrate this with an example.

EXAMPLE 1.6.9. *Let* $\mathfrak{h} = \mathfrak{r}_2(k)$ *with* $[x, y] = y$, *and* $\mathfrak{a} = ky$ *be an ideal in* $\mathfrak{h}$. *Then* $\mathfrak{a}$ *and* $\mathfrak{h}/\mathfrak{a}$ *are nilpotent, but* $\mathfrak{h}$ *is not nilpotent.*

Indeed, $\mathfrak{a}$ and $\mathfrak{h}/\mathfrak{a}$ are 1-dimensional and hence abelian and nilpotent. On the other hand we have $\mathfrak{h}^n = ky$ for all $n \geq 1$. So $\mathfrak{h}$ is not nilpotent.

LEMMA 1.6.10. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be nilpotent ideals in* $\mathfrak{g}$. *Then the ideal* $\mathfrak{a} + \mathfrak{b}$ *is nilpotent.*

PROOF. We show that

$$(\mathfrak{a} + \mathfrak{b})^{2m} \subset \mathfrak{a}^m + \mathfrak{b}^m$$

for all $m \geq 0$. Then the claim follows by choosing m so big that $\mathfrak{a}^m = \mathfrak{b}^m = 0$. The case $m = 0$ is clear. Let

$$y := [x_1, [x_2, [x_3, \dots, [x_{2m}, x_{2m+1}] \cdots]]] \in (\mathfrak{a} + \mathfrak{b})^{2m},$$

where we may assume that $x_j \in \mathfrak{a} \cup \mathfrak{b}$. If at least $m + 1$ of the x_j are in $\mathfrak{a}$, we have $y \in \mathfrak{a}^m$. If this is not the case then there are at at least $m + 1$ of the x_j in $\mathfrak{b}$, and hence $y \in \mathfrak{b}^m$. Since we can write every element of $(\mathfrak{a} + \mathfrak{b})^{2m}$ as a sum of elements of the form of y , we are done. □

The lemma shows that there exists a *maximal* nilpotent ideal in a finite-dimensional Lie algebra. Indeed, if $\mathfrak{n}$ is a nilpotent ideal of maximal dimension in $\mathfrak{g}$ and $\mathfrak{a}$ an arbitrary nilpotent ideal in $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{n}$ is again a nilpotent ideal. Then $\mathfrak{n} = \mathfrak{n} + \mathfrak{a}$ by dimension reasons. Hence $\mathfrak{a} \subset \mathfrak{n}$, so that $\mathfrak{n}$ contains every nilpotent ideal. So it is maximal. Also, it is uniquely determined. So the following definition makes sense.

DEFINITION 1.6.11. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the maximal nilpotent ideal in $\mathfrak{g}$ is called the *nilradical* of $\mathfrak{g}$, and is denoted by $\mathrm{nil}(\mathfrak{g})$.

Now we want to come to Engel's Theorem. We'll need the following lemma.

LEMMA 1.6.12. *Let* V *be a* $\mathfrak{g}$*-module and* $\mathfrak{a}$ *be an ideal in* $\mathfrak{g}$. *Then*

$$V^{\mathfrak{a}} = \{v \in V \mid \mathfrak{a}.v = 0\}$$

is a submodule of V.

PROOF. Let $w \in V^{\mathfrak{a}}$, $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$. Then $[y, x] \in \mathfrak{a}$ and

$$y.(x.w) = [y, x].w + x.(y.w) = 0.$$

Hence we have $x.w \in V^{\mathfrak{a}}$. □

LEMMA 1.6.13. *Let* V *be a nonzero vector space over a field* k *and* $\mathfrak{g} \leq \mathfrak{gl}(V)$ *be a finite-dimensional subalgebra such that every element in* $\mathfrak{g}$ *is a nilpotent endomorphism of* V. *Then there is a* $v \in V$, $v \neq 0$, *with* $\mathfrak{g}.v = 0$.

PROOF. Let $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphismus. Then also $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{gl}(V))$ is nilpotent. Indeed, $\mathrm{ad}(x)^n(y)$ is, for all $y \in \mathfrak{gl}(V)$, a linear combination of terms of the form $x^i y x^{n-i}$. Hence $x^n = 0$ implies that $\mathrm{ad}(x)^{2n} = 0$. More generally we have, for x, y in an associative algebra,

$$(\mathrm{ad}(x))^n(y) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} x^i y x^{n-i}.$$

We show the lemma by induction on $\dim \mathfrak{g}$. The case $\dim \mathfrak{g} = 0$ is clear. We may assume that the claim is true for all Lie algebras $\mathfrak{h}$ with $\dim \mathfrak{h} < \dim \mathfrak{g}$.

Claim 1: For every proper subalgebra $\mathfrak{h}$ *of* $\mathfrak{g}$ *the normalizer* $N_{\mathfrak{g}}(\mathfrak{h})$ *is strictly larger than* $\mathfrak{h}$. To see this, consider the canonical homomorphism $\rho\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ with $\rho(x)(y+\mathfrak{h}) = [x,y]+\mathfrak{h}$. This turns the quotient space $\mathfrak{g}/\mathfrak{h}$ into an $\mathfrak{h}$-module. Since $x \in \mathfrak{h}$ is nilpotent, so is $\operatorname{ad}(x)$. Hence for every $x \in \mathfrak{h}$ there exists a $n \geq 1$ such that $\operatorname{ad}(x)^n = 0$, hence also with $\rho(x)^n = 0$. Because of $\dim \mathfrak{h} < \dim \mathfrak{g}$ we can apply the induction hypothesis to the Lie subalgebra $\rho(\mathfrak{h})$ of $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$. So it exists a $\overline{v} \in \mathfrak{g}/\mathfrak{h}$, $\overline{v} \neq \overline{0}$ with $(\mathfrak{g}/\mathfrak{h}).\overline{v} = 0$. Hence there is a $v \in \mathfrak{g} \setminus \mathfrak{h}$ with $\rho(\mathfrak{h})(v+\mathfrak{h}) = 0$, and hence with $[v,\mathfrak{h}] \subset \mathfrak{h}$. This yields $v \in N_{\mathfrak{g}}(\mathfrak{h}) \setminus \mathfrak{h}$.

Claim 2: $\mathfrak{g}$ contains an ideal $\mathfrak{a}$ of codimension one. To see this, let $\mathfrak{a}$ be a proper subalgebra in $\mathfrak{g}$ of maximal dimension. Then $N_{\mathfrak{g}}(\mathfrak{a})$ is strictly larger than $\mathfrak{a}$ by Claim 1. So $N_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{g}$ and $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. For $x \in \mathfrak{g} \setminus \mathfrak{a}$ we also have that $\mathfrak{a} + kx$ is a subalgebra of $\mathfrak{g}$, hence $\mathfrak{g} = \mathfrak{a} + kx$. In particular, $\mathfrak{a}$ has codimension one.

Now we can finish the induction by applying the hypothesis on $\mathfrak{a} \subset \mathfrak{gl}(V)$. We obtain

$$V^{\mathfrak{a}} = \{v \in V \mid \mathfrak{a}.v = 0\} \neq 0.$$

By Lemma 1.6.12 this is a $\mathfrak{g}$-submodule of V. For $x \in \mathfrak{g} \setminus \mathfrak{a}$ the restriction of x to $V^{\mathfrak{a}}$ is a nilpotent endomorphism of $V^{\mathfrak{a}}$. Hence by assumption there exists a $w \in V^{\mathfrak{a}} \setminus 0$ with $x.w = 0$. Since we have $\mathfrak{g} = \mathfrak{a} + kx$, this gives then $\mathfrak{g}.w = 0$ and we are done. □

Under the assumptions of Lemma 1.6.13 we note the following corollary.

Corollary 1.6.14. *In V there exists a chain of subspaces*

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

with $\dim V_i = i$ *and* $\mathfrak{g}.V_i \subseteq V_{i-1}$ *for* $i = 1, \ldots, n$.

Hence there is a basis of V such that the matrices of elements in $\mathfrak{g}$ are all strictly upper triangular matrices. If $\mathfrak{g}$ is not yet a linear Lie algebra, we can consider a representation ρ of $\mathfrak{g}$, for example the adjoint representation. Then the image $\rho(\mathfrak{g})$ is a linear Lie algebra.

Definition 1.6.15. A representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called *nilpotent*, if there is an $n \geq 1$ such that $\rho(x_1)\cdots\rho(x_n) = 0$ for all $x_1, \ldots, x_n \in \mathfrak{g}$.

Then we write $\rho(\mathfrak{g})^n = 0$. We want to show that the adjoint representation of $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}$ is nilpotent. We note the following lemma.

Lemma 1.6.16. *A representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is nilpotent if and only if there is a basis of V, such that the matrices of all $\rho(x)$ are strictly upper-triangular.*

Here now is Engel's Theorem.

Theorem 1.6.17 (Engel). *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation, for which all $\rho(x)$ are nilpotent endomorphisms. Then ρ is a nilpotent representation.*

Proof. We'll show the result by induction on $\dim V$. The case $\dim V = 1$ is clear, because every nilpotent linear map of a 1-dimensional vector space is zero. The induction step goes as follows. By Lemma 1.6.13 we know that $V^{\mathfrak{g}} \neq 0$ and hence that $V^{\mathfrak{g}}$ is a nontrivial submodule of V with quotient module $V/V^{\mathfrak{g}}$. Let $\overline{\rho}\colon \mathfrak{g} \to \mathfrak{gl}(V/V^{\mathfrak{g}})$ the induced representation of $\mathfrak{g}$ on $V/V^{\mathfrak{g}}$. Then also the endomorphisms $\overline{\rho}(x)$ are nilpotent and we may apply the induction hypothesis on $V/V^{\mathfrak{g}}$. So the representation $\overline{\rho}$ is nilpotent, i.e., $\overline{\rho}(\mathfrak{g})^n.(V/V^{\mathfrak{g}}) = 0$. But this implies $\rho(\mathfrak{g})^n.V \subset V^{\mathfrak{g}}$ and hence $\rho(\mathfrak{g})^{n+1}.V = 0$. □

For the adjoint representation $\rho = \operatorname{ad}$ we obtain the following corollary.

COROLLARY 1.6.18 (Engel). *A finite-dimensional Lie algebra $\mathfrak{g}$ over an arbitrary field k is nilpotent if and only if every endomorphism $\operatorname{ad}(x)$ for $x \in \mathfrak{g}$ is nilpotent.*

Now we want to generalize this to *solvable* Lie algebras. The analogue to Proposition 1.6.8 is the following result.

PROPOSITION 1.6.19. *Let $\mathfrak{g}$ be a Lie algebra. Then the following assertions hold.*

(1) *If $\mathfrak{g}$ is solvable then all subalgebras and all homomorphic images of $\mathfrak{g}$ are solvable.*
(2) *Given a short exact sequence of Lie algebras*

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

with both $\mathfrak{a}$ and $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}$ solvable, we have that $\mathfrak{h}$ is solvable. Hence solvability is an extension property.
(3) *If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals in $\mathfrak{g}$, then also the ideal $\mathfrak{a} + \mathfrak{b}$ is solvable.*

PROOF. For (1): If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ then $\mathfrak{a}^{(m)} \subset \mathfrak{g}^{(m)}$. So $\mathfrak{g}^{(m)} = 0$ implies that $\mathfrak{a}^{(m)} = 0$. If $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ is a surjective homomorphism, then we obtain inductively that $\varphi(\mathfrak{g}^{(n)}) = \mathfrak{h}^{(n)}$, and thus $\mathfrak{h}^{(n)} = 0$ provided that $\mathfrak{g}^{(n)} = 0$.

For (2): Let $\pi \colon \mathfrak{h} \to \mathfrak{h}/\mathfrak{a}$ be the quotient map. Then we have $\pi(\mathfrak{h}^{(n)}) = (\mathfrak{h}/\mathfrak{a})^{(n)}$ since π is surjective. By assumption $(\mathfrak{h}/\mathfrak{a})^{(n)} = 0$, so that $\mathfrak{h}^{(n)} \subset \mathfrak{a}$ and hence $\mathfrak{h}^{(n+m)} \subset \mathfrak{a}^{(m)} = 0$ for a $m \geq 0$, since $\mathfrak{a}$ is solvable. Consequently $\mathfrak{h}$ is solvable, too.

For (3): By assumption and by (1), $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}) \cong (\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$ is solvable. By (2), also $\mathfrak{a} + \mathfrak{b}$ is solvable. □

Contrary to nilpotent Lie algebras a solvable Lie algebra may have trivial center, for example $Z(\mathfrak{r}_2(k)) = 0$.

By (3), every finite-dimensional Lie algebra $\mathfrak{g}$ has a largest solvable ideal.

DEFINITION 1.6.20. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The largest solvable ideal in in $\mathfrak{g}$ is called the *solvable radical* of $\mathfrak{g}$, and we denote it by $\operatorname{rad}(\mathfrak{g})$.

LEMMA 1.6.21. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then we have*

$$\operatorname{rad}(\mathfrak{g}/\operatorname{rad}(\mathfrak{g})) = 0.$$

PROOF. Let $\pi \colon \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ be the quotient map and let $\mathfrak{a}$ be a solvable ideal in $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$. Then $\operatorname{rad}(\mathfrak{g}) \subset \pi^{-1}(\mathfrak{a})$ is a solvable ideal with a solvable quotient

$$\pi^{-1}(\mathfrak{a})/\operatorname{rad}(\mathfrak{g}) = \pi(\pi^{-1}(\mathfrak{a})) = \mathfrak{a}.$$

Hence $\pi^{-1}(\mathfrak{a})$ itself is a solvable ideal of $\mathfrak{g}$, hence $\pi^{-1}(\mathfrak{a}) \subset \operatorname{rad}(\mathfrak{g})$. It follows that $\mathfrak{a} = \pi(\pi^{-1}(\mathfrak{a})) = 0$ in $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$. □

LEMMA 1.6.22. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then* $\operatorname{rad}(\mathfrak{g}) = 0$.

PROOF. Since ideals of semisimple Lie algebras are semisimple, $\operatorname{rad}(\mathfrak{g})$ is semisimple and hence perfect. So we have $[\operatorname{rad}(\mathfrak{g}), \operatorname{rad}(\mathfrak{g})] = \operatorname{rad}(\mathfrak{g})$. On the other hand, $\operatorname{rad}(\mathfrak{g})$ is solvable by definition. Hence there is a $n \geq 0$ with $0 = \operatorname{rad}(\mathfrak{g})^{(n)} = \operatorname{rad}(\mathfrak{g})$. □

Now we come to **Lie's Theorem**, which is the analogues statement of Engel's Theorem for solvable Lie groups. For this, we need to generalize Lemma 1.6.12 as follows (for $\chi = 0$ we obtain it back).

Lemma 1.6.23. *Let V be a finite-dimensional $\mathfrak{g}$-module over a field k of characteristic zero, $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ and $\chi \in \mathfrak{a}^* = \operatorname{Hom}(\mathfrak{a}, k)$. Then*

$$V^\chi = \{v \in V \mid h.v = \chi(h)v \ \ \forall\, h \in \mathfrak{a}\}$$

is a $\mathfrak{g}$-submodule of V. For a given χ with $V^\chi \neq 0$ we have $\chi([\mathfrak{a}, \mathfrak{g}]) = 0$.

Proof. Let $x \in \mathfrak{g}$, $h \in \mathfrak{a}$ and $0 \neq v \in V^\chi$. Define spaces

$$V_m = \operatorname{span}\{v, x.v, \ldots, x^{m-1}.v\}$$

for $m \geq 1$ and $V_0 = 0$. Then $x.V_m \subset V_{m+1}$. Since $\dim(V)$ is finite, there is a minimal n with $V_n = V_{n+1}$. Then $x.V_n \subset V_n$ and hence $V_m = V_n$ for $m \geq n$. Then $(v, x.v, \ldots, x^{n-1}.v)$ is a basis of V_n. By induction over n we want to show that

$$h.(x^j.v) - \chi(h)x^j.v \in V_j \tag{1.4}$$

for all $j \geq 0$. This implies then $h.V_{j+1} \subset V_{j+1}$ for all $j \geq 0$. The base case $j = 0$ goes as follows. We have $V_1 = \operatorname{span}\{v\}$ and (1.4) hold because of $v \in V^\chi$.
For $j \geq 1$ we have $h.(x^{j-1}.v) = \chi(h)x^{j-1}.v + V_{j-1}$ and $\mathfrak{a}.V_j \subset V_j$ by induction hypothesis. Moreover we have $x.V_{j-1} \subset V_j$. Hence we have

$$\begin{aligned} h.(x^j.v) &= x.(h.(x^{j-1}.v)) + [h, x].(x^{j-1}.v) \\ &\in (\chi(h)x^j.v + x.V_{j-1}) + \mathfrak{a}.V_j \\ &\subset \chi(h)x^j.v + V_j. \end{aligned}$$

This shows (1.4). It follows that $h \in \mathfrak{a}$ acts by endomorphisms $\rho(h)$ of V_n, which have upper-triangular form with respect to the above basis, with diagonal entries equal to $\chi(h)$. Hence we have $\operatorname{tr}(\rho(h)) = n\chi(h)$. Thus we have, for elements of the form $[x, h] \in \mathfrak{a}$,

$$\begin{aligned} n\chi([x, h]) &= \operatorname{tr}(\rho([x, h])) \\ &= \operatorname{tr}([\rho(x), \rho(h)]) \\ &= 0. \end{aligned}$$

Now we need that k has characteristic zero (or at least that $\operatorname{char}(k) > \dim V$) to conclude that $\chi([h, x]) = 0$. For $w \in V^\chi$ we still need to show that $x.w \in V^\chi$. But this follows from

$$\begin{aligned} h.(x.w) &= x.(h.w) + [h, x].w \\ &= \chi(h)x.w + \chi([h, x]).w \\ &= \chi(h)x.w. \end{aligned}$$

□

Lemma 1.6.24. *Let k be an algebraically closed field of characteristic zero, V be a finite-dimensional k-vector space and $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. If $V \neq 0$, then there exists a $v \neq 0$ in V with $\mathfrak{g}.v \subset kv$.*

Proof. We prove the result by induction over $\dim \mathfrak{g}$. For $\mathfrak{g} = 0$ there is nothing to prove. Let $\mathfrak{a} \subset \mathfrak{g}$ be a subspace of codimension one containing $\mathfrak{g}^1$. Such a subspace exists since $\mathfrak{g}$ is solvable so that $\mathfrak{g}^1 \neq \mathfrak{g}$. Every subspace containing $\mathfrak{g}^1$ is an ideal. Indeed, then $\mathfrak{a}/\mathfrak{g}^1$ is an ideal in $\mathfrak{g}/\mathfrak{g}^1$, since $\mathfrak{g}/\mathfrak{g}^1$ is abelian. Hence $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. By induction hypothesis we find a $v \in V$, $v \neq 0$ with $\mathfrak{a}.v \subset kv$. Now let $\chi \in \mathfrak{a}^*$ with $h.v = \chi(h).v$ for all $h \in \mathfrak{a}$. Then V^χ is a $\mathfrak{g}$-submodule by Lemma 1.6.23. Chose an arbitrary $x \in \mathfrak{g} \setminus \mathfrak{a}$. Then $\mathfrak{g} = \mathfrak{a} + kx$. There is an eigenvector w

for x in V^χ, since k is algebraically closed. Together we obtain $\mathfrak{g}.w \subset \mathfrak{a}.w + kx.w \subset kw$, since V^χ is a $\mathfrak{g}$-submodule. $\square$

PROPOSITION 1.6.25 (Lie's Theorem). *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field k of characteristic zero, and let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then V admits a basis such that all endomorphisms $\rho(x)$ for $x \in \mathfrak{g}$ are represented by upper-triangular matrices.*

PROOF. We'll again use induction over $\dim V$ to show that there is a $\mathfrak{g}$-invariant flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ in V such that $\dim V_j = j$. Choosing the basis elements for V as $v_i \in V_i$, the claim follows because of $\rho(x)(V_i) \subset V_i$. For $V = 0$ there is nothing to show. So let $\dim V \geq 1$. By Lemma 1.6.24 there is a $v \in V$, $v \neq 0$ with $\mathfrak{g}.v \subset kv$. It follows that $W = kv$ is a 1-dimensional $\mathfrak{g}$-submodule. By applying the induction hypothesis to the quotient module V/W we find there an $\mathfrak{g}$-invariant flag $0 = W_1 \subset \cdots \subset W_n$ with $\dim W_j = j - 1$. Let $\pi\colon V \to V/W$ be the quotient map. Then $V_0 = 0$ and $V_j = \pi^{-1}(W_j)$ defines a $\mathfrak{g}$-invariant flag in V with $\dim V_j = j$. $\square$

COROLLARY 1.6.26. *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field k of characteristic zero. Then every simple representation of $\mathfrak{g}$ is 1-dimensional.*

PROOF. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a simple representation. Then the $\mathfrak{g}$-invariant spaces V_i from above are 1-dimensional subrepresentations. However, they are no proper subrepresentations since ρ is simple. Hence we have $\dim V = 1$. $\square$

REMARK 1.6.27. Lie's Theorem does not hold in general if we omit one of the assumptions. Consider the Lie algebra $\mathfrak{sl}_2(k)$, together with the natural representation

$$\rho\colon \mathfrak{sl}_2(k) \to \mathfrak{gl}_2(k)$$

given by $\rho(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\rho(y) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\rho(h) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The $\rho(z)$ have no common eigenvector different from zero for all $z \in \mathfrak{sl}_2(k)$, so that there is no basis in which all operators are all of upper-triangular form. Here the assumption that $\mathfrak{g}$ is solvable is violated, except for characteristic $p = 2$. But in that case Lie's Theorem is also not true, as this example shows.

REMARK 1.6.28. The following example contradicts Lie's Theorem in any characteristic $p > 0$. Let $\mathfrak{g} = \mathfrak{r}_2(k)$ be the solvable Lie algebra in dimension 2 over a field k of characteristic $p > 0$, with Lie bracket $[x, y] = x$ in a basis (x, y). Define a p-dimensional representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$$

of $\mathfrak{g}$ on the vector space V with basis $(e_1, \ldots, e_p)$ by $\rho(x) = E$, $\rho(y) = F$ with

$$\begin{aligned} E(e_1) &= e_p, \\ E(e_i) &= e_{i-1},\ i \geq 2, \\ F(e_i) &= (i-1)e_i. \end{aligned}$$

It is easy to see that

$$\begin{aligned}
[E,F](e_1) &= E(F(e_1)) - F(E(e_1)) \\
&= 0 - (p-1)e_p \\
&= e_p, \\
[E,F](e_i) &= E(F(e_i)) - F(E(e_i)) \\
&= (i-1)e_{i-1} - (i-2)e_{i-1} \\
&= e_{i-1}
\end{aligned}$$

for $i \geq 2$. Hence $[\rho(x),\rho(y)] = [E,F] = E = \rho(x)$ and ρ, because of $\mathrm{char}(k) = p$, is a representation. This is not true in characteristic zero. The operators E, F are given as follows

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & p-1 \end{pmatrix}.$$

From $Ev = \lambda v$ and $Fv = \mu v$ we obtain $v = 0$, first for $\mu = 0$, and then for $\mu \neq 0$. Hence the $\rho(v)$ do not have a common nonzero eigenvector.

Let us note here that Lie's Theorem has an analogue for algebraic groups, which was proved by Ellis Kolchin (1916-1991).

THEOREM 1.6.29 (Lie-Kolchin). *Let G be a connected solvable linear algebraic group over an algebraically closed field of arbitrary characteristic. Let $\rho\colon G \to GL(V)$ be a representation on a finite-dimensional vector space V. Then there exists a common nonzero eigenvector $v \in V$ for all $\rho(g)$ with $g \in G$.*

For a proof, see [**20**]. Again we cannot omit some of the assumptions. In particular, the connectedness assumption is necessary even for closed subgroups. Also, it fails for solvable connected Lie groups in general, because these are not necessarily isomorphic to groups of upper triangular matrices.

A standard counterexample for $k = \mathbb{R}$ is as follows. Consider the connected abelian linear algebraic group over $\mathbb{R}$ given by

$$G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, a, b \in \mathbb{R} \right\}.$$

Let ρ be the natural representation. Obviously, $\rho(G)$ is not triangularizable over $\mathbb{R}$.

COROLLARY 1.6.30. *Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic zero. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$ is nilpotent.*

PROOF. Let $\mathfrak{g}^1$ be nilpotent. Then $\mathfrak{g}^1$ and $\mathfrak{g}/\mathfrak{g}^1$ are both solvable, namely nilpotent respectively abelian. By Proposition 1.6.19 then $\mathfrak{g}$ is solvable. This even holds in characteristic $p > 0$.

The converse statement only holds in characteristic zero. Let $\mathfrak{g}$ be solvable. Assume first that k is algebraically closed. Then we can apply Lie's Theorem for the adjoint representation of $\mathfrak{g}$. With respect to a suitable basis of $\mathfrak{g}$ the subalgebra $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ consists of upper-triangular matrices. Hence $[\mathrm{ad}(\mathfrak{g}),\mathrm{ad}(\mathfrak{g})] = \mathrm{ad}([\mathfrak{g},\mathfrak{g}])$ consists of *strictly* upper-triangular matrices, and hence is nilpotent. Since the kernel of $\mathrm{ad}\colon [\mathfrak{g},\mathfrak{g}] \to \mathfrak{gl}(\mathfrak{g})$ lies in the center of $[\mathfrak{g},\mathfrak{g}]$, we obtain that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent by Proposition 1.6.8. If k is not algebraically closed we may apply the method of scalar extension. Let $\mathbb{F}$ be an algebraic closure of k. For a k-vector space V we consider $V_{\mathbb{F}} = V \otimes_k \mathbb{F}$. If $\mathfrak{g}$ is solvable, so is $\mathfrak{g}_{\mathbb{F}}$. Hence $[\mathfrak{g}_{\mathbb{F}},\mathfrak{g}_{\mathbb{F}}]$ is nilpotent by the above argument, and hence $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. □

Remark 1.6.31. There are indeed examples of solvable Lie algebras in characteristic $p > 0$, whose commutator subalgebra is *not* nilpotent. We can construct such an example from Remark 1.6.28. Let $\mathfrak{g} = \mathfrak{r}_2(k)$ and V be the p-dimensional representation given there. We equip the space

$$\mathfrak{h} = \mathfrak{g} \oplus V$$

with the structure of a Lie algebra by viewing V as an abelian Lie algebra and letting $\mathfrak{g}$ act on V by ρ. More concretely, $\mathfrak{h}$ has a basis $(x, y, e_1, \ldots, e_p)$ with Lie brackets

$$\begin{aligned}
[x,y] &= x,\\
[x,e_1] &= e_p,\\
[x,e_i] &= e_{i-1},\ i \geq 2,\\
[y,e_i] &= (i-1)e_i,\ 1 \leq i \leq p.
\end{aligned}$$

Since V and the quotient $\mathfrak{h}/V$ are solvable, so is $\mathfrak{h}$. But $[\mathfrak{h},\mathfrak{h}] = kx \oplus V$ is not nilpotent. We have $[\mathfrak{h},\mathfrak{h}]^1 = [\mathfrak{h},\mathfrak{h}]^2 = \cdots = V$. This example is taken from Jacobson [**21**].

We mention still another corollary to Lie's Theorem.

Corollary 1.6.32. *Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic zero. Then $[\mathfrak{g},\mathrm{rad}(\mathfrak{g})]$ is a nilpotent ideal of $\mathfrak{g}$, i.e., satisfying*

$$[\mathfrak{g},\mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{nil}(\mathfrak{g}).$$

Proof. Let $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ and set $L := \mathfrak{r} + \langle y \rangle$ for an element $y \in \mathfrak{g}$. Then we have $[L,L] \subseteq [\mathfrak{r},\mathfrak{r}] + [\mathfrak{r},\langle y \rangle] \subseteq \mathfrak{r}$, so that $[L,L]$ is a solvable ideal of $\mathfrak{g}$, and hence L is solvable, too. It follows that $[L,L]$ is nilpotent by Corollary 1.6.30, i.e., $\mathrm{ad}(x)$ is nilpotent for all $x \in [L,L]$. Now let $x = [a,b] \in [\mathfrak{r},\mathfrak{g}]$ a pure commutator. Then there is a $y \in \mathfrak{g}$ with $b \in \mathfrak{r} + \langle y \rangle$, i.e., we have $x \in [L,L]$, so that $\mathrm{ad}(x)$ is nilpotent. Now since y is arbitrary, $x = [a,b]$ runs through whole $[\mathfrak{r},\mathfrak{g}]$, which yields that $\mathrm{ad}(x)$ is nilpotent for all $x \in [\mathfrak{r},\mathfrak{g}]$. By Engel's theorem, $[\mathfrak{r},\mathfrak{g}]$ is nilpotent. □

1.7. The classification of Lie algebras in low dimension

Our aim of this section is to classify all Lie algebras of dimension $n \leq 3$ over an *arbitrary* field k. For $n \leq 2$ we obtain just one non-abelian Lie algebra. In dimension 3, however, we obtain already infinitely many different solvable Lie algebras. We order the cases for a fixed dimension of $\mathfrak{g}$ by the dimension of $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$.

Case 1*,* $\dim \mathfrak{g} = 1$*:* We have $\mathfrak{g} \cong k$, the 1-dimensional abelian Lie algebra.

Case 2*a,* $\dim \mathfrak{g} = 2, \dim \mathfrak{g}^1 = 0$*:* We have $\mathfrak{g} \cong k^2$, the abelian Lie algebra in dimension 2.

Case 2*b,* $\dim \mathfrak{g} = 2, \dim \mathfrak{g}^1 = 1$*:* Let $\mathfrak{g} = kx + ky$. Then $[\mathfrak{g}, \mathfrak{g}] = k \cdot [x, y]$ is 0-dimensional or 1-dimensional. We may assume that $[\mathfrak{g}, \mathfrak{g}] = ky$. Then $[x, y] = \alpha y$ with some $\alpha \neq 0$. Replacing x by $\alpha^{-1}x$ we may assume that $[x, y] = y$. We denote this Lie algebra, as before, by $\mathfrak{r}_2(k)$. Every non-abelian 2-dimensional Lie algebra is isomorphic to $\mathfrak{r}_2(k)$, as we have just shown. This Lie algebra is also isomorphic to $\mathfrak{aff}(k) \cong k \ltimes k$ with $[x, y] = [(1, 0), (0, 1)] = (0, 1) = y$.

Case 3*a,* $\dim \mathfrak{g} = 3, \dim \mathfrak{g}^1 = 0$*:* We have $\mathfrak{g} \cong k^3$, the abelian Lie algebra in dimension 3.

Case 3*b,* $\dim \mathfrak{g} = 3, \dim \mathfrak{g}^1 = 1$*:* Suppose that $\mathfrak{g}^1 \subset Z(\mathfrak{g})$ and let $\mathfrak{g}^1 = kz$. We can extend z to a basis (x, y, z) of $\mathfrak{g}$. Because of $z \in Z(\mathfrak{g})$ we have $[x, z] = [y, z] = 0$. We may assume that $[x, y] = z$. Then $\mathfrak{g}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_3(k)$.
If $\mathfrak{g}^1 = ky$ is not contained in the center of $\mathfrak{g}$, then there exists an $x' \in \mathfrak{g}$ with $[x', y] \neq 0$. Since $\dim \mathfrak{g}^1 = 1$, we have $[x', y] = \alpha y$, and therefore $[x, y] = y$ with $x = \alpha^{-1}x'$. The subalgebra $\mathfrak{a} = kx + ky$ hence is isomorphic to $\mathfrak{r}_2(k)$. Moreover $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ because of $\mathfrak{g}^1 \subset \mathfrak{a}$. Let $z \in \mathfrak{g} \setminus \mathfrak{a}$. Because of $\mathrm{Der}(\mathfrak{a}) = \mathrm{ad}(\mathfrak{a})$, see Example 1.2.5, there exists a $w \in \mathfrak{a}$ with $\mathrm{ad}(z)_{|\mathfrak{a}} = \mathrm{ad}(w)$. Then $[z - w, \mathfrak{a}] = 0$. We have the direct decomposition $\mathfrak{g} = \mathfrak{a} \oplus k \cdot (z - w) \cong \mathfrak{r}_2(k) \oplus k$.

Case 3*c,* $\dim \mathfrak{g} = 3, \dim \mathfrak{g}^1 = 2$*:* We claim that $\mathfrak{g}^1$ is abelian. Otherwise we had $\mathfrak{g}^1 \cong \mathfrak{r}_2(k)$ and hence $\mathfrak{g} \cong \mathfrak{r}_2(k) \oplus k$ as above, a contradiction to $\dim \mathfrak{g}^1 = 2$. Hence $\mathfrak{g}^1$ is an abelian ideal in $\mathfrak{g}$. Choosing an $x \in \mathfrak{g} \setminus \mathfrak{g}^1$, we have

$$\mathfrak{g} \cong kx \ltimes \mathfrak{g}^1 \cong k \ltimes k^2.$$

These Lie algebras are solvable. Every such semidirect product is determined by a homomorphism $D\colon k \to \mathfrak{gl}_2(k)$, i.e., by a linear map $A = D(1) \in GL_2(k)$. The Lie bracket in $\mathfrak{g}$ is then given, for $x, x' \in k^2$ and $t, t' \in k$, by

$$[(t, x), (t', x')] = (0, tAx' - t'Ax).$$

The matrix A is invertible since $\dim \mathfrak{g}^1 = 2$. Let us denote this Lie algebra by $\mathfrak{g}_A$. We want to determine the isomorphism classes of such Lie algebras. Hence let $\varphi\colon \mathfrak{g}_A \to \mathfrak{g}_B$ be an isomorphism of two such Lie algebras. Then $\varphi(\mathfrak{g}_A^1) \subset \mathfrak{g}_B^1$. Hence there exists a $C \in GL_2(k)$, a scalar $\alpha \in k^*$ and a $y \in k^2$ with $\varphi(t, x) = (\alpha t, Cx + ty)$. On the other hand, we have for all t, t', x, x' that

$$\begin{aligned}(0, tCAx' - t'CAx) &= \varphi([(t,x),(t',x')]_A)\\ &= [\varphi(t,x),\varphi(t',x')]_B\\ &= [(\alpha t, Cx+ty),(\alpha t', Cx'+t'y)]_B\\ &= (0, \alpha tBCx' + \alpha tt'By - \alpha t'BCx - \alpha tt'By)\\ &= (0, \alpha tBCx' - \alpha t'BCx).\end{aligned}$$

This implies that $CA = \alpha BC$, or

$$A = \alpha C^{-1}BC.$$

Thus the group $G = k^* \times GL_2(k)$ acts on $GL_2(k)$ by $(\alpha, C).A = \alpha C^{-1}AC$. The Lie algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are isomorphic if and only if A and B are in the same orbit of this action. Let us first assume that k is algebraically closed. The the Jordan normal form says that a representing system for the G-orbits is given by the following matrices.

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, \lambda \in k^*.$$

This corresponds to the following Lie algebras:

$$\begin{aligned}\mathfrak{r}_3(k) &: [e_1,e_2] = e_2,\ [e_1,e_3] = e_2 + e_3,\\ \mathfrak{r}_{3,\lambda}(k) &: [e_1,e_2] = e_2,\ [e_1,e_3] = \lambda e_3,\ \lambda \neq 0.\end{aligned}$$

The only isomorphisms are given, with $\lambda, \mu \in k^*$, as follows. We have $\mathfrak{r}_{3,\lambda}(k) \cong \mathfrak{r}_{3,\mu}(k)$ if and only if $\lambda = \mu$ or $\lambda\mu = 1$.

More generally, we have the following classification result in this case over an arbitrary field, see [**13**]:

Proposition 1.7.1. *Let $\mathfrak{g}$ be a 3-dimensional Lie algebra over an arbitrary field k with* $\dim \mathfrak{g}^1 = 2$*. Then $\mathfrak{g}$ is isomorphic to one of the following solvable Lie algebras,*

$$\begin{aligned}L^1 &: [e_3,e_1] = e_1,\ [e_3,e_2] = e_2,\\ L^2_\alpha &: [e_3,e_1] = e_2,\ [e_3,e_2] = \alpha e_1 + e_2,\ \alpha \neq 0,\\ L^3_\alpha &: [e_3,e_1] = e_2,\ [e_3,e_2] = \alpha e_1,\ \alpha \neq 0.\end{aligned}$$

The only isomorphisms are, for $\alpha, \beta \in k^*$, as follows. We have $L^3_\alpha \cong L^3_\beta$ if and only if $\alpha = t^2\beta$ with some $t \in k^*$.

Case 3d, $\dim \mathfrak{g} = 3, \dim \mathfrak{g}^1 = 3$*:* In this case $\mathfrak{g}$ has to be *simple*. Otherwise $\mathfrak{g}$ had a non-trivial ideal $\mathfrak{a}$, which were necessarily solvable because of $\dim \mathfrak{a} \leq 2$. In the same way, $\mathfrak{g}/\mathfrak{a}$ were solvable, so that $\mathfrak{g}$ were a solvable extension, hence solvable, a contradiction to $\mathfrak{g}^1 = \mathfrak{g}$.

Let (e_1, e_2, e_3) be a basis of $\mathfrak{g}$ and set

$$f_1 = [e_2,e_3],\ f_2 = [e_3,e_1],\ f_3 = [e_1,e_2].$$

Since $\mathfrak{g}^1 = \mathfrak{g}$ then (f_1, f_2, f_3) is another basis of $\mathfrak{g}$. We may express one basis by the other, i.e.,

$$f_i = \sum_{j=1}^{3} a_{ij} e_j.$$

The Jacobi identity $J(x, y, z) = 0$ in $\mathfrak{g}$ with respect to the basis (e_1, e_2, e_3) then imposes conditions on the coefficients a_{ij}. In fact, the matrix

$$A := (a_{ij})_{1 \le i,j \le 3}$$

then is symmetric. The proof goes as follows. Because of skew-symmetry we have $J = 0$ as soon as two elements are equal. Thus it suffices to look at the conditions given by $J(e_1, e_2, e_3) = 0$.

$$\begin{aligned}
0 &= J(e_1, e_2, e_3) \\
&= [e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] \\
&= [e_1, f_1] + [e_2, f_2] + [e_3, f_3] \\
&= a_{12} f_3 - a_{13} f_2 - a_{21} f_3 + a_{23} f_1 + a_{31} f_2 - a_{32} f_1.
\end{aligned}$$

Conversely we obtain for every symmetric matrix $A \in M_3(k)$ a Lie algebra $\mathfrak{g}_A$ by specifying the Lie brackets $[e_i, e_j]$ accordingly to A. Thus we have described *all* 3-dimensional simple Lie algebras. It remains to classify the isomorphism classes. Let M be the matrix of an isomorphism $\varphi\colon \mathfrak{g}_A \to \mathfrak{g}_B$. As before one can check that we have

$$B = \det(M)(M^{-1})^t A M^{-1}$$

Then we can define the action of the group $G = k^* \times GL_3(k)$ on the space of symmetric matrices in $M_3(k)$ by $(\alpha, C)A = \alpha C A C^t$. We see that $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are isomorphic if and only if A and B lie in the same G-orbit. If k is algebraically closed and of characteristic different from two, then we can reconstruct every symmetric bilinear form from its quadratic form. Hence in this case the representing system only consists of the identity matrix $A = E_3$. This corresponds to the simple Lie algebra

$$\mathfrak{so}_3(k) : [e_1, e_2] = e_3, \;\; [e_2, e_3] = e_1, \;\; [e_3, e_1] = e_2.$$

It is isomorphic to $\mathfrak{sl}_2(k)$, as long as the characteristic is not two. The isomorphism $\varphi\colon \mathfrak{sl}_2(k) \to \mathfrak{so}_3(k)$ is given by

$$\varphi = \begin{pmatrix} t & t & 0 \\ 0 & 0 & 2t \\ 1 & -1 & 0 \end{pmatrix},$$

where $t \in k$ is a solution of $t^2 + 1 = 0$.

PROPOSITION 1.7.2. *Let $\mathfrak{g}$ be a simple 3-dimensional Lie algebra over an algebraically closed field k of characteristic different from two. Then $\mathfrak{g} \cong \mathfrak{sl}_2(k)$.*

The result does not hold for fields k, which are not algebraically closed. It is clear that, as an example, $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$ are not isomorphic, since $\mathfrak{sl}_2(\mathbb{R})$ admits a 2-dimensional subalgebra, but $\mathfrak{so}_3(\mathbb{R})$ does not.

We will see later that the following result holds.

PROPOSITION 1.7.3. *Let $\mathfrak{g}$ be a simple real 3-dimensional Lie algebra. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}_3(\mathbb{R})$.*

If k has characteristic two, then $\mathfrak{g} \cong \mathfrak{sl}_2(k)$ is no longer simple, since it is nilpotent. Then it is replaced by the following simple Lie algebra,

$$W(1;\underline{2})^{(2)} \ : \ [x,y]=h,\ [h,x]=x,\ [h,y]=y.$$

The following result has been proved by Strade [**32**]:

Proposition 1.7.4. *Let $\mathfrak{g}$ be a simple 3-dimensional Lie algebra over an algebraically closed field of characteristic $p=2$, or over a finite field $\mathbb{F}_{2^k}$. Then $\mathfrak{g} \cong W(1;\underline{2})^{(1)}$.*

For $p>2$ Strade proved the following result [**32**].

Proposition 1.7.5. *Let $\mathfrak{g}$ be a simple 3-dimensional Lie algebra over a finite field k of characteristic $p \geq 3$. Then $\mathfrak{g} \cong \mathfrak{sl}_2(k)$.*

In dimension 4 the classification becomes of course much more difficult. The best case is a result over the complex numbers.

Proposition 1.7.6. *Every 4-dimensional complex Lie algebra is isomorphic to one of the following list, with $\alpha, \beta \in \mathbb{C}$:*

$\mathfrak{g}$	*Lie brackets*
$\mathfrak{g}_0 = \mathbb{C}^4$	$-$
$\mathfrak{g}_1 = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$	$[e_1,e_2]=e_3$
$\mathfrak{g}_2 = \mathfrak{n}_4(\mathbb{C})$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4$
$\mathfrak{g}_3 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$	$[e_1,e_2]=e_2$
$\mathfrak{g}_4 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1,e_2]=e_2,\ [e_3,e_4]=e_4$
$\mathfrak{g}_5 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1,e_2]=e_2,\ [e_1,e_3]=-e_3,\ [e_2,e_3]=e_1$
$\mathfrak{g}_6$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_3,\ [e_1,e_4]=e_4$
$\mathfrak{g}_7(\alpha)$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_3,\ [e_1,e_4]=e_3+\alpha e_4$
$\mathfrak{g}_8$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_3,\ [e_1,e_4]=2e_4,\ [e_2,e_3]=e_4$
$\mathfrak{g}_9(\alpha,\beta)$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_2+\alpha e_3,\ [e_1,e_4]=e_3+\beta e_4$
$\mathfrak{g}_{10}(\alpha)$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_2+\alpha e_3,$ $[e_1,e_4]=(\alpha+1)e_4,\ [e_2,e_3]=e_4$

There are no isomorphisms between different types, but there are still isomorphisms within some of the infinite families. We have $\mathfrak{g}_7(\alpha) \cong \mathfrak{g}_7(\beta)$ if and only if $\alpha=\beta$, and $\mathfrak{g}_{10}(\alpha) \cong \mathfrak{g}_{10}(\alpha')$ if and only if $\alpha\alpha'=1$ or $\alpha=\alpha'$. Furthermore we have $\mathfrak{g}_9(\alpha_1,\beta_1) \cong \mathfrak{g}_9(\alpha_2,\beta_2)$ if and only if the double ratios $1:\alpha_1:\beta_1$ and $1:\alpha_2:\beta_2$ coincide up to permutation. In other words, for $\alpha,\beta \neq 0$, we have

$$\mathfrak{g}_9(\alpha,\beta) \cong \mathfrak{g}_9(\alpha',\beta')$$

if and only if (α',β') is one of the following possibilities,

$$(\alpha,\beta),\ (\beta,\alpha),\ \left(\frac{1}{\alpha},\frac{\beta}{\alpha}\right),\ \left(\frac{\beta}{\alpha},\frac{1}{\alpha}\right),\ \left(\frac{1}{\beta},\frac{\alpha}{\beta}\right),\ \left(\frac{\alpha}{\beta},\frac{1}{\beta}\right).$$

We may compose every isomorphism from the following two ones,

$$\mathfrak{g}_9(\alpha,\beta) \cong \mathfrak{g}_9(\beta,\alpha)$$

$$\mathfrak{g}_9(\alpha,\beta) \cong \mathfrak{g}_9\left(\frac{1}{\beta},\frac{\alpha}{\beta}\right), \beta \neq 0$$

Note that some of the algebras are decomposable, for example,

$$\begin{aligned}\mathfrak{g}_7(0) &\cong \mathfrak{r}_{3,1}(\mathbb{C}) \oplus \mathbb{C},\\ \mathfrak{g}_9(\alpha, 0) &\cong \mathfrak{r}_{3,\alpha}(\mathbb{C}) \oplus \mathbb{C} \text{ with } \alpha \neq 0, 1\\ \mathfrak{g}_9(0, 1) &\cong \mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}.\end{aligned}$$

1.8. Lie groups and Lie algebras

Lie groups form an important class of differentiable manifolds. Basic finite-dimensional examples over the real numbers are the general linear group $GL_n(\mathbb{R})$, the unitary group $U(n)$, the orthogonal group $O_n(\mathbb{R})$ and the special linear group $SL_n(\mathbb{R})$. Of great importance here is the close relationship between a Lie group and its Lie algebra.

Definition 1.8.1. A *Lie group* G is a group, whose elements are the points of a smooth manifold, such that the group multiplication $G \times G \to G$ is smooth.

Note that it follows that the map $x \to x^{-1}$ is smooth as well. For a given Lie group G we consider the tangent space at the identity, denoted by $T_1(G)$. This vector space admits a natural Lie bracket, yielding the Lie algebra $\mathfrak{g}$ of G. The map $G \to T_1(G)$ has very good properties. It is a functor of the category of Lie groups to the category of Lie algebras. We cannot go into too much detail here, but we want to give an illustrating example with the orthogonal group $G = O(n)$, consisting of $n \times n$ matrices A with $A^t A = E_n$. How do we compute its Lie algebra $\mathfrak{so}(n)$? We consider the differentiable families

$$A\colon]-\varepsilon, \varepsilon[\to O(n) \subset \mathbb{R}^{n^2}$$

for some $\varepsilon > 0$ with $A(0) = E_n$. All entries are differentiable functions. Now we take the derivative of $A(t)^t A(t) = E_n$ with respect to t,

$$\dot{A}(t)^t A(t) + A(t)^t \dot{A}(t) = 0,$$

and substitute $t = 0$ so that we have

$$\dot{A}(0)^t + \dot{A}(0) = 0,$$

hence $X + X^t = 0$ with $X = \dot{A}(0)$. So the Lie algebra $\mathfrak{so}(n)$ consists of the skew-symmetric $n \times n$ matrices. Of course the commutator defines a Lie bracket on this space. Hence we have determined the tangent space $T_1(O(n))$.

Consider now some some $A \in O(n)$ the conjugation $c_A\colon O(n) \to O(n)$, with $B \mapsto ABA^{-1}$. If we take instead of a fixed matrix A again a differentiable family as above, take the derivative with respect to t and set $t = 0$, we obtain

$$(A(t)BA(t)^{-1})^{\cdot}|_{t=0} = \dot{A}(0)B - B\dot{A}(0).$$

Here we have used

$$(A(t)^{-1})^{\cdot} = -A(t)^{-1}\dot{A}(t)A(t)^{-1}$$

which follows from taking the derivative of the identity $A(t)A(t)^{-1} = E_n$. Now we just have computed the adjoint representation of the Lie algebra, namely

$$\begin{aligned} \operatorname{ad}(X)\colon \mathfrak{so}(n) &\to \mathfrak{so}(n) \\ Y &\to XY - YX. \end{aligned}$$

A natural question is, whether or not we can also compute the *converse* direction. It turns out that it is indeed possible if the Lie group is connected and simply connected. Then the Lie group is up to isomorphism determined by its Lie algebra. The exponential function $\exp\colon \mathfrak{g} \to G$ then is a *local diffeomorphism*. We have the following result.

Theorem 1.8.2 (Lie's third theorem). *Every real finite-dimensional Lie algebra is isomorphic to a Lie algebra of a Lie group.*

We may summarize this as follows (in finite dimension).

THEOREM 1.8.3. *The functor $G \to T_1(G)$ defines an equivalence of categories between the category of real connected and simply connected Lie groups and the category of real Lie algebras.*

How does this look like in our example with $G = O(n)$? The exponential map

$$\exp \colon \mathfrak{so}(n) \to SO(n)$$

has as image only the subgroup $SO(n)$ of $O(n)$ consisting of orthogonal matrices with determinant one. As a topological space, $O(n)$ consists of two components, one with determinant 1, the other with determinant -1. For $A \in \mathfrak{gl}(n)$, $\exp(A)$ is defined by

$$e^A = E_n + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots .$$

This series converges uniformly on each bounded subset of $\mathfrak{gl}(n)$ and w have $\det(e^A) = e^{\operatorname{tr}(A)}$. Hence we have $e^A \in GL(n)$. Moreover we have $e^{A+B} = e^A e^B$, for $AB = BA$ and $Be^A B^{-1} = e^{BAB^{-1}}$. For $\mathfrak{so}(n)$ we obtain the Lie group $SO(n)$, and not $O(n)$. So we see that we need the connectedness in the correspondence.

EXAMPLE 1.8.4. *For the matrix $A = \left(\begin{smallmatrix} 0 & -\theta \\ \theta & 0 \end{smallmatrix}\right) \in \mathfrak{so}_2(\mathbb{R})$ we have*

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO_2(\mathbb{R}).$$

A very natural question is to asks for which Lie groups the exponential function is indeed *surjective*. Even for connected matrix groups it need not be surjective in general. For example,

$$\exp \colon \mathfrak{sl}_n(\mathbb{R}) \to SL_n(\mathbb{R})$$

is not surjective for $n \geq 2$ (note that $SL_n(\mathbb{R})$ is not simply connected, but $SL_n(\mathbb{C})$ is simply connected). To see this we may assume that $n = 2$. Let

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

be any matrix in $\mathfrak{sl}_2(\mathbb{R})$. Then $A^2 = (a^2 + bc)E_2 = -\det(A)E_2$. If $\det(A) = 0$ then $e^A = E_2 + A$, and hence $\operatorname{tr}(e^A) = 2$. For $\det(A) > 0$ we can find an $\omega > 0$ with $\det(A) = \omega^2$, hence with $\omega^2 = -(a^2 + bc)$. Then $A^2 = -\omega^2 E_2$ and

$$e^A = \cos(\omega)E_2 + \frac{\sin(\omega)}{\omega}A.$$

Then we have $\operatorname{tr}(e^A) = 2\cos(\omega) \in [-2, 2]$. Finally, we could have that $\det(A) < 0$, i.e., $a^2 + bc > 0$. Then there is an $\eta > 0$ with $\eta^2 = a^2 + bc$, and $A^2 = \eta^2 E_2$. We have

$$e^A = \cosh(\eta)E_2 + \frac{\sinh(\eta)}{\eta}A.$$

This means that $\operatorname{tr}(e^A) = 2\cosh(\omega) \in [2, \infty)$. Altogether we always have that $\operatorname{tr}(e^A) \geq -2$. Therefore it is clear now that, say,

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \notin \exp(\mathfrak{sl}_2(\mathbb{R})),$$

but $A \in SL_2(\mathbb{R})$. It is easy to see that also for all $\lambda \neq 0$ the matrices

$$A = \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix} \notin \exp(\mathfrak{sl}_2(\mathbb{R}))$$

are not contained in the image of exp. On the other hand we have the following result.

Proposition 1.8.5. *Let G be a connected compact real Lie group. Then the exponential map is surjective.*

But this is not the only criterion. For example, $\exp\colon \mathfrak{gl}_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is surjective, whereas $\exp\colon \mathfrak{sl}_n(\mathbb{C}) \to SL_n(\mathbb{C})$ and $\exp\colon \mathfrak{gl}_n(\mathbb{R}) \to GL_n(\mathbb{R})$ are not surjective. For more details see [**14**] and the references given therein.

Finally we want to mention, that Ado's Theorem need not be true for finite-dimensional Lie groups in general, i.e., not all Lie groups are matrix groups. Here is a counterexample. Let

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, n \in \mathbb{Z} \right\}.$$

Then N is a normal subgroup in G and $H = G/N$ is a 3-dimensional Lie group.

Proposition 1.8.6. *Every Lie group homomorphism $\varphi\colon H \to GL_n(\mathbb{R})$ has a non-trivial kernel.*

In other words, there is no faithful linear representation. Hence H is not a matrix group.

CHAPTER 2

Structure theory of Lie algebras

2.1. Die Jordan-Chevalley decomposition

Let k be a field of characteristic *zero* in this section. For an endomorphism $x \in \operatorname{End}(V)$ the *eigenspace* $E_\lambda(x)$ to the eigenvalue $\lambda \in k$ is given by

$$E_\lambda(x) = \ker(x - \lambda \operatorname{id}).$$

The generalized eigenspace of x to λ is given by

$$H_\lambda(x) = \bigcup_{n \geq 0} \ker(x - \lambda \operatorname{id})^n.$$

DEFINITION 2.1.1. An $x \in \operatorname{End}(V)$ is called *diagonalizable*, if V is the direct sum of the eigenspaces of x, i.e., if $V = \oplus_\lambda E_\lambda(x)$. We call x *semisimple*, if V is semisimple as module for the Lie algebra $kx \subset \operatorname{End}(V)$.

If $x \in \operatorname{End}(V)$ is nilpotent then $V = H_0(x)$. The converse is also true if V is finite dimensional. By definition, x is semisimple if for every x-invariant subspace $U \subset V$ there exists a complementary x-invariant subspace U' with $V = U \oplus U'$. It is easy to see that x is semisimple if and only if the roots of its minimal polynomial are all distinct.

LEMMA 2.1.2. *Let k be an algebraically closed field. Then $x \in \operatorname{End}(V)$ is semisimple if and only if x is diagonalizable.*

PROOF. Let x be diagonalizable. Then V is the direct sum of its eigenspaces and hence the sum of 1-dimensional x-invariant subspaces, which are simple kx-submodules. By Proposition 1.2.26, part (2) it follows that x is semisimple.

Conversely, let x be semisimple. Then V is by Proposition 1.2.26 the direct sum of simple kx-modules. hence it suffices to show that every simple kx-module is 1-dimensional, because every 1-dimensional kx-submodule is spanned by an eigenvector of x. So let $V \neq 0$ be a simple kx-module and $v \in V \setminus 0$. Then we have $V = \operatorname{span}\{x^n.v \mid n \geq 0\}$. The right hand side is a nonzero x-invariant subspace, hence equal to V. Suppose that all $x^n.v$ are linearly independent. Then $U = \operatorname{span}\{x^n.v \mid n \geq 1\}$ is a proper x-invariant submodule, contradiction our assumption. Hence there exists a $n \geq 1$ such that $x^n.v$ is a linear combination of the vectors $v, x.v, \ldots, x^{n-1}.v$. It follows that V is finite-dimensional. Since k is algebraically closed, x has an eigenvector w in V and $V = \operatorname{span}\{w\}$ is 1-dimensional. □

LEMMA 2.1.3. *Let $x, y \in \operatorname{End}(V)$ be two commuting endomorphisms. If x and y are diagonalizable, the also their sum $x + y$ is diagonalizable. If x and y are nilpotent, then also the sum $x + y$ is nilpotent.*

PROOF. Since x and y commute, they are simultaneously diagonalizable, so that $x + y$ is diagonalizable. Let us give a different proof, using Lemma 2.1.2. Assume that k is algebraically closed. Then for each eigenvector $v \in E_\lambda(x)$ we have $xy(v) = yx(v) = \lambda y(v)$. Hence $yv \in E_\lambda(x)$

and y leaves the eigenspaces of x invariant. Since y is diagonalizable, V si a semisimple ky-module. Submodules of V hence are also semisimple. Thus the restriction of y to the eigenspaces of x is diagonalizable and there is a basis of eigenvectors in which x and y are simultaneously of diagonal form. Hence $x+y$ is diagonalizable.

Assume that x and y are nilpotent with $x^n = y^m = 0$. Then, because of $xy = yx$, the binomial formula implies that

$$(x+y)^k = \sum_{i+j=k} \binom{k}{i} x^i y^j.$$

For all $k \geq n+m-1$ the summands are zero, so that $(x+y)^k = 0$. □

PROPOSITION 2.1.4 (Jordan-Chevalley). *Let V be a finite-dimensional vector space over an algebraically closed field k and $x \in \operatorname{End}(V)$. Then the following statements hold.*

1. *There exist unique $x_s, x_n \in \operatorname{End}(V)$ with $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent and x_s, x_n commute.*
2. *The eigenspaces of x_s are the generalized eigenspaces of x, i.e., $E_\lambda(x_s) = H_\lambda(x)$.*
3. *There exist polynomials $p(t), q(t) \in k[t]$ without constant term such that $x_s = p(x)$ and $x_n = q(x)$.*
4. *Any $y \in \operatorname{End}(V)$ commuting with x also commutes with x_s and x_n.*
5. *For commuting $x, y \in \operatorname{End}(V)$ we have $(x+y)_s = x_s + y_s$ and $(x+y)_n = x_n + y_n$.*

PROOF. For $(1), (3), (4)$: Let α_i be the different eigenvalues of x with multiplicities m_i, for $i = 1, \ldots, k$. Hence x has the characteristic polynomial

$$f = \prod_{i=1}^{k} (t - \alpha_i)^{m_i},$$

which splits into linear factors since k is algebraically closed. V is the direct sum of the eigenspaces $E_i = E_{\alpha_i}(x)$, and each eigenspace is x-invariant. On E_i the endomorphism x has the characteristic polynomial $(t-\alpha_i)^{m_i}$. Now we apply the CRT (Chinese Remainder Theorem) to $R = k[t]$ an. There exists a polynomial p with

$$\begin{aligned} p(t) &\equiv \alpha_i \mod (t-\alpha_i)^{m_i} \\ p(t) &\equiv 0 \mod t. \end{aligned}$$

We put $q(t) = t - p(t)$. The second congruence is only necessary if zero is not an eigenvalue of x, in which case t is relatively prime to $(t-\alpha_i)^{m_i}$. Certainly p and q have no constant term. We define $x_s := p(x)$ and $x_n := q(x)$. These are polynomials in x and hence they commute. So we have $[x_s, x_n] = 0$. The polynomials also commute with endomorphisms commuting with x, and they leave the eigenspaces E_i invariant. The first congruence shows that the restriction of $x_s - \alpha_i \operatorname{id}$ on E_i is identically zero for all i. Therefore x_s acts diagonally on E_i with single eigenvalue α_i. By definition $x_n = x - x_s$ then is nilpotent. We have shown (3), (4) and (1) except for the uniqueness. So let $x = s + n$ be another decomposition with these properties. Since s and n commute with x, they also commute with x_s and x_n. We have

$$s - x_s = n - x_n.$$

Since the difference of two commuting diagonalizable endomorphisms s and x_s is again diagonalizable, and $n - x_n$ is nilpotent again, we have that $s - x_s = n - x_n$ are both diagonalizable and nilpotent. Hence they are both zero, i.e., we have $s = x_s$ and $n = x_n$.

For (5): Assume that $x, y \in \mathrm{End}(V)$ commute. Then by Lemma 2.1.3 we have that $x_s + y_s$ is diagonalizable and $x_n + y_n$ is nilpotent. Both commute with each other and we have $x + y = (x_s + y_s) + (x_n + y_n)$. Because of the uniqueness of this decomposition we obtain the claim.

For (2): Exercise. $\square$

Definition 2.1.5. The decomposition $x = x_s + x_n$ is called the *additive Jordan-Chevalley decomposition* of $x \in \mathrm{End}(V)$. Here x_s is called the *semisimple part*, and x_n is called the *nilpotent part* of x.

Example 2.1.6. *The Jordan-Chevalley decomposition of* $x = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ *is given by*

$$x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

On the other hand,

$$x = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = x_s + x_n$$

is *not* the Jordan-Chevalley decomposition of x. It is true, though, that x_s is semisimple and x_n is nilpotent, but the two summands do not commute:

$$\left[\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right] = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}.$$

Hence $x = x_s$ is the Jordan-Chevalley decomposition of x.

Corollary 2.1.7. *Let $x = x_s + x_n$ be the Jordan-Chevalley decomposition of $x \in \mathrm{End}(V)$ and let $E \subset F \subset V$ be subspaces with $x(F) \subset E$. Then we have $x_s(F) \subset E$ and $x_n(F) \subset E$.*

Proof. For every polynomial $p(t) \in t \cdot k[t]$ we have $p(x)F \subset E$ by assumption. Then the claim follows from Proposition 2.1.4. $\square$

Lemma 2.1.8. *Let V be a finite-dimensional vector space and $x \in \mathrm{End}(V)$. If x is nilpotent respectively diagonalizable, then so is* $\mathrm{ad}(x)$.

Proof. Denote by L_x and R_x the left- respectively right multiplication by x. Then

$$\begin{aligned} \mathrm{ad}(x) &= L_x - R_x \\ [L_x, R_x] &= 0. \end{aligned}$$

Because of Lemma 2.1.3 it suffices to show that L_x, R_x inherit the nilpotency respectively diagonalizability of x. Suppose that x is nilpotent with $x^n = 0$. Then $L_x^n = L_{x^n} = 0$ and also $R_x^n = 0$. Thus L_x and R_x are nilpotent, as well as $L_x - R_x = \mathrm{ad}(x)$.

Suppose now that x is diagonalizable and $\lambda_1, \dots \lambda_n$ are the different eigenvalues of x. We have $V = \bigoplus_{j=1}^n E_{\lambda_i}(x)$. For $y \in \mathrm{End}(V)$ we write $y = \sum_{j,k=1}^n y_{jk}$ with $y_{jk} E_{\lambda_k}(x) \subset E_{\lambda_j}(x)$. Consider the block matrix of y with respect to the direct sum decomposition of V into the eigenspaces. Then we have

$$\begin{aligned} L_x y_{jk} &= \lambda_j y_{jk} \\ R_x y_{jk} &= \lambda_k y_{jk} \\ \mathrm{ad}(x) y_{jk} &= (\lambda_j - \lambda_k) y_{jk}. \end{aligned}$$

Hence $L_x, R_x \in \mathrm{End}(\mathrm{End}(V))$ are diagonalizable endomorphisms of $\mathrm{End}(V)$. $\square$

Corollary 2.1.9. *Let $x \in \operatorname{End}(V)$ and $x = x_s + x_n$ be the Jordan-Chevalley decomposition. Then* $\operatorname{ad}(x) = \operatorname{ad}(x_s) + \operatorname{ad}(x_n)$ *is the Jordan-Chevalley decomposition of* $\operatorname{ad}(x)$ *in* $\operatorname{End}(\operatorname{End}(V))$.

Proof. By Lemma 2.1.8 we know that $\operatorname{ad}(x_s)$ is diagonalizable and $\operatorname{ad}(x_n)$ is nilpotent. Both are commuting with each other because of $[\operatorname{ad}(x_s), \operatorname{ad}(x_n)] = \operatorname{ad}([x_s, x_n]) = 0$. So the claim follows by the uniqueness of the Jordan-Chevalley decomposition. $\square$

Proposition 2.1.10. *Let A be a finite-dimensional k-algebra. Then the Lie algebra* $\operatorname{Der}(A)$ *contains the semisimple and nilpotent part of all of its elements.*

Proof. Let $D = D_s + D_n$ be the Jordan-Chevalley decomposition. It suffices to show that $D_s \in \operatorname{Der}(A)$ since $\operatorname{Der}(A)$ is a vector space, so that $D_n = D - D_s \in \operatorname{Der}(A)$ follows. For $a, b \in A$ and $\lambda, \mu \in k$ we have for all $n \geq 1$

$$(D - (\lambda + \mu)\operatorname{id})^n(ab) = \sum_{k=0}^{n} \binom{n}{k} (D - \lambda \operatorname{id})^k(a)(D - \mu \operatorname{id})^{n-k}(b). \tag{2.1}$$

For $a \in E_\lambda(D_s) = H_\lambda(D)$ and $b \in E_\mu(D_s) = H_\mu(D)$ we have $ab \in E_{\lambda+\mu}(D_s) = H_{\lambda+\mu}(D)$, so that $D_s(ab) = (\lambda+\mu)ab$. On the other hand we have $D_s(a)b + aD_s(b) = \lambda ab + \mu ab = (\lambda+\mu)ab$. Since A is the direct sum of the spaces $E_\lambda(D_s)$, it follows that D_s is a derivation of A. $\square$

Let us mention the *multiplicative Jordan-Chevalley decomposition* as well. An endomorphism $x \in \operatorname{End}(V)$ is called *unipotent*, if $\operatorname{id} - x$ is nilpotent. Equivalently, all eigenvalues over an algebraic closure of k are equal to one. If $x = x_s + x_n$ is the Jordan-Chevalley decomposition of x, then $g_u = \operatorname{id} + x_s^{-1} x_n$ is unipotent. We have the following result.

Proposition 2.1.11. *Let G be an algebraic group over a perfect field k. Then for every element $g \in G(k)$ there exist unique elements $g_s, g_u \in G(k)$ with $g = g_s g_u = g_u g_s$. For all linear representations $\varphi\colon G \to GL(V)$ it holds that $\varphi(g_s)$ is semisimple and $\varphi(g_u)$ is unipotent.*

Every $g \in \operatorname{Aut}(V)$ of a finite-dimensional vector space V over an algebraically closed field has a unique multiplicative Jordan-Chevalley decomposition $g = g_s g_u = g_u g_s$, where g_s is semisimple and g_u is unipotent. For $g, h \in \operatorname{Aut}(V)$ with $gh = hg$ we have $(gh)_s = g_s h_s$ and $(gh)_u = g_u h_u$.

This may not hold for all subgroups G of $GL_n(k)$. There may be elements $g \in G$, so that g_s or g_u need not be in G. Consider the following subgroup

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\}.$$

The unique multiplicative Jordan-Chevalley of the element g of order 2 is given by

$$g = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = g_u g_s.$$

However, both g_s and g_u are not in G.

Note that this cannot happen if the subgroup is *closed*, i.e., if it is a linear algebraic group. In this case we always have $g_s, g_u \in G$ for all $g \in G$.

Back to the additive Jordan-Chevalley decomposition, one can show the following result, by using *Weyl's Theorem*, see Theorem 2.3.7:

Proposition 2.1.12. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple linear Lie algebra, where V is finite-dimensional. Then $\mathfrak{g}$ contains the semisimple and nilpotent part of all of its elements.*

For the proof see for example [**19**]. So far we only have defined a Jordan-Chevalley decomposition for *linear* Lie algebras. It is also called the *concrete* Jordan-Chevalley decomposition. For arbitrary Lie algebras we may define an *abstract* Jordan-decomposition for semisimple Lie algebras as follows.

PROPOSITION 2.1.13. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic zero. Let $x \in \mathfrak{g}$. Then there exists unique elements $s, n \in \mathfrak{g}$ with*

(1) $x = s + n$.
(2) $[s, n] = 0$.
(3) $\mathrm{ad}(s)$ *is semisimple and* $\mathrm{ad}(n)$ *is nilpotent.*

The proof depends very much on Corollary 2.2.18, saying that

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}),$$

i.e., that all derivations in this case are inner.

The element s is called the *semisimple part* of x, and n the *nilpotent part* of x. Sometimes we just call s then *ad-semisimple* and n then *ad-nilpotent*. For a linear semisimple Lie algebra $\mathfrak{g}$ the concrete and abstract Jordan-Chevalley decomposition coincide.

2.2. The Cartan criterion

Definition 2.2.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. The bilinear form $\kappa\colon \mathfrak{g}\times\mathfrak{g}\to k$ given by
$$\kappa(x,y)=\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))$$
is called the *Killing form*, or Cartan-Killing form.

More generally we have, for every finite-dimensional representation $\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)$ the bilinear form $\kappa_\rho(x,y)=\operatorname{tr}(\rho(x)\rho(y))$. Then $\kappa=\kappa_{\mathrm{ad}}$ is a special case.

Definition 2.2.2. A symmetric bilinear form $\beta\colon \mathfrak{g}\times\mathfrak{g}\to k$ is called *invariant*, if
$$\beta([x,y],z)+\beta(y,[x,z])=0$$
for all $x,y,z\in\mathfrak{g}$.

More generally, replacing the adjoint representation by an arbitrary representation ρ, we could say that β is *ρ-invariant* if $\beta(\rho(x)(v),w)+\beta(v,\rho(x)(w))=0$ for all $x\in\mathfrak{g}$ and $v,w\in V$.

Definition 2.2.3. For a bilinear form β on a vector space V and a subspace $U\subset V$ we define the *orthogonal space* with respect to β by
$$U^\perp=\{v\in V\mid \beta(v,u)=0\ \ \forall\, u\in U\}.$$

Lemma 2.2.4. *Let $\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)$ be a finite-dimensional representation. Then κ_ρ is a symmetric invariant bilinear form on $\mathfrak{g}$. If $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{g}$, then $\kappa(\mathfrak{n},\mathfrak{g})=0$ and thus $\mathfrak{n}\subset\mathfrak{g}^\perp$.*

Proof. We have $\kappa_\rho(x,y)=\operatorname{tr}(\rho(x)\rho(y))=\operatorname{tr}(\rho(y)\rho(x))=\kappa_\rho(y,x)$. The form is invariant, because we have
$$\begin{aligned}\kappa_\rho([x,y],z)&=\operatorname{tr}(\rho([x,y])\rho(z))\\&=\operatorname{tr}(\rho(x)\rho(y)\rho(z))-\operatorname{tr}(\rho(y)\rho(x)\rho(z))\\&=\operatorname{tr}(\rho(y)\rho(z)\rho(x))-\operatorname{tr}(\rho(y)\rho(x)\rho(z))\\&=\operatorname{tr}(\rho(y)\rho([z,x]))\\&=-\kappa_\rho(y,[x,z]).\end{aligned}$$
Let $x\in\mathfrak{n}$ and $y\in\mathfrak{g}$. Since $\mathfrak{n}$ is nilpotent, we may assume that $\operatorname{ad}(\mathfrak{n})$ consists of strictly upper-triangular matrices. We set $V_j=\operatorname{ad}(\mathfrak{n})^j(\mathfrak{g})$ for $j\ge 0$. Then $[\mathfrak{n},\mathfrak{g}]\subset\mathfrak{n}$ implies $\operatorname{ad}(x)\operatorname{ad}(y)V_j\subset V_{j-1}$. Hence the trace of all $\operatorname{ad}(x)\operatorname{ad}(y)$ is equal to zero, i.e., we have $\kappa(x,y)=0$. □

In particular, the Killing form of a nilpotent Lie algebra is identically zero.

Lemma 2.2.5. *For every derivation $D\in\operatorname{Der}(\mathfrak{g})$ and $x,y\in\mathfrak{g}$ we have $\kappa(D(x),y)+\kappa(x,D(y))=0$.*

Proof. Using $[D,\operatorname{ad}(x)]=\operatorname{ad}(D(x))$ and the invariance we have
$$\begin{aligned}\kappa(D(x),y)&=\operatorname{tr}(\operatorname{ad}(D(x))\operatorname{ad}(y))\\&=\operatorname{tr}([D,\operatorname{ad}(x)]\operatorname{ad}(y))\\&=-\operatorname{tr}(\operatorname{ad}(x)[D,\operatorname{ad}(y)])\\&=-\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(D(y)))\\&=-\kappa(x,D(y)).\end{aligned}$$
□

LEMMA 2.2.6. *let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation, β be a ρ-invariant bilinear form on $\mathfrak{g}$, and $U \subset V$ be a subrepresentation of V. Then $U^\perp$ is a subrepresentation of V.*

PROOF. Let $u \in U$, $v \in U^\perp$ and $x \in \mathfrak{g}$. Then $\beta(\rho(x)(v), u) = -\beta(v, \rho(x)(u)) = 0$, since $\rho(x)(u) \in U$. Hence we have $\rho(x)(v) \in U^\perp$, and thus $U^\perp$ is invariant under $\mathfrak{g}$. $\square$

Applying this lemma to the Killing form $\beta = \kappa$ we obtain the following corollary.

COROLLARY 2.2.7. *For every ideal $\mathfrak{a}$ in $\mathfrak{g}$ the orthogonal space $\mathfrak{a}^\perp$ with respect to the Killing form is an ideal in $\mathfrak{g}$.*

We'll need the following result concerning nilpotency.

PROPOSITION 2.2.8. *Let V be a finite-dimensional k-vector space over a field k of characteristic zero and $E \subseteq F$ two subspaces of $\operatorname{End}(V)$. Let $x \in \operatorname{End}(V)$ be an endomorphism with $\operatorname{ad}(x)(F) \subseteq E$. If $\operatorname{tr}(xy) = 0$ for all $y \in \operatorname{End}(V)$ with $\operatorname{ad}(y)(F) \subseteq E$, then x is nilpotent.*

PROOF. Let $M = \{y \in \operatorname{End}(V) \mid \operatorname{ad}(y)(F) \subset E\}$. Suppose first that k is algebraically closed. Let $(v_1, \ldots, v_n)$ be a basis of V consisting of eigenvectors of x_s, say $x_s v_i = \lambda_i v_i$ for suitable $\lambda_i \in k$. Let $Q = \operatorname{span}_{\mathbb{Q}}\{\lambda_1, \ldots, \lambda_n\}$ be the $\mathbb{Q}$-vector space in k spanned by the λ_i. We need to show that $Q = 0$. Then $x = x_s + x_n = x_n$ is nilpotent.

Suppose that $Q \neq 0$. Then also the dual space Q^* is nonzero. Hence there exists a non-vanishing $\mathbb{Q}$-linear map $f\colon Q \to \mathbb{Q}$. Define $y \in \operatorname{End}(V)$ by $yv_i = f(\lambda_i)v_i$ for $i = 1, \ldots, n$. We claim that $y \in M$. We have

$$\begin{aligned} \operatorname{ad}(y)(E_{ij}) &= (f(\lambda_i) - f(\lambda_j))E_{ij} \\ &= f(\lambda_i - \lambda_j)E_{ij} \end{aligned}$$

for all i and j. Hence

$$E_\mu(\operatorname{ad}(y)) = \bigoplus_{f(\lambda)=\mu} E_\lambda(\operatorname{ad}(x_s)),$$

so in particular $\operatorname{ad}(y)(F) \subset E$, since $\operatorname{ad}(x_s)(F) \subseteq E$, see Corollary 2.1.7. Hence by assumption we have

$$0 = \operatorname{tr}(xy) = \sum_{i=1}^n \lambda_i f(\lambda_i).$$

It follows that

$$0 = f(0) = f(\operatorname{tr}(xy)) = \sum_{i=1}^n f(\lambda_i)^2.$$

So $f(\lambda_i) = 0$ for all i, contradicting the assumption that $f \neq 0$.

If k is not algebraically closed we may replace V by $V \otimes_k \mathbb{F}$, where F is an algebraic closure of k. Then we consider instead of $x, y \in \operatorname{End}(V)$ their $\mathbb{F}$-linear extensions in $\operatorname{End}(V \otimes_k \mathbb{F})$. It is not difficult to see how to finish the proof. $\square$

Now we'll introduce the linear case of the so-called *Cartan-criterion*.

PROPOSITION 2.2.9. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a linear Lie algebra over a field k of characteristic zero. Then $\mathfrak{g}$ is solvable if and only if $\operatorname{tr}(xy) = 0$ for all $x \in \mathfrak{g}$ and all $y \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. We may assume that k is algebraically closed, because otherwise we may replace V by $V \otimes_k \mathbb{F}$ and $\mathfrak{g}$ by $\mathfrak{g} \otimes_k \mathbb{F}$, where F is an algebraic closure of k.

Suppose that $\mathfrak{g}$ is solvable. Then we may, by Lie's Theorem, identify $\mathfrak{g}$ with a Lie subalgebra of $\mathfrak{t}_n(k)$, and $[\mathfrak{g}, \mathfrak{g}]$ with a Lie subalgebra of $\mathfrak{n}_n(k)$. Then we have, for $x \in \mathfrak{t}_n(k)$ and $y \in \mathfrak{n}_n(k)$ that $\operatorname{tr}(xy) = 0$. The argument for this is the same as the one in Lemma 2.2.4.

Conversely suppose that $\operatorname{tr}(xy) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$. By Corollary 1.6.30 it suffices to show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent in order to conclude that $\mathfrak{g}$ is solvable. By Engel's Theorem $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent if and only if all $\operatorname{ad}(x)$ for $x \in [\mathfrak{g}, \mathfrak{g}]$ are nilpotent. It is enough to show that all $x \in [\mathfrak{g}, \mathfrak{g}]$ are nilpotent. By Lemma 2.1.8 then also all $\operatorname{ad}(x)$ are nilpotent. Fix an arbitrary $x \in [\mathfrak{g}, \mathfrak{g}]$. We want to apply Proposition 2.2.8 with $E = [\mathfrak{g}, \mathfrak{g}]$ and $F = \mathfrak{g}$ and with $\mathfrak{g} \subset M = \{y \in \operatorname{End}(V) \mid [y, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]\}$. For this we need to show that $\operatorname{tr}(xy) = 0$ for all $y \in M$. Since x is the sum of commutators, and the trace is linear it suffices to show that $\operatorname{tr}([a, b]y) = 0$ with $a, b \in \mathfrak{g}$ for all $y \in M$. By the invariance of the trace form we have $\operatorname{tr}([a, b]y) = \operatorname{tr}(a[b, y])$. So we have $[b, y] \in [\mathfrak{g}, \mathfrak{g}]$ because of $y \in M$. By assumption the trace on the right hand side is always zero and Proposition 2.2.8 yields that $x \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

Corollary 2.2.10. *Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic zero. Then $\mathfrak{g}$ is solvable if and only if $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

Proof. Suppose that $\mathfrak{g}$ is solvable. Then $\operatorname{ad}(\mathfrak{g})$ is solvable, too, as an homomorphic image. By the above proposition it follows that $\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$ for all $x \in \mathfrak{g}$ and all $y \in [\mathfrak{g}, \mathfrak{g}]$.

Conversely, suppose that $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$. Then $\operatorname{ad}(\mathfrak{g})$ is solvable by the Cartan criterion. Then also $\mathfrak{g}$ is solvable, because of $\operatorname{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$. □

Remark 2.2.11. We may rewrite the condition $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$ as $[\mathfrak{g}, \mathfrak{g}]^\perp = \mathfrak{g}$. It means that $\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$ for all $x \in \mathfrak{g}$ and all $y \in [\mathfrak{g}, \mathfrak{g}]$. In general we have $[\mathfrak{g}, \mathfrak{g}]^\perp = \operatorname{rad}(\mathfrak{g})$. Furthermore $\mathfrak{g}$ is solvable if and only if $\kappa_\rho(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$, where ρ is some finite-dimensional representation of $\mathfrak{g}$.

Now we'll come to Cartan's criterion for *semisimplicity.*

Proposition 2.2.12. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero. The the following assertions are equivalent.*

(1) $\mathfrak{g}$ *is semisimple.*
(2) *The solvable radical* $\operatorname{rad}(\mathfrak{g})$ *is zero.*
(3) *The Killing form on* $\mathfrak{g}$ *is non-degenerate.*

Proof. (1) $\Rightarrow$ (2): This is exactly the assertion of Lemma 1.6.22.

(2) $\Rightarrow$ (3): Let $\mathfrak{a} = \mathfrak{g}^\perp$ be the orthogonal space of $\mathfrak{g}$ with respect to κ. Then $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ because of Corollary 2.2.7. We claim that $\kappa_{\mathfrak{a}}(x, y) = \kappa(x, y)$ for all $x, y \in \mathfrak{a}$. The endomorphisms $\operatorname{ad}(x)$ and $\operatorname{ad}(y)$ map $\mathfrak{g}$ to $\mathfrak{a}$. Thus this holds for $\operatorname{ad}(x)\operatorname{ad}(y)$, too. Hence we have

$$\begin{aligned}\kappa_{\mathfrak{a}}(x, y) &= \operatorname{tr}(\operatorname{ad}(x)_{\mathfrak{a}}\operatorname{ad}(y)_{\mathfrak{a}}) = \operatorname{tr}((\operatorname{ad}(x)\operatorname{ad}(y))_{\mathfrak{a}})\\ &= \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = \kappa(x, y).\end{aligned}$$

So we have

$$\kappa_{\mathfrak{a}}(\mathfrak{a}, [\mathfrak{a}, \mathfrak{a}]) = \kappa(\mathfrak{a}, [\mathfrak{a}, \mathfrak{a}]) \subset \kappa(\mathfrak{a}, \mathfrak{g}) = 0.$$

It follows from Corollary 2.2.10 that $\mathfrak{a}$ is solvable. Hence $\mathfrak{a} \subset \operatorname{rad}(\mathfrak{g}) = 0$ and κ is non-degenerate.

(3) $\Rightarrow$ (2): Let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{g}$. For $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$ the space $\operatorname{im}(\operatorname{ad}(x)\operatorname{ad}(y))$ is a Lie subalgebra of $\mathfrak{a}$ and hence we have $(\operatorname{ad}(x)\operatorname{ad}(y))^2 = 0$, since $\mathfrak{a}$ is abelian. Thus $\operatorname{ad}(x)\operatorname{ad}(y)$ is nilpotent and $\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$. So we have $\kappa(\mathfrak{a}, \mathfrak{g}) = 0$. Since κ is non-degenerate we obtain $\mathfrak{a} = 0$. Hence $\operatorname{rad}(\mathfrak{g}) = 0$.

(2) $\Rightarrow$ (1): Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then $\mathfrak{b} = \mathfrak{a}^{\perp} \cap \mathfrak{a}$ is an ideal in $\mathfrak{g}$. The restriction of κ on $\mathfrak{b} \times \mathfrak{b}$ vanishes. For $a, b \in \mathfrak{b}$ and $x \in \mathfrak{g}$ we have $[b, x] \in \mathfrak{b}$, hence in particular $[b, x] \in \mathfrak{a}^{\perp}$ and $\kappa([a, b], x) = \kappa(a, [b, x]) = 0$. Then $\mathfrak{b}$ is solvable by Corollary 2.2.10, so that $\mathfrak{b} \subset \operatorname{rad}(\mathfrak{g}) = 0$ and $\mathfrak{a}^{\perp} \cap \mathfrak{a} = 0$. We already have shown that (2) implies that κ is non-degenerate. Hence we have $\dim \mathfrak{a}^{\perp} = \dim \mathfrak{g} - \dim \mathfrak{a}$, and therefore $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ is a direct sum of ideals. Hence $\mathfrak{g}$ is reductive. Because of $Z(\mathfrak{g}) \subset \operatorname{rad}(\mathfrak{g}) = 0$, by Proposition 1.4.5, $\mathfrak{g}$ is semisimple. □

REMARK 2.2.13. The implication (3) $\Rightarrow$ (1) remains true in characteristic $p > 0$. However, the converse implication (1) $\Rightarrow$ (3) need not be true in characteristic $p > 0$. For example, the classical Lie algebra $\mathfrak{psl}_n(k)$ for $p \mid n$ is simple ($\mathfrak{sl}_n(k)$ has a 1-dimensional center $\mathfrak{z}$ for $p \mid n$ with simple quotient $\mathfrak{psl}_n(k) = \mathfrak{sl}_n(k)/\mathfrak{z}$). However, the Killing form of $\mathfrak{psl}_n(k)$ is identically zero.

We want to formulate the following fact, which we have shown in the proof above, as a lemma.

LEMMA 2.2.14. *Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then the Killing form $\kappa_{\mathfrak{a}}$ of $\mathfrak{a}$ is the restriction of the Killing form κ of $\mathfrak{g}$ on $\mathfrak{a} \times \mathfrak{a}$.*

COROLLARY 2.2.15. *Let $n \geq 2$ and k be a field of characteristic zero. Then the Lie algebra $\mathfrak{sl}_n(k)$ is semisimple.*

PROOF. By Cartan's criterion it suffices to show that the Killing form on $\mathfrak{sl}_n(k)$ is non-degenerate. Let $X = (x_{ij}) \in \mathfrak{sl}_n(k)$ be given such that $\kappa(X, Y) = 0$ for all $Y \in \mathfrak{sl}_n(k)$. A direct calculation shows that $\kappa(X, Y) = 2n \operatorname{tr}(XY)$. So for $Y = E_{ij}$ with $i \neq j$ we have

$$0 = \kappa(X, E_{ij}) = 2n x_{ji}.$$

Since $2n \neq 0$, X is a diagonal matrix. For $Y = E_{ii} - E_{jj}$ we obtain

$$0 = \kappa(X, E_{ii} - E_{jj}) = 2n(x_{ii} - x_{jj})$$

for $1 \leq i, j \leq n$. Hence $X = \lambda E_n$. So $\operatorname{tr}(X) = 0$ implies that $X = 0$. Hence κ is non-degenerate. □

COROLLARY 2.2.16. *Let $\mathfrak{g}$ be a Lie algebra with solvable radical* $\operatorname{rad}(\mathfrak{g})$. *Then* $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ *is semisimple.*

PROOF. Because of Lemma 1.6.21 we have $\operatorname{rad}(\mathfrak{s}) = 0$ for $\mathfrak{s} = \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$. Then $\mathfrak{s}$ is semisimple by Proposition 2.2.12. □

PROPOSITION 2.2.17. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero, and $\mathfrak{a}$ be a semisimple ideal in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ and $\mathfrak{a}^{\perp}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.*

PROOF. Since $\mathfrak{a}$ is semisimple, the Killing form $\kappa_{\mathfrak{a}} = \kappa_{|\mathfrak{a} \times \mathfrak{a}}$ of $\mathfrak{a}$ is non-degenerate. As above we obtain that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ is the direct sum of ideals. Because of $Z(\mathfrak{a}) = 0$ the centralizer of $\mathfrak{a}$ is $\mathfrak{a}^{\perp}$, hence $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}^{\perp}$. □

COROLLARY 2.2.18. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field k of characteristic zero. Then we have* $\operatorname{Der}(\mathfrak{g}) = \operatorname{ad}(\mathfrak{g})$. *In other words, all derivations of $\mathfrak{g}$ are inner.*

PROOF. We apply Proposition 2.2.17 to the Lie algebra $\mathrm{Der}(\mathfrak{g})$. Since $Z(\mathfrak{g}) = 0$ we have $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$, so $\mathrm{ad}(\mathfrak{g})$ is a semisimple ideal in $\mathrm{Der}(\mathfrak{g})$. Hence we have $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus \mathrm{ad}(\mathfrak{g})^\perp$. But $\mathrm{ad}(\mathfrak{g})^\perp$ is the centralizer of $\mathrm{ad}(\mathfrak{g})$ in $\mathrm{Der}(\mathfrak{g})$ by Proposition 2.2.17, and this is zero. For $D \in \mathrm{ad}(\mathfrak{g})^\perp$ we have

$$\mathrm{ad}(D(x)) = [D, \mathrm{ad}(x)] = 0 \quad \forall\, x \in \mathfrak{g}.$$

Since ad is injective, we obtain $D = 0$ and hence $\mathrm{ad}(\mathfrak{g})^\perp = 0$. $\square$

2.3. Weyl's Theorem

Weyl's Theorem is a central result in the theory of representations of semisimple Lie algebras. Note that we always assume that the Lie algebras and their representations are finite-dimensional. The original proof of this result by Weyl uses integration on compact Lie groups. Afterwards a purely algebraic proof was found by van der Waerden, based on work of the physicist Hendrik Casimir, who lived from 1909 till 2000. Brauer discovered in 1937 another algebraic proof. The result also follows from the Whitehead Lemma about Lie algebra cohomology.

Let us give an elementary algebraic proof using *Casimir elements* associated to a non-degenerate bilinear form β. The radical of β is defined by

$$\operatorname{rad}(\beta) = \{x \in \mathfrak{g} \mid \beta(x, y) = 0 \; \forall \, y \in \mathfrak{g}\}.$$

So β is non-degenerate if and only if $\operatorname{rad}(\beta) = 0$. If this is the case then for a given basis $(x_1, \dots, x_n)$ of $\mathfrak{g}$ there exists a unique dual basis $(x^1, \dots, x^n)$ with respect to β, given by $\beta(x_i, x^j) = \delta_{ij}$.

LEMMA 2.3.1. *Let β be a non-degenerate invariant bilinear form on a Lie algebra $\mathfrak{g}$ with basis $(x_1, \dots, x_n)$ and $(x^1, \dots, x^n)$ be the dual base with respect to β. Let $\rho \colon \mathfrak{g} \to A$ be a homomorphism into an associative algebra, i.e., satisfying also $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x) = [\rho(x), \rho(y)]$. Then the element*

$$\Omega(\beta, \rho) = \sum_{j=1}^{n} \rho(x^j)\rho(x_j)$$

in A commutes with all $\rho(x)$.

PROOF. Let $z \in \mathfrak{g}$. The we can write

$$[z, x_j] = \sum_{k=1}^{n} a_{kj} x_k,$$
$$[z, x^j] = \sum_{k=1}^{n} a^{jk} x^k$$

with elements $a_{kj}, a^{jk} \in k$. We have

$$a_{kj} = \beta([z, x_j], x^k) = -\beta(x_j, [z, x^k]) = -a^{kj}.$$

Then

$$\rho(x^j)\rho(x_j)\rho(z) - \rho(z)\rho(x^j)\rho(x_j) = \rho(x^j)\Big(\rho(x_j)\rho(z) - \rho(z)\rho(x_j)\Big) - \Big(\rho(z)\rho(x^j) - \rho(x^j)\rho(z)\Big)\rho(x_j).$$

It follows, writing $\Omega = \Omega(\beta, \rho)$,

$$
\begin{aligned}
\Omega\rho(z) - \rho(z)\Omega &= \sum_{j=1}^{n} \rho(x^j)\rho(x_j)\rho(z) - \rho(z)\rho(x^j)\rho(x_j) \\
&= \sum_{j=1}^{n} \rho(x^j)[\rho(x_j), \rho(z)] - [\rho(z), \rho(x^j)]\rho(x_j) \\
&= \sum_{j=1}^{n} \rho(x^j)\rho([x_j, z]) - \rho([z, x^j])\rho(x_j) \\
&= \sum_{j,k=1}^{n} -a_{kj}\rho(x^j)\rho(x_k) - a^{jk}\rho(x^k)\rho(x_j) \\
&= \sum_{j,k=1}^{n} a^{kj}\rho(x^j)\rho(x_k) - a^{jk}\rho(x^k)\rho(x_j) \\
&= \sum_{j,k=1}^{n} a^{kj}\rho(x^j)\rho(x_k) - \sum_{j,k=1}^{n} a^{kj}\rho(x^j)\rho(x_k) \\
&= 0.
\end{aligned}
$$

□

DEFINITION 2.3.2. Let β be a non-degenerate invariant bilinear form on $\mathfrak{g}$ and ρ be a representation of $\mathfrak{g}$ as above. Then the element $\Omega(\beta, \rho) \in A$ is called a (quadratic) *Casimir element* with respect to β and ρ.

It is easy to see that $\Omega(\beta, \rho)$ does not depend on the choice of a basis for $\mathfrak{g}$.

LEMMA 2.3.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$. Then $\beta(x, y) = \operatorname{tr}(\rho(x)\rho(y))$ is a symmetric non-degenerate invariant bilinear form on $\mathfrak{g}$.*

PROOF. Any trace form is invariant because of $\operatorname{tr}([A, B]C) = \operatorname{tr}(A[B, C])$ for $A, B, C \in \operatorname{End}(V)$. Hence $\operatorname{rad}(\beta)$ is an ideal in $\mathfrak{g}$. Since ρ is injective we have $\rho(\operatorname{rad}(\beta)) \cong \operatorname{rad}(\beta)$. By definition of the radical the trace form vanishes there, so that $\operatorname{rad}(\beta)$ is solvable by the linear Cartan criterion. Hence we have $\operatorname{rad}(\beta) \subset \operatorname{rad}(\mathfrak{g}) = 0$ and β is non-degenerate. □

We can define the Casimir element with respect to β and ρ in particular for the algebra $A = \operatorname{End}(V)$. Then ρ is a Lie algebra representation and $\Omega(\beta, \rho)$ is an endomorphism of V, also called *Casimir operator*. We have

$$
\begin{aligned}
\operatorname{tr}(\Omega) &= \sum_{j=1}^{n} \operatorname{tr}(\rho(x_j)\rho(x^j)) \\
&= \sum_{j=1}^{n} \beta(x_j, x^j) \\
&= \sum_{j=1}^{n} 1 = \dim \mathfrak{g}.
\end{aligned}
$$

A possible faithful representation of $\mathfrak{g}$ is $\rho = \mathrm{ad}$, because $\mathfrak{g}$ is semisimple and hence $\ker(\mathrm{ad}) = Z(\mathfrak{g}) = 0$.

EXAMPLE 2.3.4. *Let $\mathfrak{g} = \mathfrak{sl}_2(k)$, $V = k^2$ and ρ be the identity map $\mathfrak{g} \to \mathfrak{gl}(V)$. Let (x, y, h) be the standard basis of $\mathfrak{g}$ and β be the trace from on $\mathfrak{g}$. Then we have*

$$\Omega(\beta, \mathrm{id}) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

To see this, first note that the dual basis with respect to β is given by $(y, x, \frac{h}{2})$. Then

$$\Omega(\beta, \mathrm{id}) = \rho(x)\rho(y) + \rho(y)\rho(x) + \frac{\rho(h)^2}{2} = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix},$$

recalling from Example 1.2.16 that

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So Ω acts here as a scalar, i.e., by $\lambda \cdot \mathrm{id}$ with $\lambda \in k$. This is clear. Indeed, the representation ρ is simple and Ω commutes with all $\rho(x)$ by Lemma 2.3.1. Then Schur's Lemma 1.2.31 gives $\Omega = \lambda \cdot \mathrm{id}$. More precisely, we have $\lambda = \frac{3}{2} = \frac{\dim \mathfrak{g}}{\dim V}$.

Let us mention another consequence of Schur's Lemma here, see [**19**].

LEMMA 2.3.5. *Let $\mathfrak{g}$ be a simple Lie algebra and $\alpha(x, y)$, $\beta(x, y)$ be two symmetric nondegenerate invariant bilinear forms on $\mathfrak{g}$. Then there is a nonzero scalar $\mu \in k^*$ with $\alpha(x, y) = \mu\beta(x, y)$ for all $x, y \in \mathfrak{g}$.*

In particular the Killing form of simple Lie algebras is a scalar multiple of the trace form. The following table shows a few examples. Note that $\mathfrak{gl}(n)$ is reductive but not simple.

$\mathfrak{g}$	$\kappa(x, y)$
$\mathfrak{gl}(n)$, $n \geq 2$	$2n \operatorname{tr}(xy) - 2 \operatorname{tr}(x) \operatorname{tr}(y)$
$\mathfrak{sl}(n)$, $n \geq 2$	$2n \operatorname{tr}(xy)$
$\mathfrak{so}(n)$, $n \geq 3$	$(n-2) \operatorname{tr}(xy)$
$\mathfrak{sp}(2n)$, $n \geq 1$	$2(n+1) \operatorname{tr}(xy)$

REMARK 2.3.6. One can ask how we may express the Killing form $\operatorname{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ and more generally also $\operatorname{tr}(\mathrm{ad}^2(x)\,\mathrm{ad}^2(y))$ by the trace form. We may write

$$\begin{aligned} \operatorname{tr}((\mathrm{ad}(x))^2(\mathrm{ad}(y))^2) &= \alpha_n \operatorname{tr}(x^2y^2) + \beta_n \operatorname{tr}(xyxy) + \gamma_n \operatorname{tr}(x^2) \operatorname{tr}(y^2) \\ &\quad + \delta_n(\operatorname{tr}(xy))^2 \end{aligned}$$

with certain scalars $\alpha_n, \beta_n, \gamma_n, \delta_n$ depending on n. To give an example, let $\mathfrak{g} = \mathfrak{sl}(n)$. Then it is easy to see that we have, for all $x, y \in \mathfrak{sl}(n)$,

$$\operatorname{tr}((\mathrm{ad}(x))^2(\mathrm{ad}(y))^2) = 2n \operatorname{tr}(x^2y^2) + 2 \operatorname{tr}(x^2) \operatorname{tr}(y^2) + 4(\operatorname{tr}(xy))^2.$$

Indeed, for $n \geq 4$ we obtain from the above the following system of linear equations

$$\begin{aligned} \beta_n + \delta_n &= 4, \\ \alpha_n + \beta_n + 2\gamma_n + 2\delta_n &= 2n + 12, \\ \alpha_n + \beta_n + 4\gamma_n + \delta_n &= 2n + 12, \\ \gamma_n &= 2. \end{aligned}$$

This system has a unique solution as above. How do we obtain the linear equations? We explicitly compute the terms with respect to a standard basis of $\mathfrak{sl}(n)$. The adjoint representation is given by the formula (1.1). For example, for $h_i = E_{ii} - E_{i+1,i+1}$ we obtain that $\operatorname{ad}(h_i)$ is a diagonal matrix with $2(n-2)$ entries 1, $2(n-2)$ entries -1, once 2, once -2 and otherwise zeros on the diagonal. Hence $\operatorname{ad}^2(h_i)$ is a diagonal matrix with $4(n-2)$ entries 1 and 2 entries 4 on the diagonal, yielding trace $4n$. So we have $\operatorname{tr}(\operatorname{ad}^4(h_i)) = 4n + 24$. Plugging into our Ansatz $x = y = h_i$, we obtain

$$4n + 24 = 2\alpha_n + 2\beta_n + 4\gamma_n + 4\delta_n.$$

This is the second linear equation listed, providing we have $2 \neq 0$. Similarly we obtain the other equations.

We also can show that, for all $x, y \in \mathfrak{so}(n)$,

$$\begin{aligned} \operatorname{tr}((\operatorname{ad}(x))^2(\operatorname{ad}(y))^2) = (n-6)\operatorname{tr}(x^2y^2) - 2\operatorname{tr}(xyxy) + \operatorname{tr}(x^2)\operatorname{tr}(y^2) \\ + 2(\operatorname{tr}(xy))^2. \end{aligned}$$

Now we are ready to prove Weyl's Theorem.

Theorem 2.3.7 (Weyl). *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra over a field k of characteristic zero. Then ρ is semisimple.*

Proof. *Step* (1)*:* Since V is semisimple as $\mathfrak{g}$-module if and only if V is semisimple as $\rho(\mathfrak{g})$-module, we may replace $\mathfrak{g}$ by $\rho(\mathfrak{g})$ and assume that $\mathfrak{g} \subset \mathfrak{gl}(V)$ and $\rho = \mathrm{id}$. Then $\beta(x, y) = \operatorname{tr}(xy)$ is a symmetric invariant bilinear form. It is non-degenerate, since $\operatorname{rad}(\beta)$ is a solvable ideal by the linear Cartan criterion. Hence $\operatorname{rad}(\beta) \subset \operatorname{rad}(\mathfrak{g}) = 0$. Thus we may define the associated Casimir element $\Omega = \sum_{j=1}^n \rho(x_j)\rho(x^j) \in \operatorname{End}(V)$. It lies in $\operatorname{End}_{\mathfrak{g}}(V) = \{A \in \operatorname{End}(V) \mid Ax = xA \;\forall\, x \in \mathfrak{g}\}$, with $\operatorname{tr}(\Omega) = \dim \mathfrak{g} = n$. Note that Ax and xA are products of endomorphisms. Suppose that ρ is a simple representation. Then Ω is an automorphism of V and we have $\operatorname{tr}(\Omega) \neq 0$, since $\mathfrak{g} \neq 0$ and k has characteristic zero. So $\Omega \neq 0$ and Schur's Lemma yields $\Omega = \lambda \cdot \mathrm{id}$ with $\lambda \neq 0$.

Step (2)*:* Let $W \subset V$ be a subrepresentation of codimension 1. We'll show that W has a complement. Since $\dim V/W = 1$ we know that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ acts trivially on V/W, because all commutators of endomorphisms of a 1-dimensional vector space vanish.

(a): We'll show that we may assume that W is simple. Indeed, suppose that the claim in (2) holds for all simple modules. Then we use induction on $\dim V$ to show that the claim holds in general. So let $0 \neq V_1 \subset V$ be a *minimal* submodule. For V being simple there is nothing to show. So assume that $V_1 \neq V$ is a proper submodule. If $V_1 \cap W = 0$, then V_1 is already a module complement for W, since we then have $W + V_1 = V$ because of $\dim V/W = 1$. Otherwise, if $V_1 \cap W \neq 0$, then we obtain $V_1 \subset W$ from the minimality of V_1. Now we can apply the induction hypothesis on V/V_1. So the submodule W/V_1 of codimension 1 has a module complement U, which we can write as

$$U = U'/V_1$$

for a submodule $U' \supset V_1$ of V. Hence we have $W/V_1 \oplus U'/V_1 = V/V_1$ with $\dim U'/V_1 = 1$. Thus $V_1 \subset U'$ is a simple submodule of codimension 1. We may apply the claim (2) to it, to find a module complement V_2 to V_1 in U', so $V_1 \oplus V_2 = U'$. Then V_2 is a module complement to W in V, hence $V = W \oplus V_2$, because $\dim W + \dim V_2 = (\dim V - 1) + 1 = \dim V$ and $W \cap V_2 = 0$.

(b): We claim that $\mathfrak{g}$ acts faithfully on W. Let $\mathfrak{a} = \{x \in \mathfrak{g} \mid x.W = 0\}$. This is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, so is $\mathfrak{a}$ and we have $\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}]$. Because of $\mathfrak{g}.(V/W) = 0$ we have $\mathfrak{g}.V \subset W$, and hence $[\mathfrak{a}, \mathfrak{a}] = 0$, because for $x, y \in \mathfrak{a}$ we have $xy.V \subset x.W = 0$. This we have $\mathfrak{a} = 0$ and therefore the representation of of $\mathfrak{g}$ on W is faithful.

(c): We can now prove the claim (2). So let W be simple. Let ρ_V be the representation of $\mathfrak{g}$ on V and ρ_W be the restriction on W. The latter is faithful, see (b). Thus we can define the *Casimir operator* $\Omega_W \in \operatorname{End}(V)$, which is associated to the bilinear form $\kappa_{\rho_W}(x, y) = \operatorname{tr}(\rho_W(x)\rho_W(y))$. Because of (1), Ω_W is injective on the simple module W, because it acts there as a nonzero scalar. Hence $\ker(\Omega_W)$ is a 1-dimensional $\mathfrak{g}$-submodule of V, having trivial intersection with W. So we have $V = W \oplus \ker(\Omega_W)$. Thus the desired complement to W in V is given by $\ker(\Omega_W)$.

Step (3)*:* Now we can prove the general case, where $W \subset V$ is an arbitrary submodule. We may equip the space $\operatorname{Hom}(V, W)$ with a structure of a $\mathfrak{g}$-module by

$$x.\varphi = \rho_W(x) \circ \varphi - \varphi \circ \rho_V(x), \quad x \in \mathfrak{g}, \varphi \in \operatorname{Hom}(V, W).$$

Then the space

$$U = \{\varphi \in \operatorname{Hom}(V, W) \mid \varphi_{|W} \in k \cdot \mathrm{id}_W\}$$

is a $\mathfrak{g}$-submodule, since for $\varphi \in U$ we even have $(x.\varphi)(W) = 0$. Indeed, we obtain, with $\varphi_{|W} = \lambda \cdot \mathrm{id}_W$ and $w \in W$, that

$$\begin{aligned}(x.\varphi)(w) &= x.\varphi(w) - \varphi(x.w)\\ &= x.(\lambda w) - \lambda x.w\\ &= 0.\end{aligned}$$

Hence $U_0 = \{\varphi \in U \mid \varphi(W) = 0\}$ is a submodule of U with $\dim U/U_0 = 1$. With (2) we find a $\psi \in U$ with $k\psi \oplus U_0 = U$. By applying a suitable scalar multiplication we may assume that $\psi_{|W} = \mathrm{id}_W$. Then the 1-dimensional $\mathfrak{g}$-module $k\psi$ is trivial, and hence we have $x.\psi = 0$ and $x.\psi(v) - \psi(x.v) = (x.\psi)(v) = 0$. Thus ψ is a module homomorphism and therefore $\ker(\psi)$ is a submodule of V. But then $\ker(\psi)$ is a complementary submodule to W in V, i.e., we have $V = \ker(\psi) \oplus W$ and V is semisimple. $\square$

REMARK 2.3.8. The "converse" of Weyl's Theorem also holds. If *every* finite-dimensional representation of $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is semisimple. In fact, since the adjoint representation is semisimple, every ideal in $\mathfrak{g}$ has a complementary ideal and hence can be viewed as a quotient of $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is not semisimple. Then $\mathfrak{g}$ has a commutative quotient, hence also a 1-dimensional quotient. But the Lie algebra $\mathfrak{g} = k$ also has non-semisimple representations, see Example 1.2.28. This is a contradiction.

2.4. Levi's Theorem

In this section all Lie algebras are finite-dimensional over a field k of characteristic zero. Levi's Theorem says that any Lie algebra $\mathfrak{g}$ has a semisimple subalgebra $\mathfrak{s}$, a so-called *Levi complement*, with $\mathfrak{g} \cong \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$. In other words, the short exact sequence of Lie algebras

$$0 \to \operatorname{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{s} \to 0$$

splits. In this case, $\mathfrak{s} \cong \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$. A Levi complement need not be unique. *Malcev's Theorem* says that all Levi complements are conjugated by special automorphisms of $\mathfrak{g}$.

The existence of Levi complements reduces the classification of Lie algebras to a large extend to the classification of semisimple and solvable Lie algebras. The semisimple Lie algebras can be classified, whereas the solvable ones cannot in general. We start with the following lemma.

Lemma 2.4.1. *Let $\alpha\colon \mathfrak{g} \to \mathfrak{h}$ be a surjective homomorphisms of Lie algebras. Then we have $\alpha(\operatorname{rad}(\mathfrak{g})) = \operatorname{rad}(\mathfrak{h})$.*

Proof. Since $\operatorname{rad}(\mathfrak{g})$ is a solvable ideal in $\mathfrak{g}$, also $\alpha(\operatorname{rad}(\mathfrak{g}))$ is a solvable ideal in $\mathfrak{h}$, since α is a surjective homomorphism. We have $\alpha(\operatorname{rad}(\mathfrak{g})) \subset \operatorname{rad}(\mathfrak{h})$. To see this, we note that $\alpha(\operatorname{rad}(\mathfrak{g}))$ is an ideal in $\mathfrak{h}$, because

$$[\mathfrak{h}, \alpha(\operatorname{rad}(\mathfrak{g}))] = [\alpha(\mathfrak{g}), \alpha(\operatorname{rad}(\mathfrak{g}))] = \alpha([\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]) \subset \alpha(\operatorname{rad}(\mathfrak{g})).$$

It is solvable since homomorphic images of solvable Lie algebras are solvable. Conversely, consider the quotient map $\pi\colon \mathfrak{h} \to \mathfrak{h}/\alpha(\operatorname{rad}(\mathfrak{g}))$ and the homomorphism $\beta = \pi \circ \alpha\colon \mathfrak{g} \to \mathfrak{h}/\alpha(\operatorname{rad}(\mathfrak{g}))$. Because of $\operatorname{rad}(\mathfrak{g}) \subset \ker(\beta)$, β factorizes to a surjective homomorphisms $\beta'\colon \mathfrak{g}/\operatorname{rad}(\mathfrak{g}) \to \mathfrak{h}/\alpha(\operatorname{rad}(\mathfrak{g}))$. Since $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is semisimple by Corollary 2.2.16, also $\mathfrak{h}/\alpha(\operatorname{rad}(\mathfrak{g}))$ is semisimple, being a homomorphic image of $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$. Hence we have $\pi(\operatorname{rad}(\mathfrak{h})) \subset \operatorname{rad}(\mathfrak{h}/\alpha(\operatorname{rad}(\mathfrak{g}))) = 0$, and $\operatorname{rad}(\mathfrak{h}) \subset \alpha(\operatorname{rad}(\mathfrak{g}))$. $\square$

Lemma 2.4.2. *Let V be a $\mathfrak{g}$-module, $\mathfrak{a}$ be an ideal in $\mathfrak{g}$, and $Z_{\mathfrak{a}}(w) = \{x \in \mathfrak{a} \mid x.w = 0\}$ for $w \in V$. Let $v \in V$ be an element with $\mathfrak{g}.v = \mathfrak{a}.v$ and $Z_{\mathfrak{a}}(v) = 0$. Then we have $\mathfrak{g} \cong Z_{\mathfrak{g}}(v) \ltimes \mathfrak{a}$.*

Proof. By assumption $Z_{\mathfrak{g}}(v) \cap \mathfrak{a} = Z_{\mathfrak{a}}(v) = 0$ and $\mathfrak{a}$ is an ideal. Moreover $Z_{\mathfrak{g}}(v)$ is a subalgebra of $\mathfrak{g}$. So we only need to show that $Z_{\mathfrak{g}}(v) + \mathfrak{a} = \mathfrak{g}$. Consider the linear map $\varphi\colon \mathfrak{g} \to V$, $x \mapsto x.v$. Because of $\mathfrak{g}.v = \mathfrak{a}.v$ we have $\varphi(\mathfrak{g}) = \varphi(\mathfrak{a})$, and hence $\mathfrak{g} = \ker(\varphi) + \mathfrak{a} = Z_{\mathfrak{g}}(v) + \mathfrak{a}$. $\square$

Theorem 2.4.3 (Levi). *Every short exact sequence of Lie algebras in characteristic zero*

$$0 \to \operatorname{rad}(\mathfrak{g}) \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\alpha} \mathfrak{s} \to 0$$

with a semisimple Lie algebra $\mathfrak{s}$ splits, i.e., there is a homomorphism $\beta\colon \mathfrak{s} \to \mathfrak{g}$ with $\alpha \circ \beta = \operatorname{id}_{\mathfrak{s}}$.

Proof. Let $\mathfrak{a} = \ker(\alpha) = \iota(\operatorname{rad}(\mathfrak{g}))$. We will prove the result by induction over $\dim \mathfrak{a}$. For $\mathfrak{a} = 0$, the homomorphism α is injective and surjective. Then the claim follows with $\beta = \alpha^{-1}$. So we may assume that $\dim(\mathfrak{a}) \geq 1$. We will make a case distinction.

Case 1: There exists a proper minimal ideal $\mathfrak{a}_1$ in $\mathfrak{a} = \ker(\alpha)$, which is different from 0 and $\mathfrak{a}$. Then α factorizes to a surjective homomorphism

$$\alpha_1\colon \mathfrak{g}/\mathfrak{a}_1 \to \mathfrak{s}$$

with $\dim \ker(\alpha_1) = \dim \mathfrak{a} - \dim \mathfrak{a}_1 < \dim \mathfrak{a} = \dim \ker(\alpha)$. By induction hypothesis there exists a homomorphism

$$\beta_1\colon \mathfrak{s} \to \mathfrak{g}/\mathfrak{a}_1 \quad \text{with} \quad \alpha_1 \circ \beta_1 = \operatorname{id}_{\mathfrak{s}}.$$

Let $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}_1$ be the quotient map and

$$\mathfrak{b} = q^{-1}(\beta_1(\mathfrak{s}))$$

the preimage of $\beta_1(\mathfrak{s})$ under q. Then $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ and the homomorphism

$$\widetilde{\alpha} = q\mid_{\mathfrak{b}}\colon \mathfrak{b} \to \beta_1(\mathfrak{s}) \cong \mathfrak{s}, \quad x \mapsto x + \mathfrak{a}_1$$

is surjective. We have $\dim\ker(\widetilde{\alpha}) = \dim\mathfrak{a}_1 < \dim\mathfrak{a} = \dim\ker(\alpha)$, so that by induction hypothesis there exists a homomorphism

$$\widetilde{\beta}\colon \beta_1(\mathfrak{s}) \to \mathfrak{b} \quad \text{ with } \widetilde{\alpha}\circ\widetilde{\beta} = \mathrm{id}_{\beta_1(\mathfrak{s})}\,.$$

But then $\beta = \widetilde{\beta}\circ\beta_1\colon \mathfrak{s} \to \mathfrak{g}$ is a homomorphism with

$$\alpha\circ\beta = \alpha_1\circ(\widetilde{\alpha}\circ\widetilde{\beta})\circ\beta_1 = \alpha_1\circ\beta_1 = \mathrm{id}_{\mathfrak{s}}$$

and we are done.

Case 2: The ideal $\mathfrak{a}$ itself is minimal and different from 0. We have $\alpha(\mathrm{rad}(\mathfrak{g})) = \mathrm{rad}(\mathfrak{s}) = 0$ because of Lemma 2.4.1 and since $\mathfrak{s}$ is semisimple. So we have $\mathrm{rad}(\mathfrak{g}) \subset \ker(\alpha) = \mathfrak{a}$. If $\mathrm{rad}(\mathfrak{g}) = 0$, then $\mathfrak{a} = \ker(\alpha) = \iota(0) = 0$, a contradiction. So we may assume that $\mathrm{rad}(\mathfrak{g}) \neq 0$. Chose a maximal $n \geq 1$ such that $\mathrm{rad}(\mathfrak{g})^{(n)} \neq 0$. This is an abelian ideal of $\mathfrak{g}$ different from zero, because $0 = \mathrm{rad}(\mathfrak{g})^{(n+1)} = [\mathrm{rad}(\mathfrak{g})^{(n)}, \mathrm{rad}(\mathfrak{g})^{(n)}]$. Since $\mathfrak{a}$ was minimal, it follows that $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{g})^{(n)}$. Thus $\mathfrak{a}$ is abelian. Then $\mathfrak{a}$ is in the kernel of the representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{a}), \quad x \mapsto \mathrm{ad}(x)\mid_{\mathfrak{a}}$$

which then factorizes by the homomorphism theorem to a representation of $\mathfrak{g}/\mathfrak{a}$ on $\mathfrak{a}$. Since $\alpha\colon \mathfrak{g} \to \mathfrak{s}$ is a surjective homomorphism, we have $\mathfrak{g}/\ker(\alpha) = \mathfrak{g}/\mathfrak{a} \cong \mathfrak{s}$. This way $\mathfrak{a}$ becomes a $\mathfrak{s}$-module, which is even simple, because $\mathfrak{a}$ is minimal. Here we need another case distinction.

Case 2a: Let $\mathfrak{a}$ be a trivial $\mathfrak{s}$-module. Then $\mathfrak{a}$ is contained in the center of $\mathfrak{g}$. Hence $\mathfrak{a}$ is contained in the kernel of the adjoint representation of $\mathfrak{g}$. By the homomorphism theorem the adjoint representation of $\mathfrak{g}$ factorizes to a representation of $\mathfrak{s}$. This way $\mathfrak{g}$ becomes a $\mathfrak{s}$-module, which is semisimple by Weyl's Theorem. Hence there exists a complement to $\mathfrak{a}$, namely an ideal which is complementary to $\mathfrak{a}$ in $\mathfrak{g}$. But then we have $\mathfrak{g} \cong \mathfrak{g}/\mathfrak{a} \oplus \mathfrak{a} \cong \mathfrak{s} \oplus \mathrm{rad}(\mathfrak{g})$. Hence the above short exact sequence is split and we are done.

Case 2b: Let $\mathfrak{a}$ be a non-trivial $\mathfrak{s}$-module. The vector space $V = \mathrm{End}(\mathfrak{g})$ becomes a $\mathfrak{g}$-module by $x.\varphi = [\mathrm{ad}(x), \varphi]$. We consider the following subspaces $P \subset Q \subset R$ of V:

$$\begin{aligned}
P &= \mathrm{ad}(\mathfrak{a}),\\
Q &= \{\varphi \in V \mid \varphi(\mathfrak{g}) \subset \mathfrak{a},\ \varphi(\mathfrak{a}) = 0\},\\
R &= \{\varphi \in V \mid \varphi(\mathfrak{g}) \subset \mathfrak{a},\ \varphi\mid_{\mathfrak{a}} \in k\cdot\mathrm{id}_{\mathfrak{a}}\}.
\end{aligned}$$

Here Q is the kernel of the linear map $\chi\colon R \to k$ with $\varphi\mid_{\mathfrak{a}} = \chi(\varphi)\cdot\mathrm{id}_{\mathfrak{a}}$. Therefore $\dim(R/Q) \leq 1$. We show that these subspaces are $\mathfrak{g}$-submodules of V. Let $y \in \mathfrak{g}$. For $\mathrm{ad}(x) \in P$ we have $y.\,\mathrm{ad}(x) = [\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}([x,y]) \in P$. Therefore P is a submodule. To see that R and Q are submodules, it suffices to show that $\mathfrak{g}.R \subset Q$. So let $x \in \mathfrak{g}$, $\varphi \in R$ and $\varphi\mid_{\mathfrak{a}} = \lambda\,\mathrm{id}_{\mathfrak{a}}$. For

$a \in \mathfrak{a}$ we have

$$\begin{aligned}(x.\varphi)(a) &= x.\varphi(a) - \varphi([x,a]) \\ &= x.(\lambda a) - \lambda[x,a] \\ &= 0,\end{aligned}$$

so $x.\varphi \in Q$. So we are done. Furthermore, we have $\mathfrak{a}.R \subset P$, because for $y \in \mathfrak{a}$ we have $\operatorname{ad}(y)(\mathfrak{a}) = 0$, since $\mathfrak{a}$ is abelian and therefore

$$y.\varphi = \operatorname{ad}(y) \circ \varphi - \varphi \circ \operatorname{ad}(y) = -\lambda \operatorname{ad}(y) \in P.$$

Hence the ideal $\mathfrak{a}$ acts trivially on the quotient module R/P, which becomes a $\mathfrak{g}/\mathfrak{a}$-module this way, and hence a $\mathfrak{s}$-module. By Weyl's Theorem there exists a module complement U to the submodule Q/P of R/P with $\dim U = 1$. It is spanned by a vector $v \in R \backslash Q$, and we may assume that $v\mid_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$. Because of $\mathfrak{s} = [\mathfrak{s},\mathfrak{s}]$ we know that U is a trivial $\mathfrak{s}$-module, so that $\mathfrak{g}.v \subset P$. We want to apply Lemma 2.4.2 to this vector v. For this we need to check the assumptions of the lemma. For $x \in \mathfrak{a}$ we have by the above computation $x.v = -\lambda \operatorname{ad}(x) = -\operatorname{ad}(x)$. Assume that $x.v = 0$. Then $x \in Z(\mathfrak{a})$. Since $\mathfrak{a}$ by assumption is a non-trivial $\mathfrak{s}$-module, $\mathfrak{a}$ is a minimal non-central ideal of $\mathfrak{g}$. Hence we have $x \in Z(\mathfrak{a}) = 0$. Thus we have

$$\begin{aligned}Z_{\mathfrak{a}}(v) &= 0, \\ \mathfrak{a}.v &= \operatorname{ad}(\mathfrak{a}) = P = \mathfrak{g}.v.\end{aligned}$$

Now we can apply the lemma to obtain the claim, and the proof is finished. □

Let us now formally define a Levi complement.

DEFINITION 2.4.4. Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ with $\mathfrak{g} \cong \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$ is called a *Levi complement* in $\mathfrak{g}$.

COROLLARY 2.4.5. *Let $\mathfrak{g}$ be a Lie algebra. Then there exists a Levi complement in $\mathfrak{g}$.*

PROOF. We claim that $\mathfrak{s} = \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is a Levi complement in $\mathfrak{g}$. First note that $\mathfrak{s}$ is semisimple by Corollary 2.2.16. Let $\alpha\colon \mathfrak{g} \to \mathfrak{s}$ be the quotient map. By Levi's Theorem there exists a homomorphism $\beta\colon \mathfrak{s} \to \mathfrak{g}$ with $\alpha \circ \beta = \mathrm{id}_{\mathfrak{s}}$. Then β is injective, so $\beta(\mathfrak{s}) \cap \iota(\operatorname{rad}(\mathfrak{g})) = 0$ and $\beta(\mathfrak{s}) + \iota(\operatorname{rad}(\mathfrak{g})) = \mathfrak{g}$. This amounts to $\mathfrak{g} \cong \beta(\mathfrak{s}) \ltimes \iota(\operatorname{rad}(\mathfrak{g}))$ and we are done. □

We obtain a further corollary to Levi's Theorem.

COROLLARY 2.4.6. *Let $\mathfrak{s}$ be a Levi complement in $\mathfrak{g}$. Then*

$$[\mathfrak{g},\mathfrak{g}] \cong \mathfrak{s} \ltimes [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})].$$

If $\mathfrak{g}$ is reductive, so that $\operatorname{rad}(\mathfrak{g}) = Z(\mathfrak{g})$, then $[\mathfrak{g},\mathfrak{g}]$ is a Levi complement in $\mathfrak{g}$.

PROOF. Since $\mathfrak{s}$ is semisimple we have $[\mathfrak{s},\mathfrak{s}] = \mathfrak{s}$. So we have, because of $\mathfrak{g} = \mathfrak{s} + \operatorname{rad}(\mathfrak{g})$,

$$\begin{aligned}[\mathfrak{g},\mathfrak{g}] &= [\mathfrak{g},\mathfrak{s}] + [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] \\ &= [\mathfrak{s},\mathfrak{s}] + [\operatorname{rad}(\mathfrak{g}),\mathfrak{s}] + [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] \\ &= \mathfrak{s} + [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})].\end{aligned}$$

Because of $\mathfrak{s} \cap \operatorname{rad}(\mathfrak{g}) = 0$ we have $\mathfrak{s} \cap [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] = 0$.

For the second claim note that $\operatorname{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ is equivalent to $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] = 0$, so to $[\mathfrak{g},\mathfrak{g}] \cong \mathfrak{s}$. □

EXAMPLE 2.4.7. *For $\mathfrak{g} = \mathfrak{gl}(V)$ we have $\operatorname{rad}(\mathfrak{g}) = Z(\mathfrak{g}) = k \cdot \mathrm{id}$, and $\mathfrak{s} = [\mathfrak{g},\mathfrak{g}] = \mathfrak{sl}(V)$ is a Levi complement in $\mathfrak{g}$.*

Now we will come to *Malcev's Theorem.* Let $\mathrm{Aut}(\mathfrak{g})$ be the group of automorphisms of $\mathfrak{g}$, consisting of all bijective Lie algebra endomorphisms of $\mathfrak{g}$.

LEMMA 2.4.8. *Let $\mathfrak{g}$ be a Lie algebra and D a nilpotent derivation of $\mathfrak{g}$. Then $e^D = \exp(D)$ is an automorphism of $\mathfrak{g}$.*

PROOF. First of all, e^D is an automorphism of the vector space of $\mathfrak{g}$. The series is finite, since D is nilpotent. The inverse is given by the series e^{-D}. For $x, y \in \mathfrak{g}$ we have

$$D^p([x,y]) = \sum_{j=0}^{p} \binom{p}{j} [D^{p-j}(x), D^j(y)],$$

which follows easily by induction. Then we have

$$\begin{aligned} e^D([x,y]) &= \sum_{p=0}^{\infty} \frac{1}{p!} D^p([x,y]) \\ &= \sum_{p=0}^{\infty} \sum_{j=0}^{p} \frac{1}{(p-j)!j!} [D^{p-j}(x), D^j(y)] \\ &= \sum_{j=0}^{\infty} \sum_{p=j}^{\infty} \frac{1}{(p-j)!j!} [D^{p-j}(x), D^j(y)] \\ &= \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{p!j!} [D^{p}(x), D^j(y)] \\ &= [e^D(x), e^D(y)]. \end{aligned}$$

Hence we have $e^D \in \mathrm{Aut}(\mathfrak{g})$. □

In particular, for $D = \mathrm{ad}(x)$, we have $e^{\mathrm{ad}(x)} \in \mathrm{Aut}(\mathfrak{g})$. Such an automorphism has a special name.

DEFINITION 2.4.9. Let $\mathfrak{g}$ be a Lie algebra. An automorphism of the form $e^{\mathrm{ad}(x)}$ with $x \in \mathrm{nil}(\mathfrak{g})$ is called *special.* Let $\mathrm{Aut}_s(\mathfrak{g})$ denote the subgroups of $\mathrm{Aut}(\mathfrak{g})$ generated by all special automorphisms.

One can show that $\mathrm{Aut}_s(\mathfrak{g})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$.

THEOREM 2.4.10 (Malcev). *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two Levi complements in $\mathfrak{g}$. Then there exists a special automorphism $\varphi = e^{\mathrm{ad}(x)}$ $\mathrm{Aut}_s(\mathfrak{g})$ with $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subset \mathrm{nil}(\mathfrak{g})$, such that $\varphi(\mathfrak{s}_1) = \mathfrak{s}_2$.*

We do not prove the result here. For a proof, see for example [**5**], where Weyl's Theorem is used in the proof. Let us give some corollaries to Malcev's Theorem.

COROLLARY 2.4.11. *Every semisimple subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is contained in a Levi complement of $\mathfrak{g}$.*

PROOF. Let $\mathfrak{a} = \mathrm{rad}(\mathfrak{g}) + \mathfrak{h}$. Then $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ and $\mathrm{rad}(\mathfrak{g})$ is a solvable ideal in $\mathfrak{a}$. Then

$$\mathfrak{a}/\,\mathrm{rad}(\mathfrak{g}) \cong \mathfrak{h}/(\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g}))$$

is a quotient of $\mathfrak{h}$ and therefore it is semisimple and has a trivial solvable radical. So we have $\mathrm{rad}(\mathfrak{g}) = \mathrm{rad}(\mathfrak{a})$. The ideal $\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})$ of $\mathfrak{h}$ is semisimple and at the same time solvable, hence equal to zero. Thus we have $\mathfrak{a} = \mathfrak{h} \ltimes \operatorname{rad}(\mathfrak{g})$ and $\mathfrak{h}$ is a Levi complement in $\mathfrak{a}$. Let $\mathfrak{s}$ be a Levi complement in $\mathfrak{g}$. Then

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes \operatorname{rad}(\mathfrak{g}).$$

Since $\mathfrak{a} \cap \mathfrak{s} \cong \mathfrak{a}/\operatorname{rad}(\mathfrak{g}) \cong \mathfrak{h}$ is semisimple, $\mathfrak{a} \cap \mathfrak{s}$ is a Levi complement in $\mathfrak{a}$. By Malcev's Theorem there exists a $x \in [\mathfrak{a}, \operatorname{rad}(\mathfrak{g})]$ with

$$e^{\operatorname{ad}(x)}(\mathfrak{a} \cap \mathfrak{s}) = \mathfrak{h}.$$

Hence $\mathfrak{h}$ is contained in the Levi complement $\mathfrak{s}' = e^{\operatorname{ad}(x)}(\mathfrak{s})$ of $\mathfrak{g}$. □

COROLLARY 2.4.12. *The Levi complements in $\mathfrak{g}$ are exactly the maximal semisimple subalgebras of $\mathfrak{g}$.*

PROOF. Every maximal semisimple subalgebra of $\mathfrak{g}$ is a Levi complement because of Corollary 2.4.11. Conversely, let $\mathfrak{s}$ be a Levi complement. For each semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we have $\operatorname{rad}(\mathfrak{g}) \cap \mathfrak{h} = 0$, hence $\mathfrak{h} \subset \mathfrak{s}$, because of $\mathfrak{g} = \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$. □

COROLLARY 2.4.13. *Let $\mathfrak{g}$ be a Lie algebra with Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$ and $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes (\mathfrak{a} \cap \operatorname{rad}(\mathfrak{g}))$ is a Levi decomposition of $\mathfrak{a}$.*

PROOF. Exercise. □

2.5. Cartan subalgebras

The adjoint representation of a Lie algebra $\mathfrak{g}$, restricted to a suitable subalgebra $\mathfrak{h}$, can give a lot of structural information on the Lie algebra $\mathfrak{g}$. For $h \in \mathfrak{g}$ and $\lambda \in k$ let

$$\mathfrak{g}_\lambda(h) = \{x \in \mathfrak{g} \mid (\operatorname{ad}(h) - \lambda \operatorname{id})^n x = 0 \text{ for some } n\}$$

be the generalized eigenspace of $\operatorname{ad}(h)$ to λ. Clearly we have $\mathfrak{g}_\lambda(h) \neq 0$ if and only if λ is an eigenvalue of $\operatorname{ad}(h)$. Because of $\operatorname{ad}(h)(h) = 0$ we have $\mathfrak{g}_0(h) \neq 0$. In case k is algebraically closed, the Jordan decomposition of $\operatorname{ad}(h)$ yields

$$\begin{aligned} \mathfrak{g} &= \bigoplus_{\lambda \in k} \mathfrak{g}_\lambda(h) \\ &= \bigoplus_{i=0}^{p} \mathfrak{g}_{\lambda_i}(h), \\ &= \mathfrak{g}_0(h) \oplus \bigoplus_{i=1}^{p} \mathfrak{g}_{\lambda_i}(h) \end{aligned}$$

where $\lambda_0 = 0, \lambda_1, \dots, \lambda_p$ are the different eigenvalues of $\operatorname{ad}(h)$.

LEMMA 2.5.1. *Let $h \in \mathfrak{g}$. Then*

$$[\mathfrak{g}_\lambda(h), \mathfrak{g}_\mu(h)] \subset \mathfrak{g}_{\lambda+\mu}(h)$$

for all $\lambda, \mu \in k$.

PROOF. Let $x \in \mathfrak{g}_\lambda(h)$ and $y \in \mathfrak{g}_\mu(h)$. Then we have, as in (2.1), for all $n \geq 1$

$$(\operatorname{ad}(h) - (\lambda+\mu)E)^n([x,y]) = \sum_{k=0}^{n} \binom{n}{k} [(\operatorname{ad}(h) - \lambda E)^k(x), (\operatorname{ad}(h) - \mu E)^{n-k}(y)].$$

So if we have $(\operatorname{ad}(h) - \lambda E)^p(x) = 0$ and $(\operatorname{ad}(h) - \mu E)^q(y) = 0$, then

$$(\operatorname{ad}(h) - (\lambda+\mu)E)^{p+q}([x,y]) = 0.$$

□

In particular, the following holds.

COROLLARY 2.5.2. *The subspace* $\mathfrak{g}_0(h)$ *in the above decomposition is a Lie subalgebra different from zero in* $\mathfrak{g}$.

Now consider the characteristic polynomial $P_h(t) = \det(tE - \operatorname{ad}(h))$ of $\operatorname{ad}(h)$. Let $n = \dim \mathfrak{g}$, then we can write this polynomial with polynomial functions $a_i(h)$ in $h \in \mathfrak{g}$ as

$$P_h(t) = \sum_{i=0}^{n} a_i(h) t^i.$$

Since zero is an eigenvalue of $\operatorname{ad}(h)$, we have $P_h(0) = 0$ and therefore $a_0 = 0$. Furthermore we have $a_n = 1$.

DEFINITION 2.5.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field k. The *rank* of $\mathfrak{g}$, denoted by $\ell = \operatorname{rank} \mathfrak{g}$, is the smallest integer $i \geq 0$ with $a_i \neq 0$. An element $h \in \mathfrak{g}$ is called *regular*, if $a_\ell(h) \neq 0$.

This number satisfies $1 \leq \operatorname{rank} \mathfrak{g} \leq \dim \mathfrak{g}$. Since the multiplicity of zero as a root of of $P_h(t)$ equals $\dim \mathfrak{g}_0(h)$, we have $\operatorname{rank} \mathfrak{g} \leq \dim \mathfrak{g}_0(h)$. Indeed, the rank of $\mathfrak{g}$ is exactly the minimal dimension of the subalgebra $\mathfrak{g}_0(h)$, if h runs through $\mathfrak{g}$, i.e.,

$$\operatorname{rank} \mathfrak{g} = \min\{\dim \mathfrak{g}_0(h) \mid h \in \mathfrak{g}\}.$$

We have equality, namely $\operatorname{rank} \mathfrak{g} = \dim \mathfrak{g}_0(h)$, if and only if h is regular.

LEMMA 2.5.4. *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* $\operatorname{rank} \mathfrak{g} = \dim \mathfrak{g}$.

PROOF. We have $\operatorname{rank} \mathfrak{g} = \dim \mathfrak{g}$ if and only if all $\operatorname{ad}(x)$ are nilpotent for $x \in \mathfrak{g}$. So the claim follows by Engel's Theorem. □

Denote by $\mathfrak{g}^{reg}$ the set of regular elements of $\mathfrak{g}$.

LEMMA 2.5.5. *The subset* $\mathfrak{g}^{reg} \subset \mathfrak{g}$ *is a non-empty dense Zariski-open set in* $\mathfrak{g}$, *invariant under all automorphisms of* $\mathfrak{g}$.

PROOF. By definition a_ℓ is not the zero polynomial, where $\ell = \operatorname{rank} \mathfrak{g}$. Hence there is an element h in $\mathfrak{g}^{reg}$. The set is Zariski open, since the condition $\dim \mathfrak{g}_0(h) > \operatorname{rank} \mathfrak{g}$ can be expressed by the vanishing of the polynomial functions $a_\ell(h)$. For $\varphi \in \operatorname{Aut}(\mathfrak{g})$ we have

$$\operatorname{ad}(\varphi(h)) = \varphi \circ \operatorname{ad}(h) \circ \varphi^{-1}.$$

Therefore we have

$$\begin{aligned} P_{\varphi(h)}(t) &= \det(tE - \operatorname{ad}(\varphi(h))) \\ &= \det(tE - \varphi \circ \operatorname{ad}(h) \circ \varphi^{-1}) \\ &= \det(\varphi \circ (tE - \operatorname{ad}(h)) \circ \varphi^{-1}) \\ &= P_h(t). \end{aligned}$$

Hence $a_\ell(\varphi(h)) = a_\ell(h)$ for all $h \in \mathfrak{g}$. It follows that $\varphi(\mathfrak{g}^{reg}) \subset \mathfrak{g}^{reg}$ for all $\varphi \in \operatorname{Aut}(\mathfrak{g})$. □

EXAMPLE 2.5.6. *Let* $\mathfrak{g} = \mathfrak{sl}_2(k)$, $\mathrm{char}(k) \neq 2$ *and*

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

be an element in $\mathfrak{g}$. *Then the characteristic polynomial of* x *is given by*

$$P_x(t) = \det \begin{pmatrix} t-2a & 0 & 2b \\ 0 & t+2a & -2c \\ c & -b & t \end{pmatrix} = t^3 + 4t\det(x).$$

Hence $a_1(x) = 4\det(x)$ *for all* $x \in \mathfrak{g}$, *and* $a_0 = 0$. *Thus* $\mathrm{rank}\,\mathfrak{g} = 1$ *and* x *is regular if and only if* $\det(x) \neq 0$.

Even better, because of $\mathrm{tr}(x) = 0$ we have that $x \in \mathfrak{g}$ is regular if and only if x is not nilpotent. So we have

$$\mathfrak{sl}_2(k)^{reg} = \mathfrak{sl}_2(k) \setminus \mathcal{N},$$

where $\mathcal{N}$ denotes the cone of nilpotent matrices in $\mathfrak{sl}_2(k)$.

LEMMA 2.5.7. *Let* $h_0 \in \mathfrak{g}^{reg}$. *Then the Lie algebra* $\mathfrak{h} = \mathfrak{g}_0(h_0)$ *is nilpotent.*

PROOF. Let $\lambda_0 = 0, \lambda_1, \dots, \lambda_p$ be the distinct eigenvalues of $\mathrm{ad}(h_0)$ and

$$\mathfrak{g}_1 = \bigoplus_{i=1}^{p} \mathfrak{g}_{\lambda_i}(h_0)$$

be the sum of the spaces $\mathfrak{g}_{\lambda_i}(h_0)$ without $\mathfrak{h} = \mathfrak{g}_0(h_0)$. Then $[\mathfrak{h}, \mathfrak{g}_1] \subset \mathfrak{g}_1$ by Lemma 2.5.1. Hence the adjoint representation of $\mathfrak{g}$, restricted to $\mathfrak{h}$, induces a representation $\rho\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}_1)$. Consider the polynomial function

$$h \mapsto d(h) = \det(\rho(h))$$

on $\mathfrak{h}$. With $q_i = \dim \mathfrak{g}_{\lambda_i}(h_0)$ we have

$$d(h_0) = \lambda_1^{q_1}\lambda_2^{q_2}\cdots\lambda_p^{q_p} \neq 0.$$

Hence d is not the zero function and there exists a Zariski open set in $\mathfrak{h}$ on which d does not vanish. Let $h \in \mathfrak{h}$ be an element with $d(h) \neq 0$. The eigenvalues of $\rho(h)$ are all different from zero. Hence we have $\mathfrak{g}_0(h) \subset \mathfrak{h}$. Since h_0 is regular, we have $\dim \mathfrak{h} = \mathrm{rank}\,\mathfrak{g}$ and $\dim \mathfrak{g}_0(h) \geq \mathrm{rank}\,\mathfrak{g}$. This implies that

$$\mathfrak{h} = \mathfrak{g}_0(h).$$

Hence, by definition of $\mathfrak{g}_0(h)$, the linear map $\mathrm{ad}_{\mathfrak{h}}(h)$ is nilpotent, i.e., $(\mathrm{ad}_{\mathfrak{h}}(h))^q = 0$ for all $q \geq \mathrm{rank}\,\mathfrak{g}$. The matrix entries of $(\mathrm{ad}_{\mathfrak{h}}(h))^q$ are polynomial functions on $\mathfrak{h}$. Because of Zariski continuity we have $(\mathrm{ad}_{\mathfrak{h}}(h))^q = 0$ for *all* $h \in \mathfrak{h}$. Therefore all $\mathrm{ad}_{\mathfrak{h}}(h)$ are nilpotent, and $\mathfrak{h}$ is nilpotent by Engel's Theorem. □

LEMMA 2.5.8. *Let* $h_0 \in \mathfrak{g}^{reg}$. *Then the Lie algebra* $\mathfrak{h} = \mathfrak{g}_0(h_0)$ *is equal to its normalizer in* $\mathfrak{g}$. *So we have* $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$.

PROOF. Let $x \in N_{\mathfrak{g}}(\mathfrak{h})$. Then we have $[h_0, x] \in \mathfrak{h} = \mathfrak{g}_0(h_0)$. Hence there is a $p \geq 0$ with

$$\mathrm{ad}(h_0)^p([h_0, x]) = \mathrm{ad}(h_0)^{p+1}(x) = 0.$$

This implies that $x \in \mathfrak{h}$. Hence we have $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. □

Such nilpotent self-normalizing subalgebras $\mathfrak{h}$ obtain a new name.

DEFINITION 2.5.9. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* in $\mathfrak{g}$, if $\mathfrak{h}$ is nilpotent and $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$.

It is not a priori clear whether or not there exists a Cartan subalgebra in a given Lie algebra $\mathfrak{g}$. We have the following result.

PROPOSITION 2.5.10. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an infinite field k. Then there exists a Cartan subalgebra in $\mathfrak{g}$. If k has characteristic zero then all Cartan subalgebras $\mathfrak{h}$ have the same dimension, namely* $\dim(\mathfrak{h}) = \operatorname{rank}(\mathfrak{g})$.

PROOF. Assume first that k is algebraically closed. Then $\mathfrak{g}_0(h)$ is a Cartan subalgebra for every $h \in \mathfrak{g}^{reg}$, as we have seen in 2.5.7 and 2.5.8. Now let k be a field of characteristic zero, K be an algebraic closure of k and $\mathfrak{g}_K = K \otimes_k \mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_K$ be the subalgebra of $\mathfrak{g}_K$ spanned by $\mathfrak{h}$ over K. Then $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$, if $\mathfrak{h}_K$ is a Cartan subalgebra in $\mathfrak{g}_K$. Hence there exists a Cartan subalgebra in characteristic zero, because there exists a Cartan subalgebra of $\mathfrak{g}_K$, which is defined over k. Furthermore one can show that a Cartan subalgebra always exists whenever the field k has more than $\dim_k(\mathfrak{g})$ elements, see [**30**]. In particular, finite dimensional Lie algebras over infinite field always have Cartan subalgebras. $\square$

The existence of Cartan subalgebras in Lie algebras over finite fields is still an open problem, see [**30**].

EXAMPLE 2.5.11. *Let $\mathfrak{g} = \mathfrak{gl}_2(k)$. Then each of the following subalgebras,*

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k \right\},$$

$$\mathfrak{h}_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in k \right\}$$

form a Cartan subalgebra in $\mathfrak{g}$.

This shows that Cartan subalgebras need not be unique. However, we have the following result.

PROPOSITION 2.5.12. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field of characteristic zero. Then all Cartan subalgebras are conjugate under automorphisms of $\mathfrak{g}$.*

For $k = \mathbb{R}$ the two Cartan subalgebras of Example 2.5.11 in $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ are not conjugate. So the proposition is not true over arbitrary fields of characteristic zero.

PROPOSITION 2.5.13. *Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Then $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$.*

PROOF. Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{g}$ with $\mathfrak{n} \supset \mathfrak{h}$. Assume that $\mathfrak{n} \neq \mathfrak{h}$. Then the adjoint representation of $\mathfrak{n}$, restricted to $\mathfrak{h}$, defines a representation $\sigma \colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{n}/\mathfrak{h})$. This representation acts by nilpotent operators by Engel's Theorem. By Lemma 1.6.13 there is a nonzero $v \in \mathfrak{n}/\mathfrak{h}$ with $\sigma(x)v = 0$ for all $x \in \mathfrak{h}$. Let $y \in \mathfrak{n}$ be a representative of the coset v. Then we have $[x, y] = \operatorname{ad}(x)(y) \in \mathfrak{h}$ for all $x \in \mathfrak{h}$. Hence $y \in N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, because $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$. Hence the coset is zero in $\mathfrak{n}/\mathfrak{h}$, d.h., $v = 0$, a contradiction. It follows that $\mathfrak{n} = \mathfrak{h}$. $\square$

COROLLARY 2.5.14. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then $\mathfrak{g}$ is the only Cartan subalgebra in $\mathfrak{g}$.*

The converse of Proposition 2.5.13 need not be true. There exist maximal nilpotent subalgebras, which are not Cartan subalgebras, see the following example.

EXAMPLE 2.5.15. *Let $\mathfrak{g} = \mathfrak{sl}_2(k)$, where k is a field of characteristic zero, and (x, y, h) be the standard basis of $\mathfrak{g}$. Then the subalgebra $\mathfrak{a} = k \cdot x$ is maximal nilpotent, but not a Cartan subalgebra in $\mathfrak{g}$.*

To see this, suppose that $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$, which contains x. Then $\dim(\mathfrak{n}) \le 2$. Thus $\mathfrak{n}$ is abelian. Let $g = \alpha x + \beta y + \gamma h \in \mathfrak{n}$. Then we have

$$0 = [x, g] = \beta h - 2\gamma x.$$

Hence $\beta = \gamma = 0$ and therefore $\mathfrak{n} = k \cdot g = k \cdot x = \mathfrak{a}$. So $\mathfrak{a}$ is maximal nilpotent. On the other hand, $\mathfrak{n}$ is not self-normalizing, since all upper-triangular matrices in $\mathfrak{g}$ normalize $\mathfrak{n}$.

Let us mention the following lemma, which is proved along the lines of Proposition 2.5.12

LEMMA 2.5.16. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Then there exists an $h \in \mathfrak{g}^{reg} \subset \mathfrak{g}$ with $\mathfrak{h} = \mathfrak{g}_0(h)$.*

From now on we want to restrict ourselves mainly to Cartan subalgebras of *semisimple* Lie algebras over a field of characteristic zero. We will see that their structure then is much simpler than in general.

PROPOSITION 2.5.17. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic zero, and $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Then $\mathfrak{h}$ is abelian.*

PROOF. We may assume that k is algebraically closed. Then there exists by Lemma 2.5.16 an $h_0 \in \mathfrak{h}$ with $\mathfrak{h} = \mathfrak{g}_0(h_0)$. Let $\lambda \ne 0$ and $x \in \mathfrak{g}_\lambda(h_0)$. For $h \in \mathfrak{h}$ and $\mu \in k$ we have

$$\operatorname{ad}(x)\operatorname{ad}(h)(\mathfrak{g}_\mu(h_0)) \subset \operatorname{ad}(x)(\mathfrak{g}_\mu(h_0)) \subset \mathfrak{g}_{\lambda+\mu}(h_0).$$

Let $\lambda_0 = 0, \lambda_1, \ldots, \lambda_p$ be the distinct eigenvalues of $\operatorname{ad}(h_0)$. Choosing a basis of $\mathfrak{g}$ corresponding to the decomposition

$$\mathfrak{g} = \bigoplus_{i=0}^{p} \mathfrak{g}_{\lambda_i}(h_0),$$

the associated block matrix of $\operatorname{ad}(x)\operatorname{ad}(h)$ has zero blocks on the diagonal. Hence the Killing form satisfies $\kappa(x, h) = 0$ for all x and h. Hence $\mathfrak{h}$ is orthogonal to all spaces $\mathfrak{g}_{\lambda_i}(h_0)$ for $1 \le i \le p$ with respect to the Killing form. Since $\mathfrak{h}$ is nilpotent and hence solvable, we have $\kappa(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$ by the Cartan criterion. So we obtain $\kappa(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]) = 0$. Since $\mathfrak{g}$ is semisimple and the characteristic of k is zero, the Killing form on $\mathfrak{g}$ is non-degenerate. So we obtain $[\mathfrak{h}, \mathfrak{h}] = 0$ and $\mathfrak{h}$ is abelian. □

Since Cartan subalgebras are maximal nilpotent, we obtain the following corollary.

COROLLARY 2.5.18. *Let $\mathfrak{g}$ be a semisimple Lie algebra in characteristic zero. Then Cartan subalgebras in $\mathfrak{g}$ are maximal abelian subalgebras.*

LEMMA 2.5.19. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic zero. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Then all $h \in \mathfrak{h}$ are semisimple.*

Proof. Let $h \in \mathfrak{h}$ and consider its Jordan-Chevalley decomposition $h = s + n$, see Proposition 2.1.13. Since $\mathfrak{h}$ is abelian, we have $\operatorname{ad}(h)(\mathfrak{h}) = 0$. Since $\operatorname{ad}(s)$ and $\operatorname{ad}(n)$ are the semisimple respectively nilpotent parts of $\operatorname{ad}(h)$, we can represent them by polynomials in $\operatorname{ad}(h)$ without constant term. In particular we have

$$\operatorname{ad}(s)(\mathfrak{h}) = \operatorname{ad}(n)(\mathfrak{h}) = 0.$$

Since $\mathfrak{h}$ is maximal abelian, we have $s, n \in \mathfrak{h}$. By Lemma 2.5.16 we obtain $\mathfrak{h} = \mathfrak{g}_0(h_0)$. As in the proof of Proposition 2.5.17 we see that $\mathfrak{h}$ is orthogonal to $\mathfrak{g}_\lambda(h_0)$, for the eigenvalues $\lambda \neq 0$ of $\operatorname{ad}(h_0)$. Let $y \in \mathfrak{h}$. Because of $[\operatorname{ad}(y), \operatorname{ad}(n)] = \operatorname{ad}([y, n]) = 0$, with $\operatorname{ad}(y)$ and $\operatorname{ad}(n)$ nilpotent we also have that $\operatorname{ad}(y)\operatorname{ad}(n)$ is a nilpotent linear map. Therefore we obtain $\kappa(y, n) = 0$ and thus n is orthogonal to $\mathfrak{g}$. Since the Killing from of $\mathfrak{g}$ is non-degenerate, it follows that $n = 0$ and that $h = s$ is semisimple. $\square$

Corollary 2.5.20. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of characteristic zero. Then all regular elements in $\mathfrak{g}$ are semisimple.*

Proof. Let $h \in \mathfrak{g}^{reg}$. Then $\mathfrak{g}_0(h)$ is a Cartan subalgebra in $\mathfrak{g}$ by Lemma 2.5.7 and Lemma 2.5.8. By Lemma 2.5.19, h is semisimple. $\square$

2.6. The root space decomposition

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k (always of characteristic zero) and $\mathfrak{h} \subset \mathfrak{g}$. For every linear map $\alpha \colon \mathfrak{h} \to k$ let

$$\mathfrak{g}_\alpha = \bigcap_{h \in \mathfrak{h}} \mathfrak{g}_{\alpha(h)}(h)$$

be the the intersection of the generalized eigenspaces for all $h \in \mathfrak{h}$. Here α is called a *weight*, if $\mathfrak{g}_\alpha \neq 0$.

PROPOSITION 2.6.1. *Let k be algebraically closed and $\mathfrak{h} \subset \mathfrak{g}$ be a nilpotent subalgebra. Then the following holds.*

$$\mathfrak{h} \subset \mathfrak{g}_0, \tag{2.2}$$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \tag{2.3}$$

$$[\mathfrak{h}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha, \tag{2.4}$$

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \tag{2.5}$$

For $\mathfrak{g}_\alpha \neq 0$ we have $\alpha([\mathfrak{h}, \mathfrak{h}]) = 0$. For $h, h' \in \mathfrak{h}$ the Killing form is given by

$$\kappa(h, h') = \sum_{\alpha \in \mathfrak{h}^*} \dim(\mathfrak{g}_\alpha)\alpha(h)\alpha(h').$$

PROOF. Since $\mathfrak{h}$ nilpotent we have $(\mathrm{ad}(h))^n x = 0$ for all $x, h \in \mathfrak{h}$ and sufficiently large $n \in \mathbb{N}$. Hence we have $\mathfrak{h} \subset \mathfrak{g}_0$. Then (2.3) and (2.4) follow by Lemma 2.5.1, even if k is not algebraically closed. The sum in (2.5) obviously is direct. It remains to show that the sum of the simultaneous generalized eigenspaces exhausts all of $\mathfrak{g}$. It is easy to see that this follows by the previous properties, by $[\mathfrak{h}, \mathfrak{g}_{\alpha(h)}(h)] \subset \mathfrak{g}_{\alpha(h)}(h)$ and because $\mathfrak{g}$ is finite-dimensional.

For the formula we will apply Lie's Theorem, where we need that k is algebraically closed. So for all $\alpha \colon \mathfrak{h} \to k$ there exists a basis of $\mathfrak{g}_\alpha$, such that the endomorphisms $\mathrm{ad}(h) \mid_{\mathfrak{g}_\alpha}$ are simultaneously represented by strictly upper-triangular matrices in $\mathfrak{gl}_m(k)$, $m = \dim \mathfrak{g}_\alpha$, and with diagonal elements equal to $\alpha(h)$. This immediately yields the formula for the Killing form. Furthermore, if we consider $\mathrm{ad}([h, h'])$, this upper-triangular matrix has the entries $\alpha([h, h'])$ on the diagonal. At the same time this coincides with $[\mathrm{ad}(h), \mathrm{ad}(h')]$, which has zero diagonal. So, if $\mathfrak{g}_\alpha \neq 0$, we obtain $\alpha([h, h']) = 0 = [\alpha(h), \alpha(h')]$, and thus $\alpha \in \mathfrak{h}^*$ and $\alpha([\mathfrak{h}, \mathfrak{h}]) = 0$. □

Now let $\mathfrak{g}$ be a semisimple *complex* Lie algebra, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then all elements $h \in \mathfrak{h}$ are ad-semisimple by Lemma 2.5.19 and $\mathfrak{h}$ is abelian. Hence we may represent the endomorphisms $\mathrm{ad}(h)$ on $\mathfrak{g}_\alpha$ by diagonal matrices $\mathrm{diag}(\alpha(h), \dots, \alpha(h))$. So we have

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \ \forall h \in \mathfrak{h}\}.$$

DEFINITION 2.6.2. The *root system* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is defined by

$$\Phi = \Phi(\mathfrak{g}, \mathfrak{h}) = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}.$$

The elements of Φ are called the *roots*. For $\alpha \in \Phi$ we call $\mathfrak{g}_\alpha$ the *root space* to the root α.

The Lie algebra $\mathfrak{g}$ decomposes into the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{2.6}$$

The decomposition is called *root space decomposition*, or *Cartan decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Here $\mathfrak{h} = \mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{h})$ is equal to the weight space to the weight zero. We will see later in Corollary 2.6.12 that the Cartan decomposition implies the following equation in positive integers

$$\dim(\mathfrak{g}) = \operatorname{rank}(\Phi) + |\Phi|.$$

PROPOSITION 2.6.3. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and (2.6) be the corresponding Cartan decomposition. Then the following holds.*

(1) *Given $\alpha, \beta \in \Phi \cup \{0\}$ with $\alpha + \beta \neq 0$, we have $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$, hence $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$.*
(2) *From $\alpha \in \Phi$ it follows that $\mathfrak{g}_\alpha \perp \mathfrak{h}$.*
(3) *The restriction of the Killing form κ on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$, and hence on $\mathfrak{h} \times \mathfrak{h}$, is non-degenerate.*
(4) *If $\alpha \in \Phi$ then $-\alpha \in \Phi$.*
(5) $\operatorname{span} \Phi = \mathfrak{h}^*$.

PROOF. For (1): Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. By Proposition 2.6.1 we have

$$(\operatorname{ad}(x)\operatorname{ad}(y))^n(\mathfrak{g}_\gamma) \subset \mathfrak{g}_{n(\alpha+\beta)+\gamma} = 0$$

for all $\gamma \in \Phi \cup \{0\}$, if n is sufficiently large and if $\alpha + \beta \neq 0$. Therefore the endomorphism $\operatorname{ad}(x)\operatorname{ad}(y) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent and the Killing form is zero, i.e., $\kappa(x, y) = 0$.

For (2): This follows immediately from (1) with $\beta = 0$.

For (3): Let $z \in \mathfrak{g}_{-\alpha}$ with $\kappa(z, \mathfrak{g}_\alpha) = 0$. We need to show that $z = 0$. By (1) we have $\kappa(\mathfrak{g}_{-\alpha}, \mathfrak{g}_\beta) = 0$ for all $\beta \neq \alpha$. Hence $\kappa(z, \mathfrak{g}_\beta) = 0$ for all β, and therefore $\kappa(z, \mathfrak{g}) = 0$ by (2.6). Since κ is non-degenerate on $\mathfrak{g} \times \mathfrak{g}$, it follows that $z = 0$.

For (4): Let $\alpha \in \Phi$. Assume that $-\alpha \notin \Phi$, so $\mathfrak{g}_{-\alpha} = 0$. Then we have $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all β and hence again $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$, which implies that $\mathfrak{g}_\alpha = 0$. This is a contradiction to $\alpha \in \Phi$.

For (5): Assume that Φ does not generate $\mathfrak{h}^*$. Then, by duality there exists an $h \in \mathfrak{h}$, $h \neq 0$ with $\alpha(h) = 0$ for all $\alpha \in \Phi$. More precisely, if $(h_1^*, \ldots, h_\ell^*)$ is a basis of $\mathfrak{h}^*$ extending the basis of Φ by, say, $h_\ell^* \notin \Phi$. Then we consider the dual basis $(h_1, \ldots, h_\ell)$ of $\mathfrak{h}$ with $h_i^*(h_j) = \delta_{ij}$. We have $h_i^*(h_\ell) = 0$ for all $i \neq \ell$ and hence $\alpha(h) = \alpha(h_\ell) = 0$ for all $\alpha \in \Phi$, since $(h_1^*, \ldots, h_{\ell-1}^*)$ is a basis of Φ. But this implies $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in \Phi$. Because of $[h, \mathfrak{g}_0] = 0$ we have $[h, \mathfrak{g}] = 0$, hence $h \in Z(\mathfrak{g}) = 0$, which is a contradiction. $\square$

LEMMA 2.6.4. *Let $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$. Then we have $[x, y] \in \mathfrak{h}$ and*

$$\kappa(h, [x, y]) = \alpha(h)\kappa(x, y). \tag{2.7}$$

In particular, $\dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \geq 1$.

PROOF. We have $[x, y] \in \mathfrak{g}_{\alpha-\alpha} = \mathfrak{g}_0 = \mathfrak{h}$ by (2.3). Furthermore we have

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \kappa(\alpha(h)x, y) = \alpha(h)\kappa(x, y).$$

Since κ is non-degenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$, there exists an $x_\alpha \in \mathfrak{g}_\alpha$ and a $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ with $\kappa(x_\alpha, x_{-\alpha}) \neq 0$. Since $\alpha \neq 0$ there is an $h \in \mathfrak{h}$ with $\alpha(h) \neq 0$, hence also with $\kappa(h, [x_\alpha, x_{-\alpha}]) \neq 0$ because of (2.7). Thus $[x_\alpha, x_{-\alpha}]$ is a nonzero element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. $\square$

LEMMA 2.6.5. *Let $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ with $[x, y] \neq 0$. Then we have $\alpha([x, y]) \neq 0$.*

PROOF. Let $h = [x, y]$. Assume that $\alpha(h) = 0$. Then we have $[h, x] = \alpha(h)x = 0$ and $[h, y] = -\alpha(h)y = 0$. Therefore x, y, h generate a nilpotent subalgebra of $\mathfrak{g}$. By Lie's Theorem we can represent $\operatorname{ad}(x), \operatorname{ad}(y)$ and $\operatorname{ad}(h)$ simultaneously by upper-triangular matrices, so that

$\mathrm{ad}(h) = [\mathrm{ad}(x), \mathrm{ad}(y)]$ is nilpotent. This is a contradiction to $h \neq 0$ and the fact that $\mathrm{ad}(h)$ is semisimple. □

LEMMA 2.6.6. *For each root $\alpha \in \Phi$ we have $\dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = 1$ and α does not vanish on the line $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.*

PROOF. We already know that $\dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \geq 1$ by Lemma 2.6.4. By Lemma 2.6.5 we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cap \ker(\alpha) = 0$. So we have

$$\begin{aligned}\dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) &= \dim\Big([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] + \ker(\alpha)\Big) + \dim\Big([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cap \ker(\alpha)\Big) - \dim \ker(\alpha) \\ &\leq \dim \mathfrak{h} + 0 - (\dim \mathfrak{h} - 1) \\ &= 1.\end{aligned}$$

□

Since the Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$, we may identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by it. More precisely, we have an isomorphism

$$\mathfrak{h}^* \to \mathfrak{h},\ \alpha \mapsto t_\alpha,$$

which is characterized by the condition

$$\kappa(h, t_\alpha) = \alpha(h) \quad \text{for all } h \in \mathfrak{h}. \tag{2.8}$$

Hence Φ corresponds to the subset $\{t_\alpha \mid \alpha \in \Phi\} \subset \mathfrak{h}$.

DEFINITION 2.6.7. For $\alpha \in \Phi$ define the element $t_\alpha \in \mathfrak{h}$ by (2.8).

LEMMA 2.6.8. *Let $\alpha \in \Phi$ be a root and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ elements with $\kappa(x, y) = 1$. Then we have $[x, y] = t_\alpha$.*

PROOF. By (2.7) we have $\kappa(h, [x, y]) = \alpha(h)$ for all $h \in \mathfrak{h}$. Using (2.8) we obtain $[x, y] = t_\alpha$. □

LEMMA 2.6.9. *For every root $\alpha \in \Phi$ we have $\kappa(t_\alpha, t_\alpha) \neq 0$.*

PROOF. Let us write $t_\alpha = [x, y]$ with $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ and $\kappa(x, y) = 1$ as above. Then we have

$$\kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha) = \alpha([x, y]) \neq 0$$

by Lemma 2.6.5. □

DEFINITION 2.6.10. Let $\alpha \in \Phi$ be a root. Then the *coroot*, or *dual root* $\alpha^\vee \in \Phi^*$ is defined by

$$h_\alpha = \alpha^\vee = \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha \in \mathfrak{h}.$$

Note that we have

$$\alpha(h_\alpha) = \frac{2\alpha(t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = 2,$$

an $(-\alpha)^\vee = -\alpha^\vee$.

Now we will see that all root spaces $\mathfrak{g}_\alpha$ are in fact *one-dimensional*, and that no integer multiple of a root α is again a root, except for $\pm\alpha$. In other words, we have

$$\mathbb{Z}\alpha \cap \Phi = \{\alpha, -\alpha\}$$

for all $\alpha \in \Phi$.

PROPOSITION 2.6.11. *For every root $\alpha \in \Phi$ we have* $\dim(\mathfrak{g}_\alpha) = 1$, *and* $\dim(\mathfrak{g}_{n\alpha}) = 0$ *for all* $n \in \mathbb{Z} \setminus \{\pm 1\}$. *For every $\alpha \in \Phi$ there exists an injective Lie algebra homomorphism* $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$ *with*

$$\mathbb{C}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \cong \mathfrak{g}_\alpha, \quad \mathbb{C}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \cong \mathfrak{g}_{-\alpha}, \quad \mathbb{C}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cong [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

PROOF. Choose elements $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$. Because of $\alpha(h_\alpha) = 2$ we then have $[h_\alpha, x_\alpha] = 2x_\alpha$ and $[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$. This implies that

$$\mathfrak{s}_\alpha = \mathbb{C}x_\alpha \oplus \mathbb{C}x_{-\alpha} \oplus \mathbb{C}h_\alpha$$

is a 3-dimensional subalgebra of $\mathfrak{g}$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In fact, the isomorphism is given by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \to x_\alpha$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \to x_{-\alpha}$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \to h_\alpha$. The subalgebra $\mathfrak{s}_\alpha$ is well-defines as soon as we know that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. Then it also follows that $\mathfrak{g}_{\pm\alpha} = \langle x_{\pm\alpha} \rangle$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle h_\alpha \rangle$. To prove it, consider the subspace

$$\mathfrak{s} = \mathbb{C}x_{-\alpha} \oplus \mathbb{C}h_\alpha \oplus \bigoplus_{n \geq 1} \mathfrak{g}_{n\alpha},$$

which is invariant under $\mathrm{ad}(x_\alpha)$ and $\mathrm{ad}(x_{-\alpha})$. Then we have, with $\mathrm{ad} = \mathrm{ad}_{\mathfrak{s}}$ and $\alpha(h_\alpha) = 2$,

$$\begin{aligned} 0 &= \mathrm{tr}([\mathrm{ad}(x_\alpha), \mathrm{ad}(x_{-\alpha})]) \\ &= \mathrm{tr}(\mathrm{ad}([x_\alpha, x_{-\alpha}])) \\ &= \mathrm{tr}(\mathrm{ad}(h_\alpha)) \\ &= -\alpha(h_\alpha) + 0 + \sum_{n \geq 1} n \cdot \dim \mathfrak{g}_{n\alpha} \cdot \alpha(h_\alpha) \\ &= 2(-1 + \dim \mathfrak{g}_\alpha + \sum_{n \geq 2} n \cdot \dim \mathfrak{g}_{n\alpha}). \end{aligned}$$

So we obtain the Diophantine equation

$$1 = \dim \mathfrak{g}_\alpha + \sum_{n \geq 2} n \cdot \dim \mathfrak{g}_{n\alpha}.$$

On the right hand side all terms are non-negative integers. Assume that $\dim \mathfrak{g}_{n\alpha} > 0$ for some $n \geq 2$. Then the sum on the RHS would be at least 2, a contradiction. So we have $\dim \mathfrak{g}_{n\alpha} = 0$ for all $n \geq 2$ and $\dim \mathfrak{g}_\alpha = 1$. We can apply the same argument for $-\alpha$. Hence we also have $\dim \mathfrak{g}_{-\alpha} = 1$ and $\dim \mathfrak{g}_{-n\alpha} = 0$ für all $n \geq 2$. □

COROLLARY 2.6.12. *For $h, h' \in \mathfrak{h}$ we have*

$$\kappa(h, h') = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h').$$

We have

$$\dim \mathfrak{g} = \mathrm{rank}(\mathfrak{g}) + |\Phi| = \mathrm{rank}(\Phi) + |\Phi|.$$

PROOF. The formula for the Killing form follows from Proposition 2.6.1 and from $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Phi$. The second formula follows from the Cartan decomposition (2.6) and from $\dim \mathfrak{g}_\alpha = 1$. □

Let $\alpha, \beta \in \Phi$. It is customary to use the following notation

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha).$$

We have $\langle \alpha, \alpha^\vee \rangle = \alpha(\alpha^\vee) = \alpha(h_\alpha) = 2$. Furthermore the following assertions hold.

Lemma 2.6.13. *Let $\alpha, \beta \in \Phi$. Then we have*

(1) $\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) \in \mathbb{Z}$.
(2) $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$.
(3) $\beta - \langle \beta, \alpha^\vee \rangle \alpha \in \Phi$.

Proof. For (1): Let $\beta = \pm\alpha$. Then $\beta(h_\alpha) = \pm 2 \in \mathbb{Z}$. Otherwise we have $\beta \neq \pm\alpha$ and we set

$$\mathfrak{s} := \bigoplus_j \mathfrak{g}_{\beta + j\alpha}.$$

Every nonzero summand here is 1-dimensional and $\mathfrak{s}$ is a $\mathfrak{s}_\alpha$-module and hence a $\mathfrak{sl}_2(\mathbb{C})$-module. By assumption and by Proposition 2.6.11, β cannot be an integer multiple of α. Thus we have $\beta + k\alpha \neq 0$ for all $k \in \mathbb{Z}$ and

$$(\beta + k\alpha)(h_\alpha) = \beta(h_\alpha) + 2k.$$

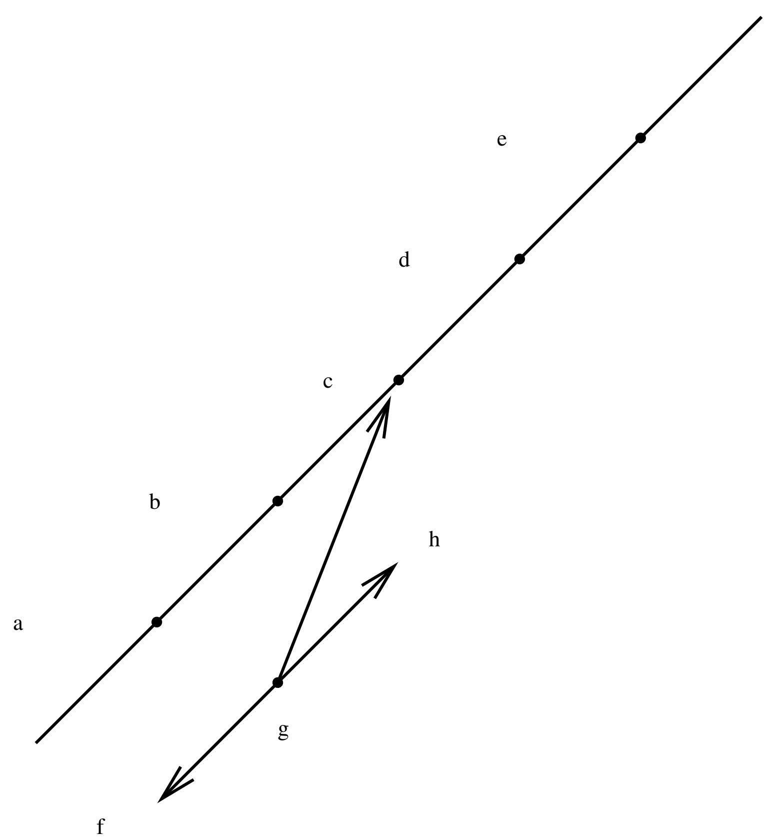
Hence all weights differ by a multiple of two and $\mathfrak{s}$ is a *simple* $\mathfrak{s}_\alpha$-module and we can use its classification. In particular we know that h_α acts with integral eigenvalues. More precisely we have the following. If $q \geq 0$ is the maximal integer with $\beta + q\alpha \in \Phi$ and $r \geq 0$ is the maximal integer with $\beta - r\alpha \in \Phi$, then the whole *string*

$$\beta - r\alpha,\ \beta - (r-1)\alpha,\ \ldots, \beta + q\alpha$$

lies in in Φ, see the picture below, and we have $\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$. In other words, we have

$$\beta(h_\alpha) = r - q \in \mathbb{Z}.$$

The numbers $\beta(h_\alpha)$ are called *Cartan numbers*.



For (2): The proof follows from the formula for the Killing form in Corollary 2.6.12.

For (3): In the root string $\beta - r\alpha, \dots, \beta + q\alpha$ we have in particular the root

$$\beta - (r-q)\alpha = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$$

This finishes the proof. $\square$

DEFINITION 2.6.14. We define a bilinear form of $\mathfrak{h}^*$ via the Killing form by

$$(\lambda, \mu) := \kappa(t_\lambda, t_\mu)$$

for $\lambda, \mu \in \mathfrak{h}^*$.

We have

$$\begin{aligned} \beta(h_\alpha) &= \langle \beta, \alpha^\vee \rangle \\ &= \kappa(t_\beta, h_\alpha) \\ &= 2\frac{\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} \\ &= \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \\ &= r - q \in \mathbb{Z}. \end{aligned}$$

The roots $\alpha \in \Phi$ generate the space $\mathfrak{h}^*$, but they are not linear independent. So we chose a basis $(\alpha_1, \dots, \alpha_\ell)$ of $\mathfrak{h}^*$. Each root $\beta \in \Phi$ then can be uniquely represented as a linear combination of the basis elements. We may assume that the coefficients are even rational (exercise). If $\mathfrak{h}^*_\mathbb{Q}$ denotes the subspace of $\mathfrak{h}^*$ generated by the roots over $\mathbb{Q}$, then we have $\dim_\mathbb{Q} \mathfrak{h}^*_\mathbb{Q} = \dim_\mathbb{C} \mathfrak{h}^*$. Let $E = \mathfrak{h}^*_\mathbb{R}$ be the *real* vector space generated by the $\alpha \in \Phi$. Then we have the following result.

PROPOSITION 2.6.15. *The restriction of the bilinear form (λ, μ) on E is a positive definite scalar product. The space E becomes a Euclidean vector space with it.*

PROOF. For $\lambda \in E$ we have $\alpha(t_\lambda) \in \mathbb{R}$ and hence

$$(\lambda, \lambda) = \kappa(t_\lambda, t_\lambda) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)^2 \geq 0$$

by Corollary 2.6.12. If $(\lambda, \lambda) = 0$ then $\alpha(t_\lambda) = 0$ for all $\alpha \in \Phi$ and hence $t_\lambda = 0$, and $\lambda = 0$. For $\lambda \neq 0$ we have $(\lambda, \lambda) > 0$, hence the scalar product is positive definite. $\square$

Instead of considering E in $\mathfrak{h}^*_\mathbb{R}$, we may view instead $E^* = \mathfrak{h}_\mathbb{R}$ in $\mathfrak{h}$. This is just the real subspace of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is generated by the coroots $\alpha^\vee$. Then the restriction of the Killing form on the real vector space $\mathfrak{h}_\mathbb{R} \times \mathfrak{h}_\mathbb{R}$ is a scalar product on $\mathfrak{h}_\mathbb{R}$. The subset $\Phi \subset E$ forms, together with this scalar product, a so-called *reduced root system.* As we have seen in Lemma 2.6.13, for $\alpha, \beta \in \Phi$, also

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

is in Φ. In our Euclidean space E this element s_α is a *reflection* at the hyperplane, which is orthogonal to α. Indeed, we have $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\lambda) = \lambda$ for $(\alpha, \lambda) = 0$. Note that these reflections leave Φ invariant.

DEFINITION 2.6.16. We denote by W the subgroup of the orthogonal group of E, generated by reflections s_α for $\alpha \in \Phi$. The group W is called the *Weyl group* of the root system Φ.

It is clear that W is a *finite* group, because all generating reflections, and hence the whole group leaves the finitely many roots invariant, and hence permute them. Since the roots generate E it follows that W is finite. In fact, W is uniquely determined, up to isomorphism, by the given semisimple complex Lie algebra $\mathfrak{g}$ and does not depend on the choice of a Cartan subalgebra, because all Cartan subalgebras are conjugated.

EXAMPLE 2.6.17. *Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then the diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$. Let $h = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}$ and $\varepsilon_i \in \mathfrak{h}^*$ for $i = 1, \ldots, n$, defined by $\varepsilon_i(h) = \lambda_i$. Then the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is given by*

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, \ i \neq j\}$$

and the Cartan decomposition is given by

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C} E_{ij}.$$

The equation $\dim(\mathfrak{g}) = \operatorname{rank}(\Phi) + |\Phi|$ *is given by* $n^2 - 1 = (n-1) + n(n-1)$ *and the Weyl group is given by* $W \cong S_n$.

To see this, recall that for $h \in \mathfrak{h}$ we have

$$\operatorname{ad}(h)(E_{ij}) = [h, E_{ij}] = (\varepsilon_i - \varepsilon_j)(h) E_{ij}.$$

Hence we have $\alpha_{ij} = \varepsilon_i - \varepsilon_j \in \Phi$. The root spaces are given by $\mathfrak{g}_\alpha = \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}$. They are 1-dimensional $\mathfrak{h}$-submodules of $\mathfrak{g}$, where $\mathfrak{g}$ is an $\mathfrak{h}$-module by the adjoint representation. By setting $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, n-1$ we obtain a basis of $\mathfrak{h}^*$ with $n - 1$ elements. If the $h_i = E_{ii} - E_{i+1,i+1}$ denote the standard basis of $\mathfrak{h}$, we have

$$\begin{aligned} \alpha_i(h_i) &= 2, \\ \alpha_i(h_{i\pm 1}) &= -1, \\ \alpha_i(h_j) &= 0, \quad |i - j| > 1. \end{aligned}$$

The coroots are just the elements $\alpha_i^\vee = h_{\alpha_i} = h_i$. So the $\alpha_i(h_j)$ are the Cartan numbers. The elements t_{α_i} are given by $t_{\alpha_i} = \frac{1}{2n} h_i$, and the Killing form is given by $\kappa(x, y) = 2n \operatorname{tr}(xy)$. The elements $E_{i,i+1}, E_{ii} - E_{i+1,i+1}, E_{i+1,i}$ form a subalgebra if $\mathfrak{g}$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

The Weyl group of $\mathfrak{sl}_n(\mathbb{C})$ is the symmetric group on n elements, S_n. Indeed, S_n acts on $\mathfrak{h}$ via conjugation by permutation matrices. This action induces an action on the dual space $\mathfrak{h}^*$. If P is a permutation matrix associated to a $\sigma \in S_n$, then this action on $\mathfrak{h}^*$ is just $P \cdot \varepsilon_i = \varepsilon_{\sigma(i)}$.

2.7. Abstract root systems

In this section we want to start the classification of simple (and hence semisimple) complex Lie algebras by classifying their root systems. Of course we need to justify that such Lie algebras are uniquely determined by their root systems and vice versa. To a given semisimple Lie algebra we can associate an *abstract* root system, which is unique up to root isomorphism and which does not depend on the choice of the Cartan subalgebra. Conversely we may construct to a given root system Φ a complex semisimple Lie algebra, up to isomorphism, whose root system is exactly Φ. This has been proved by Serre. We will treat this shortly in section 2.9.

DEFINITION 2.7.1. Let V be a finite-dimensional real vector space and $\alpha \in V$ with $\alpha \neq 0$. Then a *reflection along* α is an endomorphism s_α of V with $s_\alpha(\alpha) = -\alpha$ and $\dim(\operatorname{im}(\operatorname{id}_V - s_\alpha)) = 1$.

LEMMA 2.7.2. *Let $\alpha \in V$ with $\alpha \neq 0$. Then the following holds.*

(1) *There exists exactly one linear form $\alpha^\vee \in V^*$ with*
$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$
for all $\lambda \in V$. Then $\langle \alpha, \alpha^\vee \rangle = 2$.

(2) *We have $s_\alpha^2 = \operatorname{id}_V$ and $\det(s_\alpha) = -1$.*

(3) *The fixed point set $\{\lambda \in V \mid s_\alpha(\lambda) = \lambda\} = \ker(\alpha^\vee)$ is a hyperplane not containing α.*

(4) *Let $\beta \in V, \beta^\vee \in V^*$ with $\langle \beta, \beta^\vee \rangle = 2$. Then*
$$s_{\beta,\beta^\vee}(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta$$
defines a reflection along β.

PROOF. For (1): Because of $(\operatorname{id}_V - s_\alpha)(\alpha) = 2\alpha$ and $\dim(\operatorname{im}(\operatorname{id}_V - s_\alpha)) = 1$ we have $\operatorname{im}(\operatorname{id}_V - s_\alpha) = \mathbb{R}\alpha$. Therefore there exists exactly one linear form $\alpha^\vee \in V^*$, $\alpha^\vee \neq 0$ with
$$(\operatorname{id}_V - s_\alpha)(\lambda) = \langle \lambda, \alpha^\vee \rangle \alpha.$$
This shows the formula in (1). Furthermore we have $-\alpha = s_\alpha(\alpha) = \alpha - \langle \alpha, \alpha^\vee \rangle \alpha$, so that $\langle \alpha, \alpha^\vee \rangle = 2$.

For (2): We have, for all $\lambda \in V$,
$$\begin{aligned} s_\alpha^2(\lambda) &= s_\alpha(\lambda - \langle \lambda, \alpha^\vee \rangle \alpha) \\ &= s_\alpha(\lambda) - \langle \lambda, \alpha^\vee \rangle s_\alpha(\alpha) \\ &= \lambda - \langle \lambda, \alpha^\vee \rangle \alpha - \langle \lambda, \alpha^\vee \rangle (-\alpha) \\ &= \lambda. \end{aligned}$$
Then by (3) all eigenvalues of s_α on $\ker(\alpha^\vee)$ are equal to 1. Since $s_\alpha(\alpha) = -\alpha$ it follows that $\det(s_\alpha) = -1$.

For (3): The fixed point set is equal to $\{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \alpha = 0\} = \ker(\alpha^\vee)$. This is indeed a hyperplane in V.

For (4): This is easy to see. □

DEFINITION 2.7.3. A subset $\Phi \subset V$ of a Euclidean vector space V is called an *abstract root system* in V, if the following axioms hold.

(1) Φ is finite, generates V and does not contain the zero vector.
(2) For each root $\alpha \in \Phi$ there exists a reflection s_α along α with $s_\alpha(\Phi) = \Phi$.
(3) If $\alpha, \beta \in \Phi$ and s_α is the reflection along α with $s_\alpha(\Phi) = \Phi$, then $\beta - s_\alpha(\beta) \in \mathbb{Z}\alpha$.
(4) If $\alpha \in \Phi$, then $2\alpha \notin \Phi$.

The *trivial* root system if the root system $\Phi = \emptyset$ in $V = 0$. The *rank* of Φ is the dimension of V. Two root systems (Φ_1, V_1) and (Φ_2, V_2) are called *isomorphic*, if there exists an isomorphism $\varphi \colon V_1 \to V_2$ with $\varphi(\Phi_1) = \Phi_2$. If (Φ_1, V_1) and (Φ_2, V_2) are root systems, then $\Phi_1 \times 0 \, \cup \, 0 \times \Phi_2$ is a root system in $V_1 \oplus V_2$, which we will denote by $\Phi_1 \oplus \Phi_2$.

DEFINITION 2.7.4. A root system of the form $\Phi_1 \oplus \Phi_2$, where Φ_1 and Φ_2 are both non-trivial, is called *reducible*. A root system is called *irreducible*, if it is not reducible and non-trivial.

Every root system can be uniquely decomposed into irreducible components. Condition (3) is called the *crystallographic restriction*. It says that

$$\langle \beta, \alpha^\vee \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$$

for all $\alpha, \beta \in \Phi$, where $(\cdot\,, \cdot)$ is a suitable scalar product on V. This already implies that we even have, for $\beta \neq \pm\alpha$,

$$\langle \beta, \alpha^\vee \rangle \in \{0, \pm 1, \pm 2, \pm 3\},$$

as we will see later. This already restricts possible examples of root systems quite a lot, as we will see below. Let us consider all possible root systems of rank one and two.

$\ell = 1$: The root system A_1 of rank 1 consists of $\{\alpha, -\alpha\}$.

$\ell = 2$: We find exactly four different root systems.

Case 1: The reducible root system $A_1 \oplus A_1$ of rank 2 consists of $\{\pm\alpha, \pm\beta\}$.

Case 2: The root system A_2 consists of $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$. With $\mathfrak{h}_\mathbb{R} \cong \mathbb{R}^2$ and the canonical scalar product on $\mathbb{R}^2$ we can view the roots as in the picture below as vectors in $\mathbb{R}^2$. Indeed, for

$$\alpha = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we have $(\alpha, \alpha) = (\beta, \beta) = 1$ and

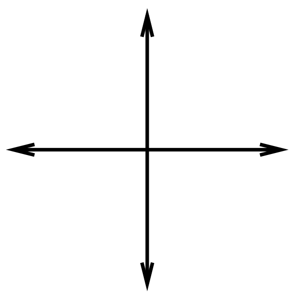
$$\begin{aligned} \langle \alpha, \alpha^\vee \rangle &= \frac{2(\alpha, \alpha)}{(\alpha, \alpha)} = 2, \\ \langle \alpha, \beta^\vee \rangle &= \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = -1, \\ \langle \beta, \beta^\vee \rangle &= 2. \end{aligned}$$

One easily checks that all conditions for a root system are satisfied. The angles between the roots are multiples of 60 degree. The Cartan numbers here are $\langle \alpha, \alpha^\vee \rangle = \langle \beta, \beta^\vee \rangle = 2$ and

$\langle\alpha,\beta^\vee\rangle = \langle\beta,\alpha^\vee\rangle = -1$. One summarizes these numbers usually by a matrix, which looks here as follows,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

This matrix is called the Cartan matrix of A_2.



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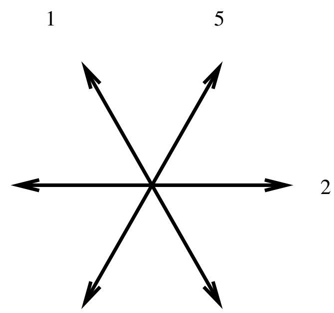

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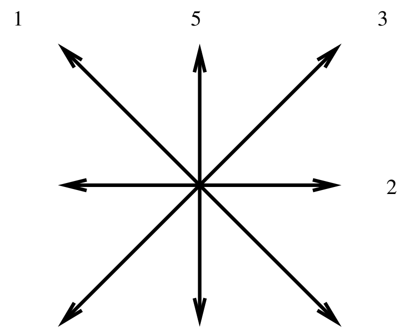

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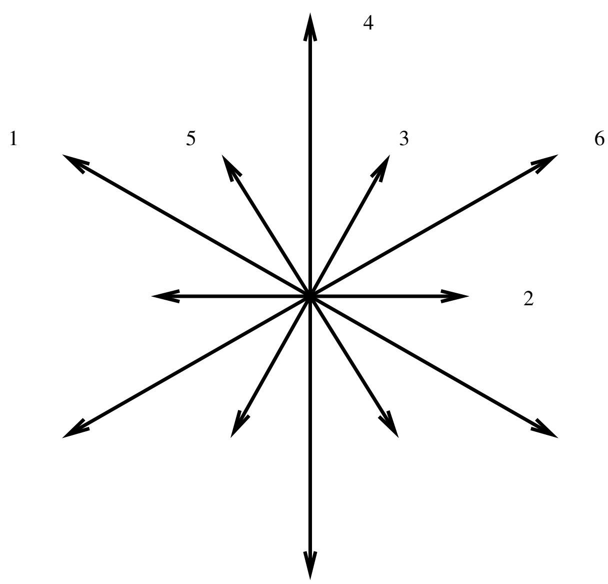


d

Case 3: The root system B_2 is given by $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta)\}$. We can represent it by the vectors

$$\alpha = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and obtain

$$\langle\alpha,\beta^\vee\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -1,$$
$$\langle\beta,\alpha^\vee\rangle = -2.$$

Hence the Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

Case 4: The root system G_2 is given by

$$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta), \pm(\alpha+3\beta), \pm(2\alpha+3\beta)\}.$$

We can represent it by choosing $\alpha = (-3/2, \sqrt{3}/2)$ and $\beta = (1,0)$. The Cartan matrix of G_2 is given by

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Of course one needs to justify that we have found all root systems of rank two. We will leave this here as an exercise.

DEFINITION 2.7.5. Let (Φ, V) be a root system. Let

$$A(\Phi) = \{\varphi \in \operatorname{Aut}(V) \mid \varphi(\Phi) = \Phi\}$$

be the group of automorphisms leaving Φ invariant. The subgroup $W = W(\Phi)$ of $A(\Phi)$, generated by the reflections s_α, $\alpha \in \Phi$, is called the *Weyl group* of the root system Φ.

Since Φ spans the vector space V, each $\varphi \in A(\Phi)$ is determined by its restriction to Φ. Hence $A(\Phi)$ is isomorphic to a subgroup of the symmetric group $\operatorname{Sym}(\Phi)$, hence finite.

LEMMA 2.7.6. *Let (Φ, V) be a root system. Then the following holds.*

(1) *We have $\Phi = -\Phi$.*
(2) *For $\alpha \in \Phi$ the reflection s_α along α with $s_\alpha(\Phi) = \Phi$ is uniquely determined.*
(3) *There exists a uniquely determined injective map $\Phi \to V^*$, $\alpha \mapsto \alpha^\vee$, such that the reflection s_α along α with $s_\alpha(\Phi) = \Phi$ is given by*

$$s_\alpha \colon \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

for all $\lambda \in V$.

(4) *For $\alpha, \beta \in \Phi$ we have $\langle \alpha, \alpha^\vee \rangle = 2$ and $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$.*

PROOF. For (1): If $\alpha \in \Phi$ then $-\alpha = s_\alpha(\alpha) \in \Phi$.

For (2): Let s'_α be another reflection along α with $s'_\alpha(\Phi) = \Phi$. Then consider $\varphi = s_\alpha s'_\alpha \in A(\Phi)$. We have

$$(\mathrm{id}_V - \varphi) = (\mathrm{id}_V - s_\alpha) s'_\alpha + (\mathrm{id}_V - s'_\alpha),$$

so that $\operatorname{im}(\mathrm{id}_V - \varphi) \subset \mathbb{R}\alpha$. Hence there is an $\alpha^* \in V^*$ with

$$\varphi(\lambda) = \lambda + \langle \lambda, \alpha^* \rangle \alpha.$$

By induction, using $\alpha \in \ker(\alpha^*)$, it follows that

$$\varphi^n(\lambda) = \lambda + n\langle \lambda, \alpha^* \rangle \alpha.$$

So we have, with $n = |A(\Phi)|$,

$$\lambda = \varphi^n(\lambda) = \lambda + n\langle \lambda, \alpha^* \rangle \alpha,$$

hence $|A(\Phi)|\langle \lambda, \alpha^* \rangle \alpha = 0$ for all $\lambda \in V$, and thus $\alpha^* = 0$ and $\varphi = \mathrm{id}$. So we have $s'_\alpha = s_\alpha^{-1} = s_\alpha$.

For (3): It remains to show the injectivity of the map $\alpha \mapsto \alpha^\vee$. This follows from Lemma 2.7.8.

For (4): Because of $\beta \neq 0$ the claim follows from $\langle \alpha, \beta^\vee \rangle \beta = \alpha - s_\beta(\alpha) \in \mathbb{Z}\beta$. $\square$

We also mention the following result, which is easy to show.

LEMMA 2.7.7. *For $\varphi \in A(\Phi)$ and $\alpha \in \Phi$ we have $\varphi \circ s_\alpha \circ \varphi^{-1} = s_{\varphi(\alpha)}$. In particular, $W(\Phi)$ is a normal subgroup of $A(\Phi)$.*

The next result is as follows.

LEMMA 2.7.8. *There exists a scalar product $(\cdot\,,\cdot)$ on V, which is $A(\Phi)$-invariant, i.e., which satisfies $(\varphi(\lambda), \varphi(\mu)) = (\lambda, \mu)$ for all $\lambda, \mu \in V$ and $\varphi \in A(\Phi)$. For $\alpha \in \Phi$ we have*

$$\langle \lambda, \alpha^\vee \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

PROOF. Let $((\cdot\,,\cdot))$ be any scalar product on V. Then

$$(\lambda, \mu) = \frac{1}{|A(\Phi)|} \sum_{\varphi \in A(\Phi)} ((\varphi(\lambda), \varphi(\mu)))$$

defines an $A(\Phi)$-invariant scalar product on V. For $\alpha \in \Phi$ ans $\lambda \in V$ we have

$$\begin{aligned} (\alpha, s_\alpha(\lambda) + \lambda) &= (s_\alpha(\alpha), s_\alpha(s_\alpha(\lambda) + \lambda)) \\ &= (-\alpha, s_\alpha(\lambda) + \lambda). \end{aligned}$$

So we obtain

$$\begin{aligned} 0 &= (\alpha, s_\alpha(\lambda) + \lambda) \\ &= (\alpha, 2\lambda - \langle \lambda, \alpha^\vee \rangle \alpha) \\ &= 2(\alpha, \lambda) - \langle \lambda, \alpha^\vee \rangle (\alpha, \alpha). \end{aligned}$$

□

In case Φ is irreducible we have only one such invariant scalar product, up to positive multiples. If Φ is reducible, then the different irreducible components are orthogonal to each other. We may normalize the scalar product on the irreducible components Ψ von Φ by $\max_{\alpha \in \Psi}(\alpha, \alpha) = 2$.

For a given root system (Φ, V) define the *dual root system* $(\Phi^\vee, V^*)$ by

$$\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}.$$

We have $(\alpha^\vee)^\vee = \alpha$ for all $\alpha \in \Phi \subset V = V^{**}$. We can identify the Weyl groups of Φ and of Φ^*.

In the next step we'll consider all possible angles $0 \leq \sphericalangle(\alpha, \beta) \leq \pi$ of two roots $\alpha, \beta \in \Phi$. Clearly we have $\sphericalangle(\alpha, \alpha) = 0$ and $\sphericalangle(\alpha, -\alpha) = \pi$. The following result shows that there are only seven different possibilities.

PROPOSITION 2.7.9. *Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Then, up to permutation of α and β, exactly one of the following seven cases can arise.*

$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	$\sphericalangle(\alpha,\beta)$	$\frac{(\alpha,\alpha)}{(\beta,\beta)}$
0	0	$\pi/2$	–
1	1	$\pi/3$	1
−1	−1	$2\pi/3$	1
1	2	$\pi/4$	2
−1	−2	$3\pi/4$	2
1	3	$\pi/6$	3
−1	−3	$5\pi/6$	3

Proof. Let $\theta = \sphericalangle(\alpha, \beta)$. Then we have

$$(\alpha, \beta) = \sqrt{(\alpha, \alpha)}\sqrt{(\beta, \beta)} \cos \theta.$$

This implies that

$$4 \cos^2 \theta = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \cdot 2\frac{(\beta, \alpha)}{(\beta, \beta)} = \langle \alpha, \beta^\vee \rangle \cdot \langle \beta, \alpha^\vee \rangle.$$

The factors on the RHS are integers. Hence we have $4\cos^2\theta \in \mathbb{Z}$ with $0 \le 4\cos^2\theta \le 4$. Because of $\beta \neq \pm\alpha$ it follows that

$$4\cos^2\theta \in \{0, 1, 2, 3\}.$$

If $\theta = 0$ or $\theta = \pi$, then $\beta = \pm\alpha$, because other multiples of α don't lie in Φ. This was excluded. So it is enough to discuss each of the above four possibilities for $4\cos^2\theta$.

Case 1: We have $4\cos^2\theta = 0$. Then $\theta = \pi/2$ and $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$.

Case 2: We have $4\cos^2\theta = 1$. Then we have either $\cos\theta = 1/2$, hence $\theta = \pi/3$ and $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 1$, or we have $\cos\theta = -1/2$ and $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$.

The other two cases are similar. □

As a corollary we obtain that our list of root systems in the rank two case is already complete. We also obtain that we can have, in an irreducible root system, at most two different root lengths. The length of a root α is given by $\|\alpha\| = \sqrt{(\alpha, \alpha)}$. For $\theta = \pi/3$ or $\theta = 2\pi/3$ the roots have equal length. But for $\theta = \pi/4, 3\pi/4$ the ratio of the length is always equal to $\sqrt{2}$. Recall that we always assume here that $\beta \neq \pm\alpha$. So we really can have different root lengths. For $\theta = \pi/6, 5\pi/6$ the ratio of the root lengths is given by $\sqrt{3}$. In the case that not all roots have equal length we speak of *long* and *short* roots.

Lemma 2.7.10. *For $\alpha, \beta \in \Phi$ with $\alpha \neq \beta$ and $(\alpha, \beta) > 0$ we have $\alpha - \beta \in \Phi$.*

Proof. If $(\alpha, \beta) > 0$ then $\langle \alpha, \beta^\vee \rangle > 0$. By our above table then either $\langle \alpha, \beta^\vee \rangle = 1$ and $\alpha - \beta = s_\beta(\alpha) \in \Phi$, or by permuting α and β, $\langle \beta, \alpha^\vee \rangle = 1$ and $\alpha - \beta = -s_\alpha(\beta) \in \Phi$. □

Each hyperplane passing though zero in V not containing a root decomposes V into two half spaces V_+ and V_-, so that Φ is decomposed into *positive roots* $\Phi_+ = \Phi \cap V_+$ and *negative roots* $\Phi_- = \Phi \cap V_-$.

Definition 2.7.11. Let Φ be a root system in V and Φ_+ be the set of positive roots. A root $\alpha \in \Phi_+$ is called *simple*, if it cannot be written as a sum of two positive roots. The set Π of simple roots is called a *basis* of Φ.

The name "basis" for Π is justified as follows.

PROPOSITION 2.7.12. *Let Φ be a root system with basis $\Pi \subset \Phi$. Then Π is an $\mathbb{R}$-linearly independent set and every root $\gamma \in \Phi$ can be written as a linear combination*

$$\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha,$$

where the coefficients n_α are either all in $\mathbb{Z}_{\geq 0}$, or all in $\mathbb{Z}_{\leq 0}$.

PROOF. Let $\alpha, \beta \in \Pi$ be simple roots with $\alpha \neq \beta$. Then $(\alpha, \beta) \leq 0$, i.e., the angle between two different roots is always obtuse. Otherwise we would have $\alpha - \beta \in \Phi$ by Lemma 2.7.10, and hence $\pm(\alpha - \beta) \in \Phi_+$. Because of

$$\begin{aligned} \alpha &= (\alpha - \beta) + \beta \\ \beta &= (\beta - \alpha) + \alpha \end{aligned}$$

this would yield a contradiction to the simplicity of α or β.

To show that the simple roots are linearly independent over $\mathbb{R}$, suppose that

$$\sum_{\alpha \in \Pi} r_\alpha \alpha = 0.$$

Then let

$$\varepsilon = \sum_{\substack{\alpha \in \Pi \\ r_\alpha > 0}} r_\alpha \alpha = \sum_{\substack{\beta \in \Pi \\ r_\beta < 0}} (-r_\beta)\beta.$$

We obtain that

$$(\varepsilon, \varepsilon) = \sum_{\substack{\alpha, \beta \in \Pi \\ r_\alpha > 0,\, r_\beta < 0}} r_\alpha(-r_\beta)(\alpha, \beta).$$

because of $(\alpha, \beta) \leq 0$ we obtain $(\varepsilon, \varepsilon) \leq 0$ and therefore $\varepsilon = 0$. By construction of Φ_+ there exists an $\eta \in V$ with $(\eta, \alpha) > 0$ for all $\alpha \in \Phi_+$. It follows that

$$0 = (\eta, \varepsilon) = \sum_{\substack{\alpha \in \Pi \\ r_\alpha > 0}} r_\alpha(\eta, \alpha) = \sum_{\substack{\beta \in \Pi \\ r_\beta < 0}} -r_\beta(\eta, \beta) > 0.$$

So the sums must be empty and we obtain $r_\alpha = 0$ for all $\alpha \in \Pi$.

For the second claim we may assume that $\gamma \in \Phi_+$, otherwise we pass to $-\gamma$. If $\gamma \in \Pi$, then we are done. Otherwise we have $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1, \gamma_2 \in \Phi_+$. As above it follows then, with η, that

$$(\eta, \gamma) > (\eta, \gamma_1) \text{ and } (\eta, \gamma) > (\eta, \gamma_2).$$

We can proceed inductively. After finitely many steps we are done, since Φ_+ is a finite set. $\square$

For a given representation $\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha$ we define the *height* of γ by

$$ht(\gamma) = \sum_{\alpha \in \Pi} n_\alpha.$$

DEFINITION 2.7.13. Let Φ be a root system with basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. Define the *Cartan matrix* $A = (A_{ij}) \in M_\ell(\mathbb{Z})$ by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

By Proposition 2.7.9 we have the following result.

Proposition 2.7.14. *The Cartan matrix of a root system has the following properties.*

(1) $A_{ii} = 2$ *for all* $i = 1, \dots, \ell$.
(2) $A_{ij} \in \{0, -1, -2, -3\}$ *for* $i \neq j$.
(3) $A_{ij} = 0 \Longleftrightarrow A_{ji} = 0$.
(4) $A_{ij} \in \{-2, -3\} \Longrightarrow A_{ji} = -1$.

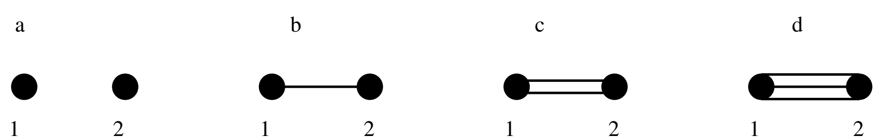
Define integers $n_{ij} := A_{ij}A_{ji}$. We have $n_{ij} \in \{0, 1, 2, 3\}$ für $i \neq j$.

Definition 2.7.15. Let Φ be a root system with basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. The *Dynkin diagram* of Φ is the graph, which is given as follows: for each simple root α_i there exists a vertex i, and each two different vertices i and j are connected by exactly n_{ij} edges.

To every Dynkin diagram we associate the quadratic form

$$Q(x_1, \dots, x_\ell) := 2\sum_{i=1}^{\ell} x_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \sqrt{n_{ij}}x_ix_j.$$

The Dynkin diagram and its quadratic form are determined by the Cartan matrix. For the four root systems of rank two they are given as follows.



Φ	A	Q
$A_1 \oplus A_1$	$\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$	$2x_1^2 + 2x_2^2$
A_2	$\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$	$2x_1^2 - 2x_1x_2 + 2x_2^2$
B_2	$\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$	$2x_1^2 - 2\sqrt{2}x_1x_2 + 2x_2^2$
G_2	$\left(\begin{smallmatrix} 2 & -1 \\ -3 & 2 \end{smallmatrix}\right)$	$2x_1^2 - 2\sqrt{3}x_1x_2 + 2x_2^2$

Note that the Dynkin diagram is connected if and only if Φ is irreducible.

Proposition 2.7.16. *The quadratic form* $Q(x_1, \dots, x_\ell)$ *of a root system* Φ *is positive definite.*

Proof. Recall that we have, for $i \neq j$, that

$$n_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \cdot 2\frac{(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)}.$$

because of $(\alpha_i, \alpha_j) \leq 0$ we obtain

$$-\sqrt{n_{ij}} = 2\frac{(\alpha_i, \alpha_j)}{\|\alpha_i\|\|\alpha_j\|}.$$

Then we can rewrite Q as follows:

$$\begin{aligned} Q(x_1, \dots, x_\ell) &= \sum_{i,j=1}^{\ell} \frac{2(\alpha_i, \alpha_j)}{\|\alpha_i\|\|\alpha_j\|} x_i x_j \\ &= 2\Big(\sum_{i=1}^{\ell} \frac{x_i \alpha_i}{\|\alpha_i\|}, \sum_{j=1}^{\ell} \frac{x_j \alpha_j}{\|\alpha_j\|}\Big) \\ &= 2(y, y). \end{aligned}$$

This shows that $Q(x_1, \dots, x_\ell) \geq 0$. If $Q(x_1, \dots, x_\ell) = 0$, then it follows that $y = 0$. Since the α_i are linearly independent, this implies $x_i = 0$ for all i and we are done. $\square$

2.8. The classification of Dynkin diagrams

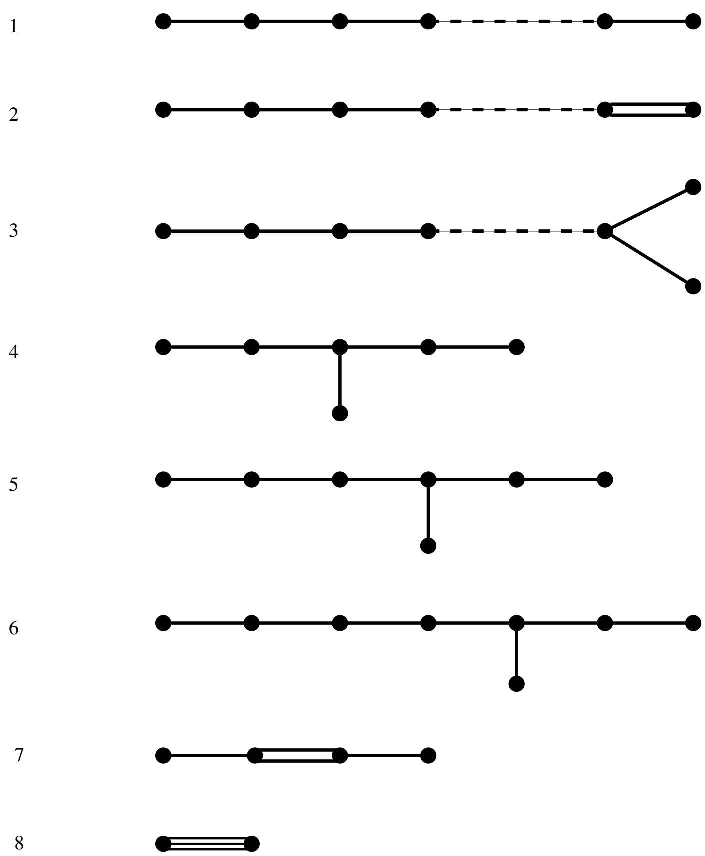
Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ. Then the Cartan matrix and hence the Dynkin diagram is independent of the choice of $\mathfrak{h}$, and also independent, up to renumbering of the fundamental roots $\alpha_1, \dots, \alpha_\ell \in \Pi$. The connected components of the Dynkin diagrams satisfy the following properties:

(A) The graph is connected.

(B) Two different vertices are connected by $0, 1, 2$ or 3 edges.

(C) The associated quadratic form is positive definite.

Theorem 2.8.1. *The graphs satisfying the conditions $(A), (B), (C)$ are given as follows:*



Here we have $\ell \geq 1$ for A_ℓ, $\ell \geq 2$ for B_ℓ and $\ell \geq 4$ for D_ℓ.

Proof. First we show that all listed graphs satisfy the three conditions. This is clear for conditions (A) and (B). Hence we will focus on (C). By Sylvester's criterion, a quadratic form $\sum_{i,j} a_{ij}x_ix_j$ is positive definite if and only if all of the leading principal minors are positive, i.e., if the principal submatrices of its symmetric matrix (a_{ij}) have positive determinant, i.e., if

$$\det(a_{11}) > 0,\ \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0, \dots, \det(a_{ij}) > 0.$$

So let Γ be a graph with ℓ vertices if the above list. We'll show by induction on ℓ, that $Q(x_1, \dots, x_\ell)$ is positive definite.

$\ell = 1$: Then $\Gamma = A_1$ and $Q(x_1) = 2x_1^2$ is positive definite.

$\ell = 2$: Then Γ is A_2, B_2 or G_2. The symmetric matrices representing $Q(x_1, x_2)$ for these types have positive leading principal minors:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}.$$

$\ell \geq 3$: Every graph Γ of the list given in Theorem 2.8.1 has at least one *outer vertex*, say ℓ, which is connected to exactly one *other* vertex, say $\ell - 1$, by exactly one edge. Denote this graph by $\boldsymbol{\Gamma_\ell}$, and by $\boldsymbol{\Gamma_{\ell-1}}$ the graph obtained from Γ_ℓ by deleting the vertex ℓ. Furthermore let $\boldsymbol{\Gamma_{\ell-2}}$ be the graph obtained from $\Gamma_{\ell-1}$ by deleting the vertex $\ell-1$. We note that the graphs $\Gamma_{\ell-1}$ and $\Gamma_{\ell-2}$ again are contained in the list. Denote by $\boldsymbol{S_\ell}$ the symmetric matrix representing the quadratic form $Q(x_1, \dots, x_\ell)$ of Γ_ℓ. The last column of S_ℓ is given by $(0, \dots, 0, -1, 2)^t$. Using Laplace expansion along this column for $\det S_\ell$ we obtain

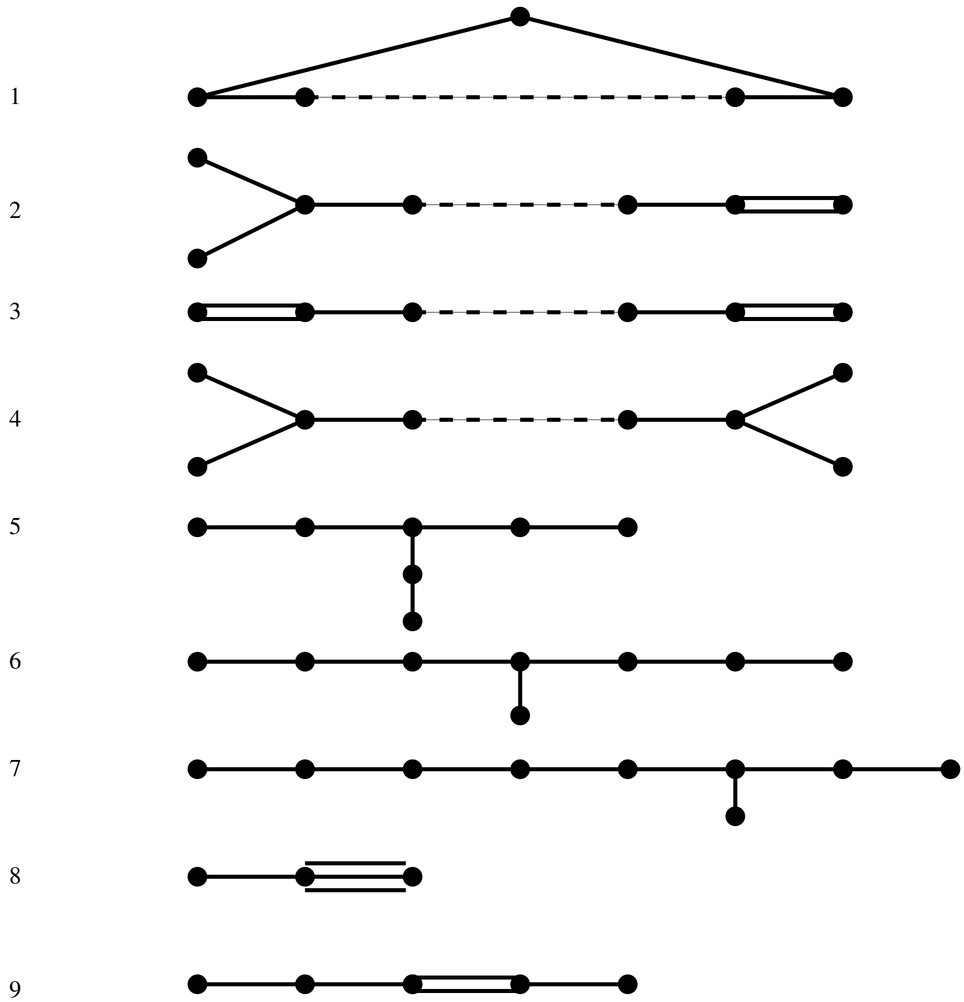
$$\det S_\ell = 2 \det S_{\ell-1} - \det S_{\ell-2}.$$

This enables us to compute $\det S_\ell$ for all graphs of the list inductively. For example, consider the graph of type A_ℓ. We already know that $\det A_1 = 2$ and $\det A_2 = 3$, see above. Then by $\det A_\ell = 2 \det A_{\ell-1} - \det A_{\ell-2}$ we obtain $\det A_\ell = \ell + 1$ by induction. Indeed, deleting an outer vertex ℓ from the graph A_ℓ, one obtains $A_{\ell-1}$, and so on. Similarly we obtain

$$\begin{aligned}
\det B_\ell &= 2, \\
\det D_\ell &= 4, \\
\det F_4 &= 2 \det B_3 - \det B_2 = 1, \\
\det E_6 &= 2 \det D_5 - \det A_4 = 3, \\
\det E_7 &= 2 \det D_6 - \det A_5 = 2, \\
\det E_8 &= 2 \det D_7 - \det A_6 = 1.
\end{aligned}$$

Note that $\Gamma_{\ell-1}$ and $\Gamma_{\ell-2}$ may not be of the same type as Γ_ℓ. For example, deleting an outer vertex ℓ of F_4 one obtains B_3. Deleting further $\ell - 1$, we obtain B_2. But in every case the determinants are positive. Now the principal minors of the symmetric matrix for Γ_ℓ are themselves symmetric matrices to certain subgraphs of Γ_ℓ. We may chose a numbering such that all subgraphs are connected. The given list has the nice property that every connected subgraph again appears in the list. Hence the determinant of *every* principal minor of the given symmetric matrix is positive. Thus the quadratic form for Γ_ℓ is positive definite.
It remains to prove the converse direction, namely that every graph satisfying $(A), (B), (C)$ is contained in our list. We need some more lemmas.

LEMMA 2.8.2. *For every graph of the following list, the determinant of the associated quadratic form $Q(x_1, \dots, x_\ell)$ equals zero.*



Here $\ell \geq 2$ for $A_\ell^{(1)}$ and $C_\ell^{(1)}$, $\ell \geq 3$ for $B_\ell^{(1)}$ and $\ell \geq 4$ for $D_\ell^{(1)}$.

Note that a graph Γ_ℓ in this list has $\ell + 1$ vertices. We might consider the graph of, say, $A_\ell^{(1)}$ as a regular $(\ell+1)$-gon, for each $\ell \geq 2$. We'll also need the next lemma, in order to prove the converse direction of the theorem.

LEMMA 2.8.3. *Let Γ be a graph satisfying $(A), (B), (C)$, and Γ' be a connected graph obtained from Γ by either deleting vertices or by decreasing the number of edges between two vertices. Then also Γ' satisfies $(A), (B), (C)$.*

Proof of Theorem 2.8.1: Let Γ be a graph satisfying $(A), (B), (C)$. Let us call the list of graphs from the list of Lemma 2.8.2 the list of *forbidden subgraphs*. This name is justified by Lemma 2.8.2 and Lemma 2.8.3, which imply that Γ must not contain such a subgraph. In particular, Γ cannot contain any cycles, because otherwise Γ would have a forbidden subgraph of type $A_\ell^{(1)}$ for some $\ell \geq 2$.

Assume that Γ contains a triple edge. Then Γ is the graph G_2, because otherwise Γ would contain a forbidden subgraph $G_2^{(1)}$. So we may assume from now on that Γ contains no triple edge. Then Γ can only contain a *single* double edge, because otherwise we'd have a forbidden subgraph $C_\ell^{(1)}$ for some $\ell \geq 2$. Moreover Γ cannot in addition a a double edge also have a ramification point, because otherwise we'd have a forbidden subgraph $B_\ell^{(1)}$ for some $\ell \geq 3$. Thus Γ looks like a chain with only a single double edge. In case this double edge sits at an end, our graph is B_ℓ. If not, we have F_4, because otherwise we'd have a forbidden subgraph $F_4^{(1)}$. Form now on we may assume that Γ has no double and no triple edges. In case Γ has no ramification point, we have the graph A_ℓ for some $\ell \geq 1$. In case there *is* a ramification point, it must be the only one, because otherwise we'd have a forbidden subgraph $D_\ell^{(1)}$ for some $\ell \geq 5$. Hence Γ has exactly one ramification point $\boldsymbol{P}$, from which exactly three edges lead off, because otherwise we'd have a subgraph $D_4^{(1)}$. Denote the number of vertices on these three edges by ℓ_1, ℓ_2, ℓ_3, where mit $\ell_1 \geq \ell_2 \geq \ell_3$. Then Γ has in total $1 + \ell_1 + \ell_2 + \ell_3$ vertices, with P as vertex in the center of Γ. We have $\ell_3 = 1$, because otherwise we would have $\ell_i \geq 2$ for all i, so that there would be a subgraph $E_6^{(1)}$. In case that $\ell_2 = 1$, we have $\Gamma = D_\ell$ for some $\ell \geq 4$. If $\ell_2 > 1$, then it follows that $\ell_2 = 2$, because otherwise $\ell_1, \ell_2 \geq 3$ and Γ would have a subgraph $E_7^{(1)}$. Thus we may assume now that $\ell_3 = 1, \ell_2 = 2$. Then $\ell_1 \leq 4$, because otherwise we'd have a subgraph $E_8^{(1)}$. So Γ is of type E_6, E_7 or E_8. We are indeed done. All graphs satisfying $(A), (B), (C)$ are contained in the list of Theorem 2.8.1. □

Proof of Lemma 2.8.2 :

PROOF. First let $\Gamma = A_\ell^{(1)}$. Each row of the symmetric matrix for the associated quadratic form has one entry equal to 2, and two entries equal to -1, and the other entries equal to zero. The sum of all column vectors of this matrix then is zero, since each component of this vector is the sum of the row entries, hence equal to $2 - 1 - 1 = 0$. Hence $\det A_\ell^{(1)} = 0$.

For all other graphs Γ we can find an outer vertex ℓ, which is connected exactly to one other vertex $\ell - 1$, by either a single or a double edge. In case of a single edge, we have as above

$$\det S_\ell = 2 \det S_{\ell-1} - \det S_{\ell-2}.$$

In case of a double edge we have

$$\det S_\ell = 2 \det S_{\ell-1} - 2 \det S_{\ell-2}.$$

This enables us to compute all determinants inductively as we have done before.

$$\begin{aligned}
\det B_3^{(1)} &= 2 \det A_3 - 2(\det A_1)^2 = 0, \\
\det C_2^{(1)} &= 2 \det B_2 - 2 \det A_1 = 0, \\
\det D_4^{(1)} &= 2 \det D_4 - (\det A_1)^3 = 0, \\
\det E_6^{(1)} &= 2 \det E_6 - \det A_5 = 0, \\
\det E_7^{(1)} &= 2 \det E_7 - \det D_6 = 0, \\
\det E_8^{(1)} &= 2 \det E_8 - \det E_7 = 0, \\
\det G_2^{(1)} &= 2 \det G_2 - \det A_1 = 0, \\
\det F_4^{(1)} &= 2 \det F_4 - \det B_3 = 0.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
\det B_\ell^{(1)} &= 2\det D_\ell - 2\det D_{\ell-1} = 0, \quad \ell \geq 4,\\
\det C_\ell^{(1)} &= 2\det B_\ell - 2\det B_{\ell-1} = 0, \quad \ell \geq 3,\\
\det D_\ell^{(1)} &= 2\det D_\ell - \det A_1 D_{\ell-2} = 0, \quad \ell \geq 5.
\end{aligned}$$

□

Proof of Lemma 2.8.3 :

PROOF. We only need to show that Γ' satisfies again (C). So let $Q(x_1, \dots, x_\ell)$ be the quadratic form of Γ, and $Q'(x_1, \dots, x_m)$ the one from Γ', with $m \leq \ell$. Then we have

$$\begin{aligned}
Q(x_1, \dots, x_\ell) &= 2\sum_{i=1}^{\ell} x_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \sqrt{n_{ij}}x_ix_j,\\
Q'(x_1, \dots, x_m) &= 2\sum_{i=1}^{m} x_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{m} \sqrt{n'_{ij}}x_ix_j,
\end{aligned}$$

with integers $n'_{ij} \leq n_{ij}$ for $1 \leq i, j \leq m$. Assume that Q' is not positive definite. Then there exist real numbers $y_1, \dots, y_m$, not all zero, such that

$$Q'(y_1, \dots, y_m) \leq 0.$$

Then we also have

$$\begin{aligned}
Q(|y_1|, \dots, |y_m|, 0, \dots, 0) &= 2\sum_{i=1}^{m} y_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{m} \sqrt{n_{ij}}|y_i||y_j|\\
&\leq 2\sum_{i=1}^{m} y_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{m} \sqrt{n'_{ij}}y_iy_j\\
&= Q'(y_1, \dots, y_m)\\
&\leq 0.
\end{aligned}$$

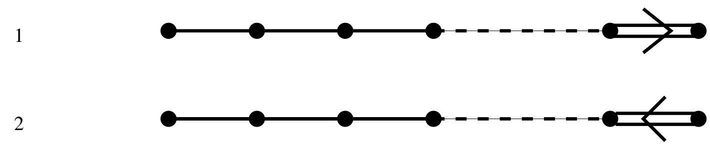
But this says that $Q(v) \leq 0$ with $v \neq 0$. Thus $Q(x_1, \dots, x_\ell)$ is not positive definite, a contradiction, and we are done. □

The next step in the classification of semisimple Lie algebras is to relate Cartan matrices bijectively to Dynkin diagrams. A Dynkin diagram is uniquely determined by the Cartan matrix, by the condition $n_{ij} = A_{ij}Aji$. However, the converse is not true in general. The Cartan matrix is not uniquely determined by the Dynkin diagram. Indeed, if $n_{ij} = 2$, then the above equation can be either $2 = 1 \cdot 2$ or $2 = -2 \cdot -1$. Similarly for $n_{ij} = 3$. This happens exactly for the graphs B_ℓ, F_4, G_2. But we can restore uniqueness here quite easily by introducing an orientation into the Dynkin diagram.

DEFINITION 2.8.4. In the Dynkin diagram we draw an arrow from the vertex i to the vertex j if and only if $\|\alpha_i\| > \|\alpha_j\|$, i.e., if $|A_{ji}| > |A_{ij}|$.

In the picture below the arrow in the left diagram means that $\|\alpha_i\| = \sqrt{2}\|\alpha_j\|$, i.e., that $A_{ij} = -1, A_{ji} = -2$. For the right diagram we have $\|\alpha_i\| = \sqrt{3}\|\alpha_j\|$, and $A_{ij} = -1, A_{ji} = -3$. So we may consider the arrow as an inequality sign for the length of the fundamental roots.

For the diagrams of type B_2, F_4, G_2 the direction of the arrow is irrelevant since these diagrams are symmetric. However, for $B_\ell, \ell \geq 3$ it makes a difference. Therefore we split up this type into two types as follows.



This finally leads to the classical list of Dynkin diagrams of simple Lie algebras. By summarizing the obtained result we can now formulate the classification of simple complex Lie algebras.

THEOREM 2.8.5 (Cartan, Killing). *Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the Lie algebras of the following table.*

type	$\mathfrak{g}$	rank($\mathfrak{g}$)	$\lvert\Phi\rvert$	dim($\mathfrak{g}$)	$\lvert W\rvert$
A_n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$n \geq 1$	$n(n+1)$	$n(n+2)$	$(n+1)!$
B_n	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$n \geq 2$	$2n^2$	$n(2n+1)$	$n! \cdot 2^n$
C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	$n \geq 3$	$2n^2$	$n(2n+1)$	$n! \cdot 2^n$
D_n	$\mathfrak{so}_{2n}(\mathbb{C})$	$n \geq 4$	$2n(n-1)$	$n(2n-1)$	$n! \cdot 2^{n-1}$
G_2	$\mathfrak{g}_2(\mathbb{C})$	2	12	14	$2^2 \cdot 3$
F_4	$\mathfrak{f}_4(\mathbb{C})$	4	48	52	$2^7 \cdot 3^2$
E_6	$\mathfrak{e}_6(\mathbb{C})$	6	72	78	$2^7 \cdot 3^4 \cdot 5$
E_7	$\mathfrak{e}_7(\mathbb{C})$	7	126	133	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8	$\mathfrak{e}_8(\mathbb{C})$	8	240	248	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

Here $|W|$ is the cardinality of the Weyl group. The associated Cartan matrices are given as follows, uniquely up to permutation of the indices of A_{ij}:

$$A_\ell = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix}$$

$$B_\ell = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -2 & 2 \end{pmatrix}$$

$$C_\ell = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -2 \\ & & & & & & & -1 & 2 \end{pmatrix}$$

$$D_\ell = \begin{pmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & \cdot & \cdot & & & & & \\ & & & \cdot & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & -1 & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & -1 & 2 & -1 & -1 \\ & & & & & & & -1 & 2 & 0 \\ & & & & & & & -1 & 0 & 2 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & -1 & \\ & & -1 & 2 & & \\ & & -1 & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & -1 & \\ & & & -1 & 2 & & \\ & & & -1 & & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & -1 & \\ & & & & -1 & 2 & & \\ & & & & -1 & & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

Two simple Lie algebras having the same Cartan matrix are isomorphic, see [**10**].

Remark 2.8.6. The following table gives an overview of all complex simple Lie algebras of dimension n, with $n \leq 1224$:

n	$\mathfrak{g}$	n	$\mathfrak{g}$	n	$\mathfrak{g}$	n	$\mathfrak{g}$
3	A_1	105	B_7, C_7	325	D_{13}	728	A_{26}
8	A_2	120	A_{10}, D_8	351	B_{13}, C_{13}	741	B_{19}, C_{19}
10	B_2, C_2	133	E_7	360	A_{18}	780	D_{20}
14	G_2	136	B_8, C_8	378	D_{14}	783	A_{27}
15	A_3	143	A_{11}	399	A_{19}	820	B_{20}, C_{20}
21	B_3, C_3	153	D_9	406	B_{14}, C_{14}	840	A_{28}
24	A_4	168	A_{12}	435	D_{15}	861	D_{21}
28	D_4	171	B_9, C_9	440	A_{20}	899	A_{29}
35	A_5	190	D_{10}	465	B_{15}, C_{15}	903	B_{21}, C_{21}
36	B_4, C_4	195	A_{13}	483	A_{21}	946	D_{22}
45	D_5	210	B_{10}, C_{10}	496	D_{16}	960	A_{30}
48	A_6	224	A_{14}	528	A_{22}, B_{16}, C_{16}	990	C_{22}
52	F_4	231	D_{11}	561	D_{17}	1023	A_{31}
55	C_5	248	E_8	575	A_{23}	1035	D_{23}
63	A_7	253	B_{11}, C_{11}	595	B_{17}, C_{17}	1081	B_{23}, C_{23}
66	D_6	255	A_{15}	624	A_{24}	1088	A_{32}
78	B_6, C_6	276	D_{12}	630	D_{18}	1128	D_{24}
80	A_8	288	A_{16}	666	B_{18}, C_{18}	1155	A_{33}
91	D_7	300	B_{12}, C_{12}	675	A_{25}	1176	B_{24}, C_{24}
99	A_9	323	A_{17}	703	D_{19}	1224	A_{34}

2.9. Serre's structure theorem

Having classified all complex simple and semisimple Lie algebras by Dynkin diagrams and Cartan matrices, one would like to associate to every Dynkin diagram a unique simple Lie algebra, whose Dynkin diagram is the given one. J. Tits was the first, who proved in 1966 that this can be done. However, his construction does not work for all types simultaneously. In particular, the construction for each exceptional type is different. Jean-Pierre Serre found a *uniform* construction of all simple Lie algebras by generators and relations, which is directly derived from the root system. This gives an elegant way of realizing the simple, exceptional Lie algebras. Let us shortly describe this construction.

Let α, β be roots and define

$$\langle \beta, \alpha \rangle := 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

The the root string goes $\beta + j\alpha$ goes from $\beta - r\alpha$ till $\beta + q\alpha$, where $r - q = \langle \beta, \alpha \rangle$. Let Π be the basis of the root system Φ. If $\alpha, \beta \in \Pi$, then $\beta - \alpha$ is not a root, since we have a positive and a negative coefficient. The the root string is

$$\beta, \beta + \alpha, \dots, \beta + q\alpha$$

with $q = -\langle \beta, \alpha \rangle$. Therefore we have

$$(\operatorname{ad} x_\alpha)^{-\langle \beta, \alpha \rangle + 1} x_\beta = 0$$

for $x_\alpha \in \mathfrak{g}_\alpha$ and $x_\beta \in \mathfrak{g}_\beta$, but

$$(\operatorname{ad} x_\alpha)^k x_\beta \neq 0, \quad 0 \leq k \leq -\langle \beta, \alpha \rangle,$$

for $x_\alpha \neq 0, x_\beta \neq 0$. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. Then we may take $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ with $h_i = [e_i, f_i]$, such that

$$e_1, \dots, e_\ell, \ f_1, \dots, f_\ell, \ h_1, \dots, h_\ell$$

generates the Lie algebra, and we have the following relations:

$$[h_i, h_j] = 0, \tag{2.9}$$

$$[e_i, f_i] = h_i, \tag{2.10}$$

$$[e_i, f_j] = 0, \ i \neq j, \tag{2.11}$$

$$[h_i, e_j] = \langle \alpha_j, \alpha_i \rangle e_j, \tag{2.12}$$

$$[h_i, f_j] = -\langle \alpha_j, \alpha_i \rangle f_j, \tag{2.13}$$

$$(\operatorname{ad} e_i)^{-\langle \alpha_j, \alpha_i \rangle + 1} e_j = 0, \ i \neq j, \tag{2.14}$$

$$(\operatorname{ad} f_i)^{-\langle \alpha_j, \alpha_i \rangle + 1} f_j = 0, \ i \neq j. \tag{2.15}$$

Serre's Structure Theorem now is as follows.

THEOREM 2.9.1 (Serre). *Let Φ be a root system of rank ℓ with basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. Then the complex Lie algebra, which is generated by the 3ℓ elements e_i, f_i, h_i for $i = 1, \dots, \ell$ and satisfies the relations* (2.9), … ,(2.15), *is semisimple and its roots system is isomorphic to Φ.*

Proof. We will only give an idea for the proof. One starts with the free Lie algebra $\mathfrak{f}$ in 3ℓ generators

$$X_1, \dots, X_\ell,\ Y_1, \dots, Y_\ell,\ Z_1, \dots, Z_\ell.$$

Let $\mathfrak{g}$ be a semisimple Lie algebra defined by generators and relations with (2.9)–(2.15). Then there exists a unique Lie algebra homomorphism $\varphi\colon \mathfrak{f} \to \mathfrak{g}$ with $\varphi(X_i) = e_i$, $\varphi(Y_i) = f_i$ and $\varphi(Z_i) = h_i$. Let $\mathfrak{a}$ be the ideal in $\mathfrak{f}$ generated by the elements

$$[Z_i, Z_j],\ [X_i, Y_j] - \delta_{ij} Z_i,\ [Z_i, X_j] - \langle \alpha_j, \alpha_i \rangle X_j,\ [Z_i, Y_j] + \langle \alpha_j, \alpha_i \rangle Y_j.$$

Consider the quotient Lie algebra

$$\mathfrak{m} = \mathfrak{f}/\mathfrak{a}.$$

It satisfies the first five relations from above. Denote by x_i, y_i, z_i the images of X_i, Y_i, Z_i in $\mathfrak{m}$. We realize $\mathfrak{m}$ as Lie subalgebra of the Lie algebra $\mathrm{End}(A)$ with a tensor algebra $A = T(V)$ for a suitable vector space V with basis $v_1, \dots, v_\ell$. We let act $\mathfrak{f}$ on A, whose kernel contains $\mathfrak{a}$, so that we obtain an induced action of $\mathfrak{m}$ on A, $\psi\colon \mathfrak{m} \to \mathrm{End}(A)$. The elements x_i, y_i, z_i are linearly independent and the z_i generate an ℓ-dimensional abelian Lie subalgebra $\mathfrak{z}$ of $\mathfrak{m}$. This yields a decomposition

$$\mathfrak{m} = \mathfrak{m}_- \oplus \mathfrak{z} \oplus \mathfrak{m}_+,$$

where $\mathfrak{m}_+$ is the Lie subalgebra generated by the x_i, and $\mathfrak{m}_-$ the Lie subalgebra generated by the y_i. Writing $c_{ij} := \langle \alpha_i, \alpha_j \rangle$ we define for all $i \neq j$ in $\{1, 2, \dots, \ell\}$ the elements

$$\begin{aligned} x_{ij} &= \mathrm{ad}(x_i)^{-c_{ji}+1}(x_j), \\ y_{ij} &= \mathrm{ad}(y_i)^{-c_{ji}+1}(y_j). \end{aligned}$$

The sixth and seventh relation from above,, namely (2.14) and (2.15) hold if and only if the ideal $\mathfrak{k}$ generated by the x_{ij} and y_{ij}, is the zero ideal. We have, for all k and $i \neq j$,

$$\mathrm{ad}(x_k)(y_{ij}) = 0, \tag{2.16}$$

$$\mathrm{ad}(y_k)(x_{ij}) = 0. \tag{2.17}$$

The claim is now that the quotient

$$\mathfrak{g} := \mathfrak{m}/\mathfrak{k}$$

is a finite-dimensional complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} = \mathfrak{z}/\mathfrak{k}$ and root system Φ.

To see this, let $\mathfrak{i}$ be the ideal in $\mathfrak{m}_+$ generated by the x_{ij} and $\mathfrak{j}$ be the ideal in $\mathfrak{m}_-$ generated by the y_{ij}. Then we have

$$\mathfrak{i} + \mathfrak{j} \subseteq \mathfrak{k}.$$

We claim that $\mathfrak{i}$ and $\mathfrak{j}$ are ideals in $\mathfrak{m}$. Indeed, every y_{ij} is a weight vector for $\mathfrak{z}$, and $[\mathfrak{z}, \mathfrak{m}_-] \subset \mathfrak{m}_-$. Therefore we have $[\mathfrak{z}, \mathfrak{j}] \subset \mathfrak{j}]$. On the other hand, we have $[x_k, \mathfrak{m}_-] \subset \mathfrak{z} + \mathfrak{m}_-$ and $[x_k, y_{ij}] = 0$ by (2.16). The Jacobi identity now implies that $\mathrm{ad}(x_k)(\mathfrak{j}) \subset \mathfrak{j}$. Since the x_k generate $\mathfrak{m}_+$, it follows that $[\mathfrak{m}_+, \mathfrak{j}] \subset \mathfrak{j}$, again with the Jacobi identity. Therefore $\mathfrak{j}$ is an ideal of $\mathfrak{m}$. In the same way also $\mathfrak{i}$ is an ideal in $\mathfrak{m}$, and hence their sum $\mathfrak{i} + \mathfrak{j}$. This deal contains the generators of $\mathfrak{k}$, so that

$$\mathfrak{k} = \mathfrak{i} + \mathfrak{j}.$$

In particular we have $\mathfrak{z} \cap \mathfrak{k} = 0$, so that $\mathfrak{z}$ is isomorphic to an ℓ-dimensional abelian subalgebra of $\mathfrak{g} = \mathfrak{m}/\mathfrak{k}$ by projection. As vector space sum we obtain, because of $\mathfrak{j} \cap \mathfrak{m}_+ = 0$ and $\mathfrak{i} \cap \mathfrak{m}_- = 0$, then

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

with $\mathfrak{n}_- = \mathfrak{m}_-/\mathfrak{j}$ and $\mathfrak{n}_+ = \mathfrak{m}_+/\mathfrak{i}$. The x_i, y_i, z_i generate a Lie algebra $\mathfrak{sl}(2)$, hence a simple Lie algebra. So the projection map is an isomorphism on each $\mathfrak{sl}(2)$. Denote the images of x_i, y_i, z_i by e_i, f_i, h_i. Then $\mathfrak{g}$ is generated by the 3ℓ elements

$$e_1, \ldots, e_\ell,\ f_1, \ldots, f_\ell,\ h_1, \ldots, h_\ell,$$

and all relations (2.9)–(2.15) are satisfied. Furthermore $\mathfrak{g}$ has no nonzero abelian ideals, hence is semisimple. The root system of $\mathfrak{g}$ is given by Φ. □

CHAPTER 3

Representations of semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field K of characteristic zero. Every finite-dimensional representation of $\mathfrak{g}$ is semisimple by Weyl's Theorem, so it suffices to study the *simple* representations of $\mathfrak{g}$. We will often assume that $K = \mathbb{C}$. For a reference, see [**31**].

3.1. Classification by the highest weight

Let $\mathfrak{h}$ be an abelian Lie algebra. The elements of $\mathfrak{h}^*$ are called *weights*.

Definition 3.1.1. Let V be a representation of an abelian Lie algebra $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$ be a weight. The *weight space* V_λ to λ is defined by the subspace

$$V_\lambda = \{v \in V \mid Hv = \lambda(H)v \quad \forall H \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, then λ is called a *weight of* V. We denote the set of all weights of V by

$$P(V) = P_{\mathfrak{h}}(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}.$$

In case of a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, we have considered $V = \mathfrak{g}$ as a representation of $\mathfrak{h}$ via the adjoint operation. Then the nonzero weights of V just form our root system Φ, which often is also denoted by R in the literature, i.e.,

$$R = \Phi = P_{\mathfrak{h}}(\mathfrak{g}) \setminus 0.$$

The weight space V_α to $\alpha \in R$ then was denoted by $\mathfrak{g}_\alpha$. For each root $\alpha \in \mathfrak{h}^*$ there is a coroot $\alpha^\vee \in \mathfrak{h}$ and a reflection $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$. These reflection generate the Weyl group, a finite subgroup,

$$W = W(R) \subset GL(\mathfrak{h}^*).$$

Definition 3.1.2. Let $R \subset \mathfrak{h}^*$ be a root system. The maximal convex subsets in the complement of the union of the reflection hyperplanes

$$\langle R\rangle_{\mathbb{Q}} \setminus \bigcup_{\alpha \in R} \ker(\alpha^\vee)$$

are called *Weyl chambers*.

The Weyl group acts freely and transitively on the set of Weyl chambers. We denote by R^+ the set of positive roots. It has a basis $\Pi = \Pi(R^+)$, whose elements are called *simple roots*. We obtain a bijection between the set of all systems of positive roots and the set of all Weyl chambers by the assignment

$$R^+ \to C(R^+) = \{\lambda \in \langle R\rangle_{\mathbb{Q}} \mid \langle \lambda, \alpha^\vee\rangle > 0 \quad \forall \alpha \in R^+\}.$$

The Weyl chamber $C(R^+)$ is called the *dominant Weyl chamber* associated to the system R^+ of positive roots. The reflections s_α with $\alpha \in \Pi(R^+)$ are called *simple reflections*. They are just the reflections at the walls of the dominant Weyl chamber, and they generate the Weyl group. The image of a chamber under such a reflection at one of its walls is just separated from its reflection image by this wall, i.e., we have

$$s_\alpha(R^+) = R^+ \setminus \{\alpha\} \cup \{-\alpha\}$$

for every simple root $\alpha \in \Pi(R^+)$.

Definition 3.1.3. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system $R = R(\mathfrak{g}, \mathfrak{h})$. For each system of positive roots $R^+ \subset R$ we define a partial order on the set $\mathfrak{h}^*$ of all weights by

$$\lambda \geq \mu \Longleftrightarrow \lambda \in \mu + \underline{R},$$

where $\underline{R}$ denotes the submonoid in $\mathfrak{h}^*$ generated by R^+, i.e., the set of all finite sums of positive roots including the empty sum, the zero. Let V be a representation of $\mathfrak{g}$ and assume that there is a maximal element μ in the set of weights $P_\mathfrak{h}(V)$ with respect to this partial order. Then μ is called the *highest weight of* V with respect to R^+, and every nonzero element in V_μ is called a *highest weight vector*.

Remark 3.1.4. It can happen, for some infinite-dimensional representations V of $\mathfrak{g}$, that V does not have any $\mathfrak{h}$-weights at all, i.e., with $V \neq 0$ but $P_\mathfrak{h}(V) = \emptyset$. It can also happen that $P_\mathfrak{h}(V)$ is not empty, but doesn't possess a maximal element. We'll see that all simple finite-dimensional representation always have a highest weight. In fact, they are even *classified* by this highest weight.

Denote the set of isomorphism classes of finite dimensional simple representations of $\mathfrak{g}$ by $\mathrm{Rep}(\mathfrak{g})$.

Theorem 3.1.5 (Classification by the highest weight). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and $R^+ \subset R(\mathfrak{g}, \mathfrak{h})$ by a system of positive roots. Denote by*

$$X^+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \quad \forall \alpha \in R^+\}$$

the set of dominant integral weights, with respect to R^+. Then we have a bijection

$$\mathrm{Rep}(\mathfrak{g}) \xrightarrow{\sim} X^+,$$

$$V \mapsto \mu,$$

where μ is the highest weight of V with respect to R^+.

In order to give a proof we'll need several preparations. We will first prove a part by elementary methods, i.e., not using the universal enveloping algebra. Later we'll give a full proof making heavy use of it. Let us start with an example.

Example 3.1.6. *The classification result is true for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with $R^+ = \{\alpha\}$. Then $m \cdot \frac{\alpha}{2}$ is the highest weight of the simple representation $V(m)$ in dimension $m+1$, and we have $X^+ = \frac{\alpha}{2} \cdot \mathbb{N}$, which is in bijection to $\mathbb{N}$ via $\frac{\alpha}{2} \cdot m \mapsto m$.*

This follows from Theorem 1.5.2.

Denote the lattice of *integral weights* for a root system R by

$$X = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall \alpha \in R\}.$$

Then by definition all roots are integral weights, i.e., $R \subset X$ and the lattice of integral weights is invariant by the action of the Weyl group.

Let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R^+$ be the basis of R. Then the coroots $\alpha_1^\vee, \ldots, \alpha_r^\vee$ form a basis of the vector space $\mathfrak{h}$. The elements of its dual basis are denoted by

$$\varpi_1, \ldots, \varpi_r.$$

They are called the *fundamental dominant weights* and are characterized by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. They form a $\mathbb{Z}$-basis of the lattice of integral weights X. The set of dominant integral weights

$$X^+ = \mathbb{N}\varpi_1 + \ldots + \mathbb{N}\varpi_r$$

is just the intersection of X with the closure of the dominant Weyl chamber.

LEMMA 3.1.7. *Let V be a representation of a semisimple Lie algebra $\mathfrak{g}$ with root system R. Then*

$$\mathfrak{g}_\alpha V_\lambda \subset V_{\alpha+\lambda} \quad \forall \alpha \in R, \lambda \in \mathfrak{h}^*.$$

PROOF. This follows immediately from the definition of a weight space and the formula

$$HXv = [H, X]v + XHv \quad \forall H \in \mathfrak{h}, X \in \mathfrak{g}, v \in V.$$

□

LEMMA 3.1.8. *Let $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$ be a decomposition of a Lie algebra $\mathfrak{a}$ as a vector space sum of two subalgebras $\mathfrak{b}$ and $\mathfrak{c}$. Let V be a representation of $\mathfrak{a}$ and $U \subset V$ be a subspace invariant by $\mathfrak{b}$. Then the $\mathfrak{c}$-subrepresentation W of V, which is generated by U, is invariant by $\mathfrak{a}$.*

PROOF. We need to show that $XW \subset W$ for all $X \in \mathfrak{b}$. Fix an $r \in \mathbb{N}$ and consider the subspace

$$W(r) = \langle Y_1 \cdots Y_i v \mid i \leq r, Y_j \in \mathfrak{c} \rangle.$$

We have $W = \bigcup_r W(r)$ and it suffices to show that $XW(r) \subset W(r)$ for all $r \geq 0$ and all $X \in \mathfrak{b}$. We will do this by induction over r. For $r = 0$ we have $W(0) = U$, which is invariant by $\mathfrak{b}$ by assumption. For the induction step we'll use the identity

$$XY_1Y_2 \cdots Y_r v = Y_1 X Y_2 \cdots Y_r v + [X, Y_1] Y_2 \cdots Y_r v$$

We apply the induction hypothesis on the RHS on the first term, and then writing $[X, Y_1] = x + y$ with $x \in \mathfrak{b}$ and $y \in \mathfrak{c}$, again on the second term. □

PROPOSITION 3.1.9. *Let V be a simple representation of $\mathfrak{g}$. Assume that the set $P(V)$ has a maximal element with respect to the partial order on $\mathfrak{h}^*$ associated to R^+. Then this maximal element is already the highest weight of V. In particular, every finite-dimensional simple representation of $\mathfrak{g}$ has an highest weight.*

PROOF. It is enough to prove the first part. We have the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$, where

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}.$$

If $\lambda \in P(V)$ is maximal then V_λ is an $\mathfrak{b}$-invariant subspace. By Lemma 3.1.8 it follows that the $\mathfrak{n}$-subrepresentation W of V, generated by V_λ, is already a $\mathfrak{g}$-subrepresentation. Since V is simple, this implies $W = V$. Hence the weight space V_λ generates the whole space V by the action of $\mathfrak{n}$. Then the claim follows from Lemma 3.1.7 by weight translation. □

LEMMA 3.1.10. *Let V be a finite-dimensional representation of $\mathfrak{g}$. Then all weights of V in $\mathfrak{h}^*$ are integral and the set of weights $P(V)$ is invariant under the Weyl group action. In other words, $P(V) \subset X$ and $W \cdot P(V) = P(V)$.*

Proof. Let $\alpha \in R$ and let
$$\mathfrak{g}^\alpha := \mathfrak{g}_\alpha \oplus K\alpha^\vee \oplus \mathfrak{g}_{-\alpha}$$
be the subalgebra of $\mathfrak{g}$, which is isomorphic to $\mathfrak{sl}_2(K)$. In fact, the coroot $\alpha^\vee$ has the property that there is an isomorphism $\mathfrak{sl}_2(K) \cong \mathfrak{g}^\alpha$, $\operatorname{diag}(1,-1) \mapsto \alpha^\vee$. The eigenvalues of $\operatorname{diag}(1,-1)$ on V are integers, as we know. Thus we have $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\lambda \in P(V)$. For $0 \neq v \in V_\lambda$, $m = \langle \lambda, \alpha^\vee \rangle$, $Kx_\alpha = \mathfrak{g}_\alpha$, $Ky_\alpha = \mathfrak{g}_{-\alpha}$ we have $y_\alpha^m v \neq 0$ for $m \geq 0$ and $x_\alpha^{-m} v \neq 0$ for $m \leq 0$. Hence we always have $V_{\lambda - m\alpha} \neq 0$, and therefore $s_\alpha(\lambda) = \lambda - m\alpha \in P(V)$ for all generators s_α of W. □

Corollary 3.1.11. *The maximal weights of a finite-dimensional representation V of $\mathfrak{g}$ are all integral and dominant.*

Proof. Let $\lambda \in P(V)$ be a weight, which is not dominant, i.e., there exists a positive root $\alpha \in R^+$ with $\langle \lambda, \alpha^\vee \rangle < 0$. Then $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \in P(V)$ and we have $s_\alpha(\lambda) > \lambda$. So λ is not maximal, a contradiction. □

Lemma 3.1.12. *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Suppose that V and W are two simple representations of $\mathfrak{g}$ having the same highest weight. Then $V \cong W$.*

Proof. Let λ be the highest weight of V and W. Chose nonzero vectors $v \in V_\lambda$ and $w \in W_\lambda$ and let $U \subset V \oplus W$ be the subrepresentation generated by (v, w). We claim that U is simple. Consider the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$ as in the proof of Proposition 3.1.9. Then the line L through (v, w) is invariant by $\mathfrak{b}$. By Lemma 3.1.8, U is generated by (v, w) as representation of $\mathfrak{n}$. In particular, U is the direct sum of its weight spaces by Lemma 3.1.7 and $L = U_\lambda$. Every subrepresentation of U is invariant under the Cartan subalgebra and hence itself the direct sum of its weight spaces. Hence every proper subrepresentation A of U lies in $\bigoplus_{\mu \neq \lambda} U_\mu$. So we have $\pi_1(A) \neq V$ and $\pi_2(A) \neq W$ for the projections. Since V and W are simple, we obtain $\pi_1(A) = 0$ and $\pi_2(A) = 0$, i.e., $A = 0$. Hence U is simple and the nonzero maps $\pi_1 \colon U \to V$, $\pi_1 \colon U \to W$ have to be isomorphisms, because kernel and image of these maps are subrepresentations of the simple representations U, respectively V and W. Hence we obtain $V \cong U \cong W$. □

Remark 3.1.13. We have proved Theorem 3.1.5 except for showing the *surjectivity*, i.e., that for each dominant integral weight λ there exists a finite-dimensional simple representation with highest weight λ. Here there seems to be no general argument available avoiding the universal enveloping algebra. Of course we might prove surjectivity directly for, say, the case $\mathfrak{sl}_{n+1}(\mathbb{C})$. For each i the representation $\Lambda^i \mathbb{C}^{n+1}$ has highest weight $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ and highest weight vector $e_1 \wedge \cdots \wedge e_i$. For each integral dominant weight $\lambda \in X^+$ we may construct a representation with highest weight λ by forming suitable tensor products of the representations $\Lambda^i \mathbb{C}^{n+1}$, where a suitable simple summand must be the desired representation with highest weight λ.

3.2. The universal enveloping algebra

Let A be an associative algebra and denote by A_L the Lie algebra with Lie bracket $[x, y] = xy - yx$.

DEFINITION 3.2.1. A *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ over K is a pair (U, c) consisting of a K-algebra U and a Lie algebra homomorphism $c\colon \mathfrak{g} \to U_L$ such that the universal property holds: for every K-algebra A and every Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to A_L$ there exists a unique homomorphism $\widetilde{\varphi}\colon U \to A$ such that $\varphi = \widetilde{\varphi} \circ c$.

A universal enveloping algebra, if it exists, is *unique* by the usual argument. The universal property can be reformulated by saying that for every K-algebra A the pre-composition of the map c induces a bijection

$$\mathrm{Hom}_K(U, A) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, A_L)$$

between the K-algebra homomorphisms $U \to A$ and the Lie algebra homomorphisms $\mathfrak{g} \to A_L$. In the language of category theory, taking the universal enveloping algebra is the left adjoint functor of the functor $A \mapsto A_L$, from the category of K-algebras to the category of Lie algebras over K. As K-algebra, $U(\mathfrak{g})$ is already generated by the image of $\mathfrak{g}$.

EXAMPLE 3.2.2. *For* $\mathfrak{g} = 0$ *we have* $U(\mathfrak{g}) \cong K$ *and for* $\mathfrak{g} \cong K$ *we have* $U(\mathfrak{g}) \cong K[X]$.

If $\{X\}$ is a basis of $\mathfrak{g}$, then the polynomial ring $U = K[X]$ in one variable is a universal enveloping algebra for $\mathfrak{g}$ with canonical map $c\colon \mathfrak{g} \to K[X]$ given by $aX \mapsto aX$.

LEMMA 3.2.3. *Let V be an abelian group and (U, c) a universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over K. Then there is a bijection of U-module structures on V as ring modules and $\mathfrak{g}$-module structures on V as Lie algebra representations.*

PROOF. A structure on V as U-module is by definition a ring homomorphism $\varphi\colon U \to \mathrm{End}(V)$. The restriction of φ to $K \subset U$ turns V into a K-vector space, and induces a homomorphism of K-algebras $\varphi\colon U \to \mathrm{End}_K(V)$, and then a Lie algebra homomorphism $\varphi\colon U_L \to \mathfrak{gl}(V)$. Composing with c we obtain a Lie algebra homomorphism $\varphi \circ c\colon \mathfrak{g} \to \mathfrak{gl}(V)$, i.e., a representation of $\mathfrak{g}$. Altogether we have assigned a $\mathfrak{g}$-module structure on V to a given U-module structure on V. To show that the assignment is bijective we give the inverse assignment. A representation V of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to (\mathrm{End}(V))_L$. By the universal property of U we can extend this homomorphism uniquely to a homomorphism $\widetilde{\rho}\colon U \to \mathrm{End}(V)$ of K-algebras. So we obtain a U-module structure on V. It is easy to see that these two assignments are inverse to each other. $\square$

For $X \in \mathfrak{g}$ we often write X again for its image $c(X) \in U(\mathfrak{g})$. Note that the natural map $\mathfrak{g} \to U(\mathfrak{g})$ is injective, by the so-called PBW-Theorem:

THEOREM 3.2.4 (Poincaré-Birkhoff-Witt). *Every Lie algebra $\mathfrak{g}$ possesses a universal enveloping algebra $U(\mathfrak{g})$. If $(X_\lambda)_{\lambda \in \Lambda}$ is a basis of $\mathfrak{g}$ and $\leq$ a total order on Λ, then the ordered monomials $X_{\lambda(1)} \cdots X_{\lambda(r)}$ with $\lambda(1) \leq \cdots \leq \lambda(r)$ form a basis of $U(\mathfrak{g})$.*

We will give a proof in several parts. The empty monomial, for $r = 0$, is the unit $1 \in U(\mathfrak{g})$. For the existence of a universal enveloping algebra, we first recall the following definition.

DEFINITION 3.2.5. Let V be a K-vector space. A *free* K-algebra over V is a pair (T, c) consisting of a K-algebra V and a linear map $c\colon V \to T$ such that the universal property holds: for every K-algebra A and every linear map $\varphi\colon V \to A_L$ there exists a unique homomorphism $\widetilde{\varphi}\colon T \to A$ such that $\varphi = \widetilde{\varphi} \circ c$.

A free algebra, if it exists, is unique by the usual argument.

LEMMA 3.2.6. *Let V be a K-vector space. Then there exists a free K-algebra $T(V)$ over V.*

PROOF. Consider the tensor algebra $T(V)$ over V, i.e., the K-algebra

$$T(V) = \bigoplus_{r\geq 0} V^{\otimes r} = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

together with its universal property. It has a K-bilinear multiplication, uniquely determined by the rule

$$(v_1 \otimes \cdots \otimes v_m)(w_1 \otimes \cdots \otimes w_n) = (v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n).$$

The embedding "as second summand" $c\colon V \hookrightarrow T(V)$ then has the required universal property. $\square$

Now we can prove the first part of the PBW-Theorem.

PROPOSITION 3.2.7. *Let $\mathfrak{g}$ be a Lie algebra over a field K and I be the ideal of $T(V)$ generated by the elements $x\otimes y - y\otimes x - [x, y]$ for $x, y \in \mathfrak{g}$. Then the K-algebra $U(\mathfrak{g}) = T(\mathfrak{g})/I$ together with the map $c\colon \mathfrak{g} \hookrightarrow T(\mathfrak{g}) \to U(\mathfrak{g})$ is a universal enveloping algebra of $\mathfrak{g}$.*

PROOF. Let $p\colon T(\mathfrak{g}) \to U(\mathfrak{g})$ be the projection and $\iota\colon \mathfrak{g} \to T(\mathfrak{g})$ be the canonical map, so that $c = p \cdot \iota$.We have

$$\begin{aligned} c([x, y]) &= p([x, y]) \\ &= p(x \otimes y - y \otimes x) \\ &= c(x)c(y) - c(y)c(x) \\ &= [c(x), c(y)], \end{aligned}$$

since by construction $x\otimes y - y\otimes x - [x, y] \in I = \ker(p)$. Hence c is a Lie algebra homomorphism. For the universal property of $U(\mathfrak{g})$ consider the following diagram.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[rr, bend left, "c"] \arrow[dr, "\varphi"'] & T(\mathfrak{g}) \arrow[r, "p"] \arrow[d, "\widehat{\varphi}"] & U(\mathfrak{g}) \arrow[dl, "\widetilde{\varphi}"] \\ & A & \end{tikzcd}$$

Let $\varphi\colon \mathfrak{g} \to A$ be a homomorphism from $\mathfrak{g}$ into a K-algebra A. Since φ is linear, there is a unique extension to a ring homomorphism $\widehat{\varphi}\colon T(\mathfrak{g}) \to A$. Since φ is a Lie algebra homomorphism from $\mathfrak{g}$ to A_L, we have $\widehat{\varphi}(x \otimes y - y \otimes x - [x, y]) = 0$, hence $\widehat{\varphi}(I) = 0$. Hence $\widehat{\varphi}$ factorizes over a ring homomorphism $\widetilde{\varphi}\colon U(\mathfrak{g}) \to A$ as desired. $\square$

LEMMA 3.2.8. *The monomials from the PBW-Theorem span the universal enveloping algebra.*

PROOF. Consider the subspace U_r in $U(\mathfrak{g})$ generated by all monomials of length $\leq r$, i.e., the image of

$$\bigoplus_{0\leq s\leq r} \mathfrak{g}^{\otimes s}$$

in $U(\mathfrak{g})$. Then we show by induction that U_r is spanned by the ordered monomials of length $\leq r$. Indeed, let $X_{\lambda(1)} \cdots X_{\lambda(r)}$ be a monomial. We have

$$X_{\lambda(i)} X_{\lambda(i+1)} = X_{\lambda(i+1)} X_{\lambda(i)} + [X_{\lambda(i)}, X_{\lambda(i+1)}].$$

We may write the commutator as a finite linear combination of monomials, so that the coset of a monomial of length r in U_r/U_{r-1} does not depend on the order of the factors. This enables us to carry out the induction over r. □

For the last step, we use a shorter notation for the Lie algebra element X_λ, with $\lambda \in \Lambda$. So let us write $\lambda' := X_\lambda$.

LEMMA 3.2.9. *Let $\mathfrak{g}$ be a Lie algebra over K with basis $(\lambda')_{\lambda \in \Lambda}$ and $\leq$ be a partial order on Λ. Denote by $K[\widehat{\lambda}]_{\lambda \in \Lambda}$ or just $K[\widehat{\lambda}]$ the polynomial ring in the variables $\widehat{\lambda}$ for $\lambda \in \Lambda$. Then there is an action $\mathfrak{g} \times K[\widehat{\lambda}] \to K[\widehat{\lambda}]$ such that*

$$\lambda' \widehat{\lambda}_1 \widehat{\lambda}_2 \cdots \widehat{\lambda}_r = \widehat{\lambda} \widehat{\lambda}_1 \widehat{\lambda}_2 \cdots \widehat{\lambda}_r$$

whenever $\lambda \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_r$.

For a proof, see [**31**], Lemma 4.3.24 and Lemma 4.3.26.

Proof of the PBW-Theorem 3.2.4*:* The construction of $U(\mathfrak{g})$ in Proposition 3.2.7 shows that $\mathfrak{g}$ has a universal enveloping algebra, which is unique by the universal property. The monomials given at the PBW-Theorem span the universal enveloping algebra by Lemma 3.2.8. We are left to show that the ordered monomials are linearly independent. Consider $K[\widehat{\lambda}]$ as $U(\mathfrak{g})$-module, see Lemma 3.2.9. If $\lambda'_1 \cdots \lambda'_r$ is an ascending monomial in $U(\mathfrak{g})$, then

$$\lambda'_1 \cdots \lambda'_r \cdot 1_{K[\widehat{\lambda}]} = \widehat{\lambda}_1 \widehat{\lambda}_2 \cdots \widehat{\lambda}_r.$$

However, since the ascending monomials are linearly independent in $K[\widehat{\lambda}]$, they must have been already linearly independent in $U(\mathfrak{g})$. □

Recall that the *opposite algebra* $(A^{op}, \circ)$ of an algebra $(A, \cdot)$ is defined by the K-bilinear product $b \circ a := a \cdot b$. If $\mathfrak{g} \to U$ is a universal enveloping algebra, then so is $\mathfrak{g}^{op} \to U^{op}$. The multiplication by $(-1)\colon \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^{op}$ extends to a K-algebra isomorphism

$$S\colon U \xrightarrow{\cong} U^{op}, \quad u \mapsto u^t,$$

which is called the *principal antiautomorphism* of U. If V is a representation of $\mathfrak{g}$, then V^* is the contragredient representation defined by $(uf)(v) = f(u^t v)$ for all $f \in V^*$, $v \in V$ and $u \in U$.

DEFINITION 3.2.10. Let V be a vector space. Define the *symmetric algebra* of V by

$$S(V) = T(V)/\langle x \otimes y - y \otimes x \rangle.$$

The symmetric algebra is the universal enveloping algebra of the abelian Lie algebra. It inherits a grading from $T(V)$. For a Lie algebra $\mathfrak{g}$ the universal enveloping algebra also inherits the filtration from $T(\mathfrak{g})$. Denote the associated graded algebra by $\operatorname{gr}(U(\mathfrak{g}))$. We obtain a version of the PBW-Theorem without coordinates as follows.

THEOREM 3.2.11. *Let $\mathfrak{g}$ be a Lie algebra over a field K. The two surjections $T(\mathfrak{g}) \to S(\mathfrak{g})$ and $T(\mathfrak{g}) \xrightarrow{\cong} \operatorname{gr}(T(\mathfrak{g})) \to \operatorname{gr}(U(\mathfrak{g}))$ have the same kernel and hence define an isomorphism of graded K-algebras*

$$\operatorname{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g}).$$

Corollary 3.2.12. *The universal enveloping algebra of a Lie algebra has no zero-divisors. The universal enveloping algebra of a finite-dimensional Lie algebra is Noetherian.*

Proof. A K-algebra with an exhausting filtration beginning with zero has no zero-divisors, respectively is Noetherian if this is true for the associated graded ring. However, the graded ring is a polynomial ring over a field by the above theorem, hence has no zero-divisors. If $\mathfrak{g}$ is finite-dimensional, the polynomial ring has only finitely many variables. So it is Noetherian by Hilbert's Basissatz. $\square$

3.3. The construction of highest weight modules

In section 3.1 on the study of finite-dimensional representations of semisimple Lie algebras we haven't answered the question whether or not every integral dominant weight arises as the highest weight of some finite-dimensional simple representation. We can now give a positive answer using the universal enveloping algebra. As before, let $\mathfrak{g}$ be a semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and R^+ be a system of positive roots in $R = R(\mathfrak{g}, \mathfrak{h})$.

Definition 3.3.1. Let $\lambda \in \mathfrak{h}^*$ be a weight. Denote by I_λ the left ideal in $U(\mathfrak{g})$ generated by all $X \in \mathfrak{g}_\alpha$ with $\alpha \in R^+$ and all $H - \lambda(H)$ with $H \in \mathfrak{h}$. The quotient

$$\Delta(\lambda) = U(\mathfrak{g})/I_\lambda$$

is called the *Verma module of highest weight* λ. The coset of $1 \in U(\mathfrak{g})$ is called the *canonical generator* of $\Delta(\lambda)$ and is denoted by $v_\lambda \in \Delta(\lambda)$.

Verma modules are named after Daya-Nand Verma, who wrote his his Ph.D. thesis (1968) about this topic as a student of Nathan Jacobson at Yale University. We have the following structure theorem for Verma modules.

Proposition 3.3.2. *Let $\lambda \in \mathfrak{h}^*$ be a weight.*

1. *Let $\alpha, \dots, \beta \in R^+$ be the positive roots, listed in a fixed order, and $y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be the generators of the roots spaces for negative roots. Then the vectors $y_\alpha^{m(\alpha)} \cdots y_\beta^{m(\beta)} v_\lambda$, index by all multiindices $m\colon R^+ \to \mathbb{N}$, form a basis of the Verma module $\Delta(\lambda)$.*

2. *Every Verma module $\Delta(\lambda)$ has a weight space decomposition of the form*

$$\Delta(\lambda) = \bigoplus_{\mu \leq \lambda} \Delta(\lambda)_\mu$$

and its highest weight space $\Delta(\lambda)_\lambda$ is one-dimensional with basis v_λ.

3. *We have*

$$\dim(\Delta(\lambda)_\mu) = \mathcal{P}(\lambda - \mu),$$

where $\mathcal{P}\colon \mathfrak{h}^* \to \mathbb{N}$ is the **Kostant partition function***, counting the number of different non-negative decompositions of a weight into a sum of positive roots.*

Proof. Given scalars $\lambda_1, \dots, \lambda_r \in K$, the polynomial ring $K[H_1, \dots, H_r]$ has a basis of polynomials of the form $(H_1 - \lambda_1)^{n(1)} \cdots (H_r - \lambda_r)^{n(r)}$ for multiindices $n\colon \{1, \dots, r\} \to \mathbb{N}$. In particular, if $H_1, \dots, H_r$ is a basis of $\mathfrak{h}$, and $\alpha, \dots, \beta$ are the positive roots in a fixed order, and $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ are basis vectors, then the PBW-theorem implies that the products

$$y_\alpha^{m(\alpha)} \cdots y_\beta^{m(\beta)} \cdot (H_1 - \lambda_1)^{n(1)} \cdots (H_r - \lambda_r)^{n(r)} \cdot x_\alpha^{l(\alpha)} \cdots x_\beta^{l(\beta)}$$

are a basis of $U(\mathfrak{g})$, for $m, l\colon R^+ \to \mathbb{N}$ and $n\colon \{1, \dots, r\} \to \mathbb{N}$. By considering this basis with $\lambda_i := \lambda(H_i)$, the span of the basis vectors with $n \neq 0$ or $\ell \neq 0$ yields the left ideal I_λ in $U(\mathfrak{g})$. Hence the cosets of the $y_\alpha^{m(\alpha)} \cdots y_\beta^{m(\beta)}$ for $m\colon R^+ \to \mathbb{N}$ yield a basis for the Verma module $\Delta(\lambda) = U(\mathfrak{g})/I_\lambda$. By definition we have $Hv_\lambda = \lambda(H)v_\lambda$ for all $H \in \mathfrak{h}$. Thus v_λ is a weight vector for the weight λ, and the first part is proved. Now the other two parts are obvious. $\square$

REMARK 3.3.3. We may reformulate the first part as follows: $\Delta(\lambda)$ is a free $U(\mathfrak{n})$ submodule of rank one with basis v_λ, where $\mathfrak{n} = \bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}$. In formulas, the multiplication yields a bijection $U(\mathfrak{n}) \xrightarrow{\sim} \Delta(\lambda)$ given by $u \mapsto uv_\lambda$.

We consider an example for Kostant's partition function.

EXAMPLE 3.3.4. *Let R be of type A_2 with $R^+ = \{\alpha, \beta, \alpha+\beta\}$. If an element μ can be expressed as a non-negative integer linear combination of the positive roots, we have*

$$\mathcal{P}(n_1\alpha + n_2\beta) = 1 + \min(n_1, n_2).$$

If not, then $\mathcal{P}(\mu) = 0$.

Indeed, if μ is a non-negative integer linear combination of the positive roots $\mu = n_1\alpha + n_2\beta + n_3(\alpha+\beta)$, we also obtain presentations as a linear combination of α and β. Conversely we obtain representations by replacing $\alpha + \beta$ a number of times.

LEMMA 3.3.5. *Let L be a representation of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$ be a weight and $v \in L_\lambda$ be a weight vector satisfying $\mathfrak{g}_\alpha v = 0$ for all $\alpha \in R^+$. Then there exists a unique homomorphism of representations $\Delta(\lambda) \to L$ with $v_\lambda \mapsto v$.*

PROOF. For every R-module M, the evaluation at 1_R induces a bijection $\mathrm{Hom}_R(R, M) \xrightarrow{\sim} M$. The universal property of quotients shows that for every left ideal $I \subset R$ the evaluation at $1_R + I$ induces a bijection

$$\mathrm{Hom}_R(R/I, M) \xrightarrow{\sim} \{m \in M \mid Im = 0\}.$$

Now apply this for $R = U(\mathfrak{g})$ and $I = I_\lambda$ in the above situation. Since by assumption $I_\lambda v = 0$, the claim follows. □

The classification of simple highest weight modules is as follows.

PROPOSITION 3.3.6. *We have the following assertions.*

1. *For every weight $\lambda \in \mathfrak{h}^*$ the Verma module $\Delta(\lambda)$ has a largest proper submodule $\operatorname{rad}\Delta(\lambda)$.*
2. *The corresponding quotient module $L(\lambda) = \Delta(\lambda)/\operatorname{rad}\Delta(\lambda)$ is simple. So we obtain a bijection between the elements of $\mathfrak{h}^*$ and simple representations with a highest weight, up to isomorphism, by $\lambda \mapsto L(\lambda)$.*
3. *If a simple representation L has a maximal weight, then this weight already is the highest weight of L.*

PROOF. The weight space decomposition of $\Delta(\lambda)$ induces a weight space decomposition $N = \bigoplus_{\mu\in\mathfrak{h}^*} N_\mu$ for every $\mathfrak{h}$-submodule N of $\Delta(\lambda)$. In case that N is a $\mathfrak{g}$-submodule, $N_\lambda \neq 0$ already implies that $N = \Delta(\lambda)$. Thus if N is a proper $\mathfrak{g}$-submodule we obtain $N \subset \bigoplus_{\mu\neq\lambda} \Delta(\lambda)_\mu$. Hence the sum of all proper submodules is again a proper submodule. So claim 1. follows.

Concerning the second claim, the quotient module $L(\lambda)$ is certainly simple with highest weight λ. Conversely, every simple representation with highest weight λ is a quotient of $\Delta(\lambda)$, and the kernel of the associated surjection has to be the largest proper submodule of $\Delta(\lambda)$. □

DEFINITION 3.3.7. Let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ be the half-sum of the positive roots. Define an action of the Weyl group W on $\mathfrak{h}^*$ by

$$x \cdot \lambda = x(\lambda + \rho) - \rho.$$

This action is called the *dot action*.

Note that we conjugate the usual action of the Weyl group by a fixed translation. Recall that the reflection map s_α satisfies $s_\alpha(\rho) = \rho - \alpha$.

LEMMA 3.3.8. *For every simple root $\alpha \in \Pi$ and every weight $\lambda \in \mathfrak{h}^*$ with $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N}$ there is a module monomorphims*

$$\Delta(s_\alpha \cdot \lambda) \hookrightarrow \Delta(\lambda).$$

PROOF. For a simple root α we have $\langle \rho, \alpha^\vee \rangle = 1$ and hence $s_\alpha \cdot \lambda < \lambda$ is equivalent to $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$. Now let $\alpha \in R^+$ with $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$. For $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ we obtain

$$x_\alpha y_\alpha^{n+1} v_\lambda = 0$$

by a computation which is analogous to the one in the proof of Theorem 1.5.2. If in addition $\alpha \in \Pi$, then we even have $x_\beta y_\alpha^i v_\lambda = 0$ for all $\beta \in R^+ \setminus \{\alpha\}$ and all $i \in \mathbb{N}$, because $i\alpha - \beta$ then is never a sum of positive roots. Since we have $s_\alpha \cdot \lambda = \lambda - (n+1)\lambda$, it follows that $0 \neq y_\alpha^{n+1} v_\lambda \in \Delta(\lambda)_{s_\alpha \cdot \lambda}$ and we obtain by the universal property of the Verma module, as coinduced representation a nonzero homomorphism $\Delta(s_\alpha \cdot \lambda) \to \Delta(\lambda)$, mapping the canonical generator of $\Delta(s_\alpha \cdot \lambda)$ to $y_\alpha^{n+1} v_\lambda$. Since all Verma modules are free of rank one over the integral domain $U(\mathfrak{n})$, this homomorphism must be injective. $\square$

PROPOSITION 3.3.9. *Let $\lambda \in \mathfrak{h}^*$. Then the simple module $L(\lambda)$ of highest weight λ is finite-dimensional if and only if $\lambda \in X^+$, i.e., if λ is integral and dominant.*

PROOF. If $L(\lambda)$ is finite-dimensional then $\lambda \in X^+$ by Corollary 3.1.11. Conversely, let $\lambda \in X^+$. Lemma 3.3.8 shows that, for a simple root α with $\langle \lambda, \alpha^\vee \rangle$ integral and non-negative, a highest weight vector of $L(\lambda)$ generates a finite-dimensional $\mathfrak{g}^\alpha$-subrepresentation of $L(\lambda)$. Now for every representation V of $\mathfrak{g}$ the sum W of all finite-dimensional $\mathfrak{g}^\alpha$-subrepresentations, for arbitrary fixed $\alpha \in R$, is a $\mathfrak{g}$-subrepresentation of V. If we have $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$ for every simple root α, then $L(\lambda)$ is the sum of its finite-dimensional $\mathfrak{g}^\alpha$-subrepresentations for every simple root α. Furthermore we have $s_\alpha P(L(\lambda)) = P(L(\lambda))$, see Lemma 3.1.7, for every simple reflection $s_\alpha \in W$. But then $P(L(\lambda))$ is necessarily stable under the Weyl group, and hence finite. It follows that $\dim(L(\lambda)) < \infty$. $\square$

3.4. The Weyl formulas

We recall the notations of this chapter. Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system $R = R(\mathfrak{g}, \mathfrak{h})$, let X be the lattice of integral weights, and R^+ be a system of positive roots. Let X^+ be the set of dominant integral weights with respect to R^+. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots.

We will start with the Weyl dimension formula, which is good for examples. The proof will follow later from a more general result.

THEOREM 3.4.1 (The Weyl dimension formula). *For every $\lambda \in X^+$ the dimension of the simple representation $L(\lambda)$ with highest weight λ is given by*

$$\dim(L(\lambda)) = \frac{\prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in R^+} \langle \rho, \alpha^\vee \rangle} = \frac{\prod_{\alpha \in R^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in R^+} (\rho, \alpha)}.$$

Since $\langle \beta, \alpha^\vee \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ it is clear that the second equality holds.

EXAMPLE 3.4.2. *Let $\mathfrak{g}$ be of type A_1, with $R^+ = \{\alpha\}$. If $\lambda(H) = m$, then $\lambda = m\rho$, so that the Weyl formula gives*

$$\dim(L(\lambda)) = \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} = \frac{(m+1)(\rho, \alpha)}{(\rho, \alpha)} = m + 1.$$

EXAMPLE 3.4.3. *Let $\mathfrak{g}$ be of type A_2. Then $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ and $\rho = \alpha_1 + \alpha_2$. Then $\lambda = m_1\varpi_1 + m_2\varpi_2$ with $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ and $(\varpi_i, \alpha_j) = \delta_{ij}$. We compute $(\lambda + \rho, \alpha_1) = m_1 + 1$, $(\lambda+\rho, \alpha_2) = m_2+1$ and $(\lambda+\rho, \alpha_1+\alpha_2) = m_1+m_2+2$. Also, $\prod_{\alpha \in R^+}(\rho, \alpha) = (\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2$. Hence the Weyl formula gives*

$$\dim(L(\lambda)) = \frac{(m_1+1)(m_2+1)(m_1+m_2+2)}{2}.$$

EXAMPLE 3.4.4. *Let $\mathfrak{g}$ be of type B_2, with $R^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$. We have $(\alpha, \alpha) = -(\alpha, \beta) = \frac{1}{2}(\beta, \beta) = 1$ and $\lambda = m_1\varpi_1 + m_2\varpi_2$, and the Weyl formula gives*

$$\dim(L(\lambda)) = \frac{(m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2)}{6}.$$

Indeed, we have $\rho = 2\alpha + \frac{3}{2}\beta$ and an easy calculation shows that $\varpi_1 = \alpha + \frac{1}{2}\beta$, $\varpi_2 = \alpha + \beta$. So we have

$$\lambda = m_1\varpi_1 + m_2\varpi_2 = (m_1+m_2)\alpha + \left(\frac{m_1}{2} + m_2\right)\beta$$

with positive integers m_1, m_2. Then

$$\begin{aligned}
\dim(L(\lambda)) &= \frac{(\lambda+\rho,\alpha)(\lambda+\rho,\beta)(\lambda+\rho,\alpha+\beta)(\lambda+\rho,2\alpha+\beta)}{(\rho,\alpha)(\rho,\beta)(\rho,\alpha+\beta)(\rho,2\alpha+\beta)} \\
&= \frac{((m_1+1)/2)(m_2+1)((m_1+3)/2+m_2)(m_1+m_2+2)}{(1/2)(1)(3/2)(2)} \\
&= \frac{(m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2)}{6}.
\end{aligned}$$

EXAMPLE 3.4.5. *Let $\mathfrak{g}$ be of type G_2, with $R^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$. With $\lambda = m_1\varpi_1 + m_2\varpi_2$ the Weyl formula gives*

$$\dim(L(\lambda)) = \frac{1}{120}(m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2)$$
$$(m_1+3m_2+4)(2m_1+3m_2+5).$$

Let $\mathbb{Z}\mathfrak{h}^*$ be the group ring of the additive group $(\mathfrak{h}^*, +)$. Viewing $\lambda \in \mathfrak{h}^*$ as an element of $\mathbb{Z}\mathfrak{h}^*$ we'll write e^λ instead of λ so that the sum $\lambda + \mu$ is not ambiguous. Hence $\{e^\lambda \mid \lambda \in \mathfrak{h}^*\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\mathfrak{h}^*$ with $e^\lambda e^\mu = e^{\lambda+\mu}$. The ring $\mathbb{Z}\mathfrak{h}^*$ is an integral domain, because every two elements lie in a subring $\mathbb{Z}E$ for a finitely generated subgroup $E \subset \mathfrak{h}^*$. Since E is a free abelian group the group ring $\mathbb{Z}E$ is isomorphic to a ring of Laurent polynomials. So we don't have zero divisors.

DEFINITION 3.4.6. Let V be a finite-dimensional representation of $\mathfrak{g}$. We define the *character* $\mathrm{ch}(V) \in \mathbb{Z}\mathfrak{h}^*$ of V by

$$\mathrm{ch}(V) = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu.$$

Note that $\mathrm{ch}(V)$ is invariant under the Weyl group W. This follows from the representation theory of $\mathfrak{sl}_2(K)$. Indeed, suitable powers of the generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ provide isomorphisms between the weight spaces for λ and $s_\alpha(\lambda)$.

Recall that the *length* of an element w in a Weyl group W, denoted by $l(w)$, is the smallest number k so that w is a product of k reflections by simple roots. So, the notion depends on the choice of a positive Weyl chamber. In particular, a simple reflection has length one. The Weyl dimension formula is a consequence of the following *Weyl character formula.*

THEOREM 3.4.7 (The Weyl character formula). *For every integral dominant weight $\lambda \in X^+$ we have for the character of the finite-dimensional simple representation $L(\lambda)$ of highest weight λ the formula*

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w\rho}}$$

in the quotient field of $\mathbb{Z}\mathfrak{h}^*$.

EXAMPLE 3.4.8. *For $\mathfrak{g}$ of type A_1 we have $\rho = \frac{1}{2}\alpha$, $X^+ = \mathbb{N}\rho$ and we obtain for all $n \geq 1$ that*

$$\mathrm{ch}(L(n\rho)) = \frac{e^{(n+1)\rho} - e^{-(n+1)\rho}}{e^\rho - e^{-\rho}} = e^{n\rho} + e^{(n-2)\rho} + e^{(n-4)\rho} + \cdots + e^{-n\rho}.$$

REMARK 3.4.9. The Weyl character formula and its generalization by Kac yields several famous combinatorial identities. Let

$$e(t) = \prod_{n \geq 1} (1 - t^n).$$

As an example, we obtain the *Euler pentagonal identity*

$$e(t) = \sum_{j \in \mathbb{Z}} (-1)^j t^{j(3j+1)/2},$$

we obtain the *Gauß identity*

$$\frac{e^2(t)}{e(t^2)} = \sum_{j\in\mathbb{Z}} (-1)^j t^{j^2},$$

we obtain the *Jacobi identity*

$$e^3(t) = \sum_{j\geq 0}(-1)^j(2j+1)^{j(j+1)/2},$$

and *Jacobi's triple product identity*

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} z^n q^{n^2}.$$

Consider the set $\mathrm{Abb}(\mathfrak{h}^*, \mathbb{Z})$ of all maps $f\colon \mathfrak{h}^* \to \mathbb{Z}$. We write such maps as infinite formal sums

$$f = \sum_{\lambda\in\mathfrak{h}^*} f(\lambda)e^{\lambda}.$$

We can extend the definition of a character of a representation of $\mathfrak{g}$ with finite-dimensional weight spaces with $\mathrm{ch}(V) \in \mathrm{Abb}(\mathfrak{h}^*, \mathbb{Z})$. We want to do computations with characters of Verma modules. For this reason we introduce the following extended character ring.

Definition 3.4.10. Denote by $\mathbb{Z}\langle\mathfrak{h}^*\rangle \subset \mathrm{Abb}(\mathfrak{h}^*, \mathbb{Z})$ the set of all maps $f\colon \mathfrak{h}^* \to \mathbb{Z}$, whose support is contained in a finite union of sets of the form $\{\lambda - \sum n_\alpha \alpha \mid n \in \mathrm{Abb}(R^+, \mathbb{N})\}$.

We may view $\mathbb{Z}\mathfrak{h}^* \subset \mathbb{Z}\langle\mathfrak{h}^*\rangle$ as the subset of all maps with finite support. The multiplication in $\mathfrak{h}^*$ extends to a commutative, associative multiplication in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ by

$$(fg)(\nu) = \sum_{\lambda+\mu=\nu} f(\lambda)g(\mu),$$

because the support condition ensures that only finitely many terms do not vanish in these sums.

Lemma 3.4.11. *The character of a Verma module $\Delta(\lambda)$ is given by*

$$\mathrm{ch}(\Delta(\lambda)) = e^{\lambda} \prod_{\alpha\in R^+} (1 + e^{-\alpha} + e^{-2\alpha} + \ldots)$$

In particular, in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ we have

$$\left(\prod_{\alpha\in R^+} (1 - e^{-\alpha})\right) \mathrm{ch}(\Delta(\lambda)) = e^{\lambda}.$$

Proof. Rewriting the results of Proposition 3.3.2 on the structure of Verma modules in our new terminology gives

$$\begin{aligned}\mathrm{ch}(\Delta(\lambda)) &= \sum_{\mu} P(-\mu) \\ &= e^{\lambda} \prod_{\alpha\in R^+} (1 + e^{-\alpha} + e^{-2\alpha} + \ldots).\end{aligned}$$

This shows the first formula. The second one follows directly from the first one. □

Remark 3.4.12. The character satisfies a very nice property, namely

$$\mathrm{ch}(M \otimes N) = \mathrm{ch}(M) \cdot \mathrm{ch}(N).$$

Here it is enough to assume that M, N are $\mathfrak{h}$-modules with finite-dimensional weight spaces, and that they are the sum of their weight spaces, and that $\mathrm{ch}(M), \mathrm{ch}(N)$ in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$.

In the next step we want to study the eigenvalues of the Casimir operator for a Verma module. The Killing form κ of $\mathfrak{g}$ induces an isomorphism $\overline{\kappa}\colon \mathfrak{h} \to \mathfrak{h}^*$, characterized by $\langle \overline{\kappa}(h), h' \rangle = \kappa(h, h')$ for all $h, h' \in \mathfrak{h}$. Denote by (λ, μ) the bilinear form on $\mathfrak{h}^*$ corresponding to the Killing form on $\mathfrak{h}$ under the isomorphism $\overline{\kappa}$. If $\overline{\kappa}$ sends h to λ, then $\mu(h) = (\lambda, \mu)$ for all $\mu \in \mathfrak{h}^*$. We know that this bilinear form is positive definite on the subspace $\langle R \rangle_{\mathbb{Q}}$. It is invariant under the Weyl group.

LEMMA 3.4.13. *Every endomorphism of a Verma module is the multiplication with a scalar.*

PROOF. Consider the maps

$$K \hookrightarrow \mathrm{End}_{\mathfrak{g}}(\Delta(\lambda)) \hookrightarrow \mathrm{End}_K(\Delta(\lambda)_\lambda).$$

The second map is injective, because $\Delta(\lambda)_\lambda$ generates $\Delta(\lambda)$ by Proposition 3.3.2. The composition of both maps must be a bijection, since $\dim(\Delta(\lambda)_\lambda) = \dim\langle v_\lambda \rangle = 1$ by Proposition 3.3.2. Thus all maps are bijections and we are done. □

LEMMA 3.4.14. *Let $C = C_\kappa$ be the Casimir operator of $\mathfrak{g}$ with respect to the Verma module $\Delta(\lambda)$. Then C acts on $\Delta(\lambda)$ as multiplication with the scalar*

$$c_\lambda = (\lambda + \rho, \lambda + \rho) - (\rho, \rho).$$

PROOF. By Lemma 3.4.13 we know that C acts as multiplication with a scalar and we only have to find out by which scalar it acts on the highest weight space $\Delta(\lambda)_\lambda$. Let $\alpha \in R^+$. Chose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) = 1$. Let $h_1, \ldots, h_n$ be an orthonormal basis of $\mathfrak{h}$ with respect to the Killing form κ. Then we have

$$\begin{aligned} C &= \sum_{\alpha \in R^+} y_\alpha x_\alpha + x_\alpha y_\alpha + \sum_{i=1}^n h_i^2 \\ &= \sum_{\alpha \in R^+} 2 y_\alpha x_\alpha + [x_\alpha, y_\alpha] + \sum_{i=1}^n h_i^2, \end{aligned}$$

which acts on $\Delta(\lambda)_\lambda$ by the scalar

$$c_\lambda = \sum_{\alpha \in R^+} \lambda([x_\alpha, y_\alpha]) + \sum_{i=1}^n \lambda(h_i)^2.$$

Writing $\lambda = \overline{\kappa}(h)$ we obtain

$$c_\lambda = \sum_{\alpha \in R^+} \kappa(h, [x_\alpha, y_\alpha]) + \sum_{i=1}^n \kappa(h, h_i)^2.$$

Because of $\kappa(h, [x_\alpha, y_\alpha]) = \kappa([h, x_\alpha], y_\alpha) = \alpha(h)\kappa(x_\alpha, y_\alpha)$ we obtain

$$\begin{aligned} c_\lambda &= 2\rho(h) + \kappa(h, h) \\ &= (2\rho, \lambda) + (\lambda, \lambda) \\ &= (\lambda + \rho, \lambda + \rho) - (\rho, \rho). \end{aligned}$$

□

Remark 3.4.15. One can also show the formula of *Freudenthal*, which is a recursive formula for the dimensions of the weight spaces $L(\lambda)_\mu$ in a simple representation $L(\lambda)$. We have

$$\dim(L(\lambda)_\mu)(|\lambda+\rho|^2-|\mu+\rho|^2)=2\sum_{\alpha\in R^+}\sum_{j\geq 1}\dim(L(\lambda)_{\mu+j\alpha})(\mu+j\alpha,\alpha).$$

The next lemma is a result concerning composition series of Verma modules.

Lemma 3.4.16. *Every Verma module $\Delta(\lambda)$ is of finite length and every simple subquotient of $\Delta(\lambda)$ is a simple highest weight module $L(\mu)$ with $\mu\leq\lambda$ and $(\mu+\rho,\mu+\rho)=(\lambda+\rho,\lambda+\rho)$.*

Proof. The second claim follows from Proposition 3.3.6, part 3 and Lemma 3.4.14, because the Casimir operator acts on each subquotient of $\Delta(\lambda)$ again by the scalar c_λ. In particular we know that there are only finitely many μ, which can be a highest weight of a simple subquotient of $\Delta(\lambda)$. Indeed, $\lambda\leq\mu$ implies that $\mu=\lambda+\nu$ with $\nu\in\langle R\rangle$, and there are only finitely many elements of the root lattice $\nu\in\langle R\rangle$ with $(\lambda+\rho,\lambda+\rho)=(\lambda+\nu+\rho,\lambda+\nu+\rho)$, because this equation is equivalent to the equation $(\nu,\nu)+2(\lambda+\rho,\nu)=0$. And the bilinear form $(\cdot,\cdot)$ is positive definite on $\langle R\rangle_{\mathbb{Q}}$, so that our equation can have only finitely many solutions in the lattice $\langle R\rangle$.
Furthermore every nonzero subquotient S of $\Delta(\lambda)$ has a simple subquotient. Indeed, every nonzero module over a ring has a simple subquotient. So for each such simple subquotient S there is a weight μ with $(\mu+\rho,\mu+\rho)=(\lambda+\rho,\lambda+\rho)$ and $S_\mu\neq 0$. Taking a strictly decreasing filtration of $\Delta(\lambda)$ we can estimate the length $l(\Delta(\lambda))$ by

$$l(\Delta(\lambda))\leq\sum_{\substack{\mu\leq\lambda\\(\mu+\rho,\mu+\rho)=(\lambda+\rho,\lambda+\rho)}}\dim(\Delta(\lambda)_\mu).$$

□

Proposition 3.4.17 (Kostant's character formula). *Let $\lambda\in X^+$ be a dominant integral weight. Then the character of a simple representation $L(\lambda)$ with highest weight λ is the alternating sum over the characters of the Verma modules $\Delta(w\cdot\lambda)$ with $w\in W$, where $w\cdot\lambda=w(\lambda+\rho)-\rho$, i.e.,*

$$\operatorname{ch}(L(\lambda))=\sum_{w\in W}(-1)^{l(w)}\operatorname{ch}(\Delta(w\cdot\lambda)).$$

Proof. Let us write $|\lambda|=\sqrt{(\lambda,\lambda)}$ for $\lambda\in\langle R\rangle_{\mathbb{Q}}$. By Lemma 3.4.16 we can write

$$\operatorname{ch}(\Delta(\lambda))=\sum_{\substack{\mu\leq\lambda\\|\mu+\rho|=|\lambda+\rho|}}a_\lambda^\mu\operatorname{ch}(L(\mu))$$

for suitable $a_\lambda^\mu\in\mathbb{N}$ with $a_\lambda^\lambda=1$. The corresponding matrix is unitriangular, hence invertible. So we can invert the formula and can also write

$$\operatorname{ch}(L(\lambda))=\sum_{\substack{\mu\leq\lambda\\|\mu+\rho|=|\lambda+\rho|}}b_\lambda^\mu\operatorname{ch}(\Delta(\mu))$$

for suitable $b_\lambda^\mu\in\mathbb{Z}$ with $b_\lambda^\lambda=1$. So far, everything is valid for all $\lambda\in\mathfrak{h}_{\mathbb{Q}}^*$. Now we know in addition that λ is integral and dominant so that $L(\lambda)$ is finite-dimensional by Proposition 3.3.9

and that $\mathrm{ch}(L(\lambda))$ is invariant under the Weyl group W. Multiplying both sides of the last equation by

$$\prod_{\alpha\in R^+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)=e^{\rho}\prod_{\alpha\in R^+}\left(1-e^{-\alpha}\right)$$

we obtain

$$\prod_{\alpha\in R^+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)\mathrm{ch}(L(\lambda))=\sum_{\mu}b_\lambda^\mu e^{\mu+\rho}=\sum_{\nu}d_\nu e^{\nu},$$

where $d_\nu=b_\lambda^{\nu-\rho}$, $d_{\lambda+\rho}=1$ and $d_\nu=0$ if $|\nu|\neq|\lambda+\rho|$ or $\nu\not\leq\lambda+\rho$. It is easy to see that the LHS changes sign if we apply a simple reflection s_β on it. Hence the same is true for the RHS, so that $d_\nu=(-1)^{l(w)}d_{w\nu}$ for all $w\in W$. In particular we have $d_\nu=0$ if not $|\nu|=|\lambda+\rho|$ and $w\nu\leq\lambda+\rho$ for all $w\in W$. By Lemma 3.4.18 below it follows that $d_\nu=0$ if not $\nu\in W(\lambda+\rho)$. Together with $d_{\lambda+\rho}=1$ and reparametrization we obtain Kostant's formula. $\square$

We still have to state and prove Lemma 3.4.18, which we've used in the last proof.

LEMMA 3.4.18. *Let $\lambda\in X^+$ be an integral dominant weight and $\nu\in X$ an integral weight. Then $|\nu|=|\mu|$ and $w\nu\leq\mu$ for all $w\in W$ imply that $\nu\in W\mu$.*

PROOF. We may consider a conjugate of ν, which lies in X^+. This is always possible. So we may assume that $\nu\in X^+$. Hence we only have to show that $\nu\leq\mu$ and $|\mu|=|\nu|$ imply that $\mu=\nu$. Since the scalar product of a vector from the dominant Weyl chamber with a positive root is always nonnegative, $\mu-\nu$ and ν enclose an obtuse angle. Hence the sum must have at least the length of each of the two summands, and the equality of their lengths is only possible when the corresponding summand coincides with the sum. $\square$

Proof of Theorem 3.4.7*, i.e., of Weyl's character formula*: consider the equation

$$\prod_{\alpha\in R^+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)\mathrm{ch}(L(\lambda))=\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)},$$

which comes from the proof of Kostant's formula in Proposition 3.4.17 and divide it by its specialization at $\lambda=0$, which is the so-called *Weyl's denominator formula*

$$e^{\rho}\prod_{\alpha\in R^+}\left(1-e^{-\alpha}\right)=\prod_{\alpha\in R^+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)=\sum_{w\in W}(-1)^{l(w)}e^{w\rho}.$$

This gives exactly Weyl's character formula. $\square$

REMARK 3.4.19. Weyl's denominator formula can be used among other things to show that

$$\mathrm{ch}(L(n\rho))=e^{n\rho}\prod_{\alpha\in R^+}\left(1+e^{-\alpha}+e^{-2\alpha}+\cdots+e^{-n\alpha}\right),$$

for any semisimple Lie algebra.

Proof of Theorem 3.4.1*, i.e., of Weyl's dimension formula*: consider the ring homomorphism $a\colon\mathbb{Z}\mathfrak{h}^*\to\mathbb{Z}$ with $a(e^\lambda)=1$ for all $\lambda\in\mathfrak{h}^*$. We would like to apply Weyl's character formula. For this we need to apply an abstract version of L'Hopital's rule. Consider the subring $\mathbb{Z}X$ in the group ring $\mathbb{Z}\mathfrak{h}^*$ and the group homomorphism $\partial_\alpha\colon\mathbb{Z}X\to\mathbb{Z}X$ with $\partial_\alpha(e^\mu)=\langle\mu,\alpha^\vee\rangle e^\mu$. It is easy to see that the $\partial\alpha$ are commuting ring derivations. For $D=\prod_{\alpha\in R^+}\partial_\alpha\in\mathrm{End}(\mathbb{Z}X)$ we have $aDe^\mu=\prod_{\alpha\in R^+}\langle\mu,\alpha^\vee\rangle$. It follows that $aDe^{w\mu}=(-1)^{l(w)}aDe^\mu$ for simple reflections w and then for arbitrary $w\in W$. By Weyl's character formula and his denominator formula we have

$$\left(e^{\rho}\prod_{\alpha\in R^+}\left(1-e^{-\alpha}\right)\right)\mathrm{ch}(L(\lambda))=\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)}$$

Applying aD on both sides we obtain

$$aD\left(e^{\rho}\prod_{\alpha\in R^+}\left(1-e^{-\alpha}\right)\right)a(\mathrm{ch}(L(\lambda)))=|W|\prod_{\alpha\in R^+}\langle\lambda+\rho,\alpha^\vee\rangle.$$

Now $a(\mathrm{ch}(L(\lambda)))=\dim(L(\lambda))$, and factors $1-e^{-\alpha}$, which are not hit by a derivation, vanish under a. So dividing the equation by the one obtained for $\lambda=0$ gives the result. $\square$

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