ÉTALE REPRESENTATIONS FOR REDUCTIVE ALGEBRAIC GROUPS WITH ONE-DIMENSIONAL CENTER

DIETRICH BURDE AND WOLFGANG GLOBKE

ABSTRACT. A complex vector space V is a prehomogeneous G-module if G acts rationally on V with a Zariski-open orbit. The module is called étale if $\dim V = \dim G$. We study étale modules for reductive algebraic groups G with one-dimensional center. For such G, we show that even though every étale module is a regular prehomogeneous module, its irreducible submodules have to be non-regular. For these non-regular prehomogeneous modules, we determine some strong constraints on the ranks of their simple factors. This allows us to show that there do not exist étale modules for $G = \mathrm{GL}_1 \times S \times \cdots \times S$, with S simple.

CONTENTS

1. INTRODUCTION

Affine étale representations of Lie groups arise in many contexts. For a given connected Lie group G, the existence of such a representation is equivalent to the existence of a left-invariant affine structure on G (see [2, 4]). In 1977 Milnor [15] discussed the importance of such structures for the study of fundamental groups of complete affine manifolds, and for the study of affine crystallographic groups, which initiated generalizations of the Bieberbach theorems for Euclidean crystallographic groups to affine crystallographic groups, see [10]. Milnor asked the existence question for left-invariant affine structures on a given Lie group G, and suggested that all solvable Lie groups G admit such a structure. This question received a lot of attention, and was eventually answered negatively by Benoist [3]. For a survey on the the results and the history see [6, 7, 10].

Affine étale representations of G and left-invariant affine structures on G both define a bilinear product on the Lie algebra $\mathfrak{g}$ of G that gives $\mathfrak{g}$ the structure of a *left-symmetric algebra* (*LSA-structure* for short), and conversely an LSA-structure determines an affine structure on G (see Paragraph 1.2 below). The existence question then can be formulated on the Lie algebra level, and has been studied for several classes of Lie algebras, e.g., for semisimple, reductive,

Date: September 26, 2018.
2010 *Mathematics Subject Classification.* Primary 32M10; Secondary 20G05, 20G20.

nilpotent and solvable Lie algebras, see [6]. LSA-structures on Lie algebras also correspond to non-degenerate involutive set-theoretical solutions of the Yang-Baxter equation, and to certain left brace structures, see [8, 1]. A natural generalization of LSA-structures is given by post-Lie algebra structures on pairs of Lie algebras [7].

Étale representations also appear in the classification of adjoint orbits on graded semisimple Lie algebras $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. The classification of G_0-orbits of nilpotent elements can be reduced to determining certain graded semisimple subalgebras $\mathfrak{s}$ associated to such elements which contain an étale representation for the grade-preserving subalgebra $\mathfrak{s}_0$ on the module $\mathfrak{s}_1$, see [20].

It follows from the Whitehead Lemma in Lie algebra cohomology that a semisimple Lie algebra over a field K of characteristic zero does not admit an LSA-structure. The reductive Lie algebra $\mathfrak{gl}_n(K)$, however, admits a canonical LSA-structure, and the group $\mathrm{GL}_n(K)$ admits a bi-invariant affine structure. Indeed, it is natural to consider the *reductive* case, where we have the powerful tools of invariant theory and representation theory for reductive groups at hand. Furthermore, we can use the theory of prehomogeneous modules for reductive groups as developed by Sato and Kimura. Still, it turns out that the existence question is already very difficult in the reductive case, and is still open in general.

On the other hand there are several results for reductive Lie algebras – respectively reductive groups – with one-dimensional center. The first author showed in [5, Theorem 2] that a reductive Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ with $\mathfrak{s}$ simple and $\dim \mathrm{Z}(\mathfrak{g}) = 1$ admits an LSA-structure if and only if $\mathfrak{s} = \mathfrak{sl}_n(K)$. Baues [2, Section 5] classified all LSA-structures on $\mathfrak{gl}_n(K)$.

It is the aim of this article to make further progress for the reductive case with one-dimensional center.

1.1. **Reductive prehomogeneous modules.** A *prehomogeneous module* (G, ϱ, V) consists of a linear algebraic group G and a rational representation $\varrho : G \to \mathrm{GL}(V)$ on a finite-dimensional complex vector space V, such that G has a Zariski-open orbit in V. The vector space V is called a *prehomogeneous vector space*. We always assume that the representation ϱ is faithful up to a finite subgroup. From now on G is assumed to be reductive. Recall that in this case, the Lie algebra $\mathfrak{g}$ of G is a direct sum $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$, where $\mathfrak{a}$ is the center of $\mathfrak{g}$, and $\mathfrak{s}$ is semsimple. We will call a prehomogeneous module (G, ϱ, V) for a reductive group G a *reductive prehomogeneous module*. A reductive group G is called *k-simple* if its semisimple factor has k simple factors.

There are several classification results on reductive prehomogeneous modules by a group of Japanese mathematicians around Mikio Sato and Tatsuo Kimura from the 1970s up to the present. However, a complete classification of prehomogeneous modules is not available.

The first classification result on prehomogeneous modules is due to Sato and Kimura [19]. They classified *irreducible* and *reduced* prehomogeneous modules for reductive algebraic groups (the terminology will be explained in Section 2). In addition, they determined the stabilizer subgroups of the open orbits and the relative invariants for all cases. We will label each module in this class by SK n, where n is its number in [19, §7]. This classification can also be found in Kimura's book [11].

Kimura [12, §3] classified prehomogeneous modules of one-simple reductive groups,

$$(\mathrm{GL}_1^k \times S, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$$

where S is a simple group. We will label them Ks n, where n is the number of the module in [12, §3]. In each case, the generic isotropy subgroup is determined. This classification included non-irreducible modules.

Furthermore, Kimura et al. [13, §3], [14, §5] studied the prehomogeneity of modules for two-simple groups,

$$(\mathrm{GL}_1^k \times S_1 \times S_2, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k),$$

where S_1 and S_2 are simple groups, under the assumption that one independent scalar multiplication acts on each irreducible component. This assumption is a non-trivial simplification of the problem, especially for the modules studied in [14], as it is far from obvious if one of these modules could be prehomogeneous with less than k factors GL_1 acting on the module. They studied two types of two-simple modules, I and II, and we will label them KI n and KII n, where n is their number in [13] or [14], respectively.

1.2. **Étale representations and LSA-structures.** It is clear that $\dim G \geq \dim V$ holds for any prehomogeneous module. If equality holds, $\dim G = \dim V$, we say that the representation ϱ (the module V) is an *étale representation* (an *étale module*). More generally, one considers *affine* étale representations for arbitrary algebraic groups. For reductive groups they can always be assumed to be linear [2, Corollary 3.9].

The existence of étale representations for a reductive algebraic group G implies the existence of LSA-structures on the reductive Lie algebra $\mathfrak{g}$ of G. More precisely, if $\varrho' = (\mathrm{d}\varrho)_1$ denotes the induced representation of $\varrho' : \mathfrak{g} \to \mathfrak{gl}(V)$ (also called an étale representation) and $v \in V$ is a point in the open orbit of $\varrho(G)$, then

$$x \cdot y = \mathrm{ev}_v^{-1}(\varrho'(x)\mathrm{ev}_v(y)), \quad x, y \in \mathfrak{g}$$

defines an LSA-structure on $\mathfrak{g}$. Here, $\mathrm{ev}_v : \mathfrak{g} \to V$ denotes the evaluation map $x \mapsto \varrho'(x)v$ at the point v. It is invertible since $\dim \mathfrak{g} = \dim V$ for an étale representation. In addition the LSA-structure determines a left-invariant flat torsion-free affine connection ∇ on G, by setting

$$\nabla_x y = x \cdot y.$$

Conversely, an LSA-structure on $\mathfrak{g}$, or a left-invariant flat torsion-free affine connection on G, gives rise to an étale representation of $\mathfrak{g}$.

1.3. **Overview and results.** The aim is, as said, to make progress on the structure of étale modules for reductive algebraic groups with one-dimensional center. We briefly recall the theory of prehomogeneous modules as developed by Sato and Kimura [19] in Section 2. We study some combinatorial aspects of castling transforms of irreducible reductive prehomogeneous modules in Section 3. We find a rather strong constraint on which groups can appear as castling transforms:

Theorem A. *Let (G, ϱ, V) be an irreducible prehomogeneous module for a reductive algebraic group. Then:*

$$(G, \varrho, V) = (L \times \mathrm{SL}_{m_1} \times \cdots \times \mathrm{SL}_{m_k},\ \sigma \otimes \omega_1 \otimes \cdots \otimes \omega_1,\ V^n \otimes \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_k}),$$

where L is a reductive algebraic group with one simple factor, σ is irreducible, and $n, m_1, \ldots, m_k \geq 1$ such that

(1) $\gcd(m_i, m_j) = 1$ *for* $1 \leq i < j \leq k$.
(2) $\gcd(n, m_i) = 1$ *for all but at most one index* $i_0 \in \{1, \ldots, k\}$.

Moreover, if (G, ϱ, V) is castling-equivalent to a one-simple irreducible module, then part (2) holds for all $i \in \{1, \ldots, k\}$.

Some general properties of étale representations (not just for reductive groups) are reviewed in Section 4. We show that every reductive étale module is regular, and that unipotent and semisimple algebraic groups do not admit (linear) étale representations. In Section 5 we identify the étale modules among certain classifications of prehomogeneous modules due to Sato and Kimura [19] and Kimura et al. [12, 13]. In Section 6 we derive criteria for reductive algebraic groups G with one-dimensional center to admit étale representations. By Lemma 4.3, an étale module for G is a regular prehomogeneous module (in the sense of Section 2). A main tool for the investigation of reducible modules is the following theorem proved by Baues.

Theorem B (Baues). *Let $G = \mathrm{GL}_1 \times S$ with S semisimple and let (G, ϱ, V) be an étale module. Suppose (G, ϱ, W) is a proper submodule of (G, ϱ, V). Then (G, ϱ, W) is a non-regular prehomogeneous module.*

We can combine Theorems A and B to find a non-existence result for a certain class of reductive groups. We show:

Theorem C. *Let $G = \mathrm{GL}_1 \times S \times \overset{k}{\cdots} \times S$, where S is a simple algebraic group and $k \geq 2$. Then G has no étale representations.*

Notations and conventions. We write V^m to emphasize that the dimension of a vector space V is m.

The unit element of a group G is denoted by 1 or 1_G. For matrix groups, we also use I_n to denote the identity matrix. When writing GL_n (resp. SL_n, Sp_n, SO_n, Spin_n), we always assume the complex numbers as the coefficient field.

For convenience, we will often denote a module (ϱ, V) by the representation ϱ only. In this case, we also write $\dim \varrho$ for $\dim V$. The dual representation (or module) is denoted by (ϱ^*, V^*). The notation $\varrho^{(*)}$ means either ϱ or its dual ϱ^*.

It is well-known that an irreducible representation ϱ of a semisimple Lie algebra $\mathfrak{g}$ is uniquely determined by its highest weight ω. After the choice of a Cartan subalgebra of $\mathfrak{g}$, ω is a unique integral linear combination $m_1\omega_1 + \ldots + m_n\omega_n$ of the fundamental weights $\omega_1, \ldots, \omega_n$ of $\mathfrak{g}$. For brevity we often write ω when we mean the "representation ϱ with highest weight ω". The representation of $\mathfrak{gl}_1$ (or GL_1) by scalar multiplication on a vector space is denoted by μ. The trivial representation for any group is denoted by 1.

Acknowledgements. Dietrich Burde acknowledges support by the Austrian Science Foundation FWF, grant P28079 and grant I3248. Wolfgang Globke is supported by the Australian Research Council grant DE150101647. He would also like to thank Oliver Baues for introducing him to this subject several years ago.

2. Basics of prehomogeneous modules

2.1. **Prehomogenous modules and relative invariants.** Let (G, ϱ, V) be a prehomogeneous module. The points v in the open orbit of G are called *generic points*, and the stabilizer $G_v = \{g \in G \mid gv = v\}$ at a generic point v is called the *generic isotropy subgroup*, its Lie algebra $\mathfrak{g}_v$ is called the *generic isotropy subalgebra*. The *singular set* $V_0 = V \backslash \varrho(G)v$ is the complement of the open orbit in V.

Prehomogeneity is equivalent to

$$\dim G_v = \dim G - \dim V$$

and to $V = \{\mathrm{d}\varrho(A)v \mid A \in \mathfrak{g}\}$. In particular, if ϱ is étale, then G_v is a finite (since algebraic) subgroup and $\mathrm{d}\varrho(\cdot)v : \mathfrak{g} \to V$ is a vector space isomorphism.

Prehomogeneous modules are to a large extent characterized by their *relative invariants*, that is, those rational functions $f : V \to \mathbb{C}$ satisfying

$$f(gv) = \chi(g)f(v),$$

where $g \in G$ and $\chi \in \mathsf{X}(G) = \{\chi : G \to \mathbb{C}^\times \mid \chi \text{ is a rational homomorphism}\}$. Prehomogeneity of (G, ϱ, V) is equivalent to the fact that any absolute invariant (that is, with character $\chi = 1$) is a constant function.

Given a relative invariant f of (G, ϱ, V), define a map

$$\varphi_f : V \backslash V_0 \to V^*, \quad x \mapsto \operatorname{grad} \log f(x).$$

If the image of φ_f is Zariski-dense in V^*, then we call f a *non-degenerate* relative invariant, and (G, ϱ, V) a *regular* prehomogeneous module. For reductive algebraic groups G, we have the following characterization of regular prehomogeneous modules (see Kimura [11, Theorem 2.28]).

Theorem 2.1. *Let G be a reductive algebraic group and (G, ϱ, V) a prehomogeneous module. Then the following are equivalent:*

(1) *(G, ϱ, V) is a regular prehomogeneous module.*
(2) *The singular set V_0 is a hypersurface.*
(3) *The open orbit $\varrho(G)v = V \backslash V_0$ is an affine variety.*
(4) *Each generic isotropy subgroup G_v for $v \in V \backslash V_0$ is reductive.*
(5) *Each generic isotropy subalgebra $\mathfrak{g}_v$ for $v \in V \backslash V_0$ is reductive in $\mathfrak{g} = \mathfrak{Lie}(G)$.*

2.2. **Castling and promotion.** Two modules (G_1, ϱ_1, V_1) and (G_2, ϱ_2, V_2) (or representations ϱ_1 and ϱ_2) are called *equivalent* if there exists an isomorphism of groups $\psi : \varrho_1(G_1) \to \varrho_2(G_2)$ and a linear isomorphism $\varphi : V_1 \to V_2$ such that $\psi(\varrho_1(g))\varphi(x) = \varphi(\varrho_1(g)x)$ for all $x \in V_1$ and $g \in G_1$.

Remark 2.2. If G is reductive, then the dual representation $\varrho^* : G \to \mathrm{GL}(V^*)$ of any given representation $\varrho : G \to \mathrm{GL}(V)$ is equivalent to ϱ. This follows from a result by Mostow [16].

Let $m > n \geq 1$ and $\varrho : G \to \mathrm{GL}(V^m)$ be a finite-dimensional rational representation of an algebraic group G. Then we say the modules

$$\left(G \times \mathrm{GL}_n,\ \varrho \otimes \omega_1,\ V^m \otimes \mathbb{C}^n\right) \quad \text{and} \quad \left(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^{m*} \otimes \mathbb{C}^{m-n}\right)$$

are *castling transforms* of each other. More generally, we say two modules (G_1, ϱ_1, V_1) and (G_2, ϱ_2, V_2) are *castling-equivalent* if (G_1, ϱ_1, V_1) is equivalent to a module obtained after a finite number of castling transforms from (G_2, ϱ_2, V_2).

We say a module (G, ϱ, V) is *reduced* (or *castling-reduced*) if $\dim V \leq \dim V'$ for every castling transform (G, ϱ', V') of (G, ϱ, V).

Theorem 2.3 (Sato & Kimura [19])**.** *Let $m > n \geq 1$ and $\varrho : G \to \mathrm{GL}(V^m)$ be a finite-dimensional rational representation of an algebraic group G. Then*

$$\left(G \times \mathrm{GL}_n,\ \varrho \otimes \omega_1,\ V^m \otimes \mathbb{C}^n\right)$$

is a prehomogeneous module (with generic isotropy subgroup $H^{(n)}$) if and only if its castling transform

$$\left(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^{m*} \otimes \mathbb{C}^{m-n}\right)$$

is prehomogeneous (with generic isotropy subgroup $H^{(m-n)}$). Furthermore, $H^{(n)}$ and $H^{(m-n)}$ are isomorphic.

Addendum: If G is reductive and its center acts by scalar multiplication on $V^m \otimes \mathbb{C}^n$, then we can replace every occurrence of GL_n by SL_n in the above statement, and prehomogeneity of

$$\big(G \times \mathrm{SL}_n,\ (\sigma \otimes 1) \oplus (\varrho \otimes \omega_1),\ V^k \oplus (V^m \otimes \mathbb{C}^n)\big)$$

is equivalent to prehomogeneity of

$$\big(G \times \mathrm{SL}_{m-n},\ (\sigma^* \otimes 1) \oplus (\varrho \otimes \omega_1),\ V^{k*} \oplus (V^m \otimes \mathbb{C}^{m-n})\big).$$

Remark 2.4. Castling transforms regular prehomogeneous modules into regular prehomogeneous modules, and étale modules into étale modules (because $H^{(n)} \cong H^{(m-n)}$).

Example 2.5. Castling allows to add additional factors to the group. Let (G, ϱ, V^m) be a reductive prehomogeneous module. We can interprete it as

$$(G, \varrho, V^m) = (G \times \mathrm{SL}_1, \varrho \otimes \omega_1, V^m \otimes \mathbb{C}^1).$$

The castling transform of this module is

$$(G \times \mathrm{SL}_{m-1}, \varrho \otimes \omega_1, V^m \otimes \mathbb{C}^{m-1}).$$

We call a castling transform of this particular type a *promotion* of the module (G, ϱ, V^m).

3. Castling transforms of irreducible prehomogeneous modules

In this section we do not assume that the prehomogeneous modules are étale modules. Our aim is to prove Theorem A.

Lemma 3.1. *Let L be a reductive algebraic group with one simple factor and let*

$$(L \times \mathrm{SL}_m,\ \sigma \otimes \omega_1,\ V^n \otimes \mathbb{C}^m)$$

be a module with $(L, \sigma) \neq (\mathrm{SL}_n, \omega_1)$, σ irreducible and $m, n \geq 1$. Let

$$(L \times \mathrm{SL}_{m_1} \times \cdots \times \mathrm{SL}_{m_k},\ \sigma \otimes \omega_1 \otimes \cdots \otimes \omega_1,\ V^n \otimes \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_k})$$

be castling-equivalent to the first module, with $k \geq 2$. Then:

1. $\gcd(m_i, m_j) = 1$ *for $1 \leq i < j \leq k$.*
2. $\gcd(n, m_i) = 1$ *for all but at most one index $i_0 \in \{1, \ldots, k\}$.*
3. $\gcd(n, m_{i_0}) = \gcd(n, m)$ *for this index i_0.*

Proof. Any sequence of castling transforms of the original module will start with a promotion, adding a factor $(\mathrm{SL}_{nm-1}, \omega_1, \mathbb{C}^{nm-1})$ to the module. But clearly, $\gcd(m, nm-1) = 1 = \gcd(n, nm-1)$, and for $m_1 = m$, $\gcd(n, m_1) = \gcd(n, m)$.

Suppose the claim holds after $\ell \geq 1$ castling transforms of the original module. We may assume the groups are ordered such that $i_0 = 1$. Apply another castling transform. If the transform is a promotion, then we obtain a new factor $\mathrm{SL}_{nm_1 \cdots m_k - 1}$, and the claim clearly holds for the new module. Otherwise, consider two cases:

First, suppose that SL_{m_1} is replaced by $\mathrm{SL}_{m_1'}$ with $m_1' = nm_2 \cdots m_k - m_1$. By the induction hypothesis, for $i = 2, \ldots, k$, we have $\gcd(n, m_i) = 1$ and also $\gcd(m_1', m_i) = 1$, since every divisor of m_i divides $nm_2 \cdots m_k$ but not m_1. Suppose d is a common divisor of n and m_1. Then d divides both $nm_2 \cdots m_k$ and m_1, hence d divides m_1'. Similarly, every divisor of n and m_1' divides m_1. Hence $\gcd(n, m_1) = \gcd(n, m_1')$.

Now consider the case that a factor other than SL_{m_1}, say SL_{m_k}, is replaced by the castling transform. The new factor is $\mathrm{SL}_{m'_k}$ with $m'_k = nm_1 \cdots m_{k-1} - m_k$. By the induction hypothesis, every divisor of m_i with $i \neq k$ divides $nm_1 \cdots m_{k-1}$ but not m_k, hence not m'_k. It follows that $\gcd(m_i, m'_k) = 1$ for all $i \neq k$. Similarly, $\gcd(n, m'_k) = 1$. Moreover, $\gcd(n, m_i) = 1$ for all $2 \leq i < k$ and $\gcd(n, m_1) = \gcd(n, m)$ by the induction hypothesis.

So the claim holds after $\ell + 1$ castling transforms, and the lemma follows. □

Lemma 3.2. *For $m, n \geq 1$, let*

$$(L \times \mathrm{GL}_m,\ \sigma \otimes \omega_1,\ V^n \otimes \mathbb{C}^m)$$

be a module with $(L, \sigma) = (\mathrm{SL}_n, \omega_1)$ or $(L, \sigma) = (\mathrm{SL}_3 \times \mathrm{SL}_3, \omega_1 \otimes \omega_1)$. In the latter case, assume additionally that $\gcd(3, m) = 1$. Let

$$(\mathrm{SL}_{m_1} \times \cdots \times \mathrm{SL}_{m_k},\ \omega_1 \otimes \cdots \otimes \omega_1,\ \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_k})$$

be castling-equivalent to the first module, with $k \geq 2$. Then $\gcd(m_i, m_j) = 1$ for all $1 \leq i < j \leq k$ but at most one pair of indices $i_0, j_0 \in \{1, \ldots, k\}$.

The proof is mutatis mutandis identical to the proof of Lemma 3.1.

Theorem A. *Let (G, ϱ, V) be an irreducible prehomogeneous module for a reductive algebraic group. Then:*

$$(G, \varrho, V) = (L \times \mathrm{SL}_{m_1} \times \cdots \times \mathrm{SL}_{m_k},\ \sigma \otimes \omega_1 \otimes \cdots \otimes \omega_1,\ V^n \otimes \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_k}),$$

where L is a reductive algebraic group with one simple factor, σ is irreducible, and $n, m_1, \ldots, m_k \geq 1$ such that

(1) $\gcd(m_i, m_j) = 1$ *for* $1 \leq i < j \leq k$.
(2) $\gcd(n, m_i) = 1$ *for all but at most one index* $i_0 \in \{1, \ldots, k\}$.

Moreover, if (G, ϱ, V) is castling-equivalent to a one-simple irreducible module, then part (2) holds for all $i \in \{1, \ldots, k\}$.

Proof. Every irreducible reductive prehomogeneous module is castling-equivalent to one of those classified by Sato and Kimura [19, §7], so it is enough to prove the theorem for those modules. Lemma 3.2 proves the theorem for the irreducible modules SK I-1 (with $(\mathrm{SL}_m, \omega_1)$), SK I-12, SK III-1 and SK III-2. From the classification it is clear that every other reduced irreducible module is of the form assumed in Lemma 3.1, and so the theorem follows from this lemma in these cases. □

Remark 3.3. Every reductive prehomogeneous module decomposes into irreducible ones, but since such a decomposition can be obtained by taking direct sums (that is, $(G_1, \varrho_1, V_1) \oplus (G_2, \varrho_2, V_2) = (G_1 \times G_2, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$), it is not true that every castling transform of non-irreducible prehomogeneous modules is of the form described in Theorem A.

4. General properties of étale representations

Étale modules were introduced in Section 1. Here we present some structural results on étale modules, whereas in the next section we provide many new examples of étale modules for reductive groups.

Proposition 4.1. *The following conditions are equivalent:*

(1) $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ *is an étale module.*

(2) (G, ϱ_1, V_1) *is prehomogeneous and* $(H, \varrho_2|_H, V_2)$ *is an étale module, where* H *denotes the connected component of the generic isotropy subgroup of* (G, ϱ_1, V_1).

Equivalence also holds if each "étale" is replaced by "prehomogeneous".

Proof. By Kimura [11, Lemma 7.2], $\varrho_1 \oplus \varrho_2$ is prehomogeneous if and only if ϱ_1 and $\varrho_2|_H$ are. The representation $\varrho_1 \oplus \varrho_2$ is étale if and only if it is prehomogeneous and $\dim G = \dim V_1 + \dim V_2$.

Suppose $\varrho_1 \oplus \varrho_2$ is étale. So (G, ϱ_1, V_1) and $(H, \varrho_2|_H, V_2)$ are prehomogeneous, and since $\dim G - \dim H = \dim V_1$, we have $\dim H = \dim V_2$, so $(H, \varrho_2|_H, V_2)$ is étale.

Conversely, if we assume (G, ϱ_1, V_1) to be prehomogeneous and $(H, \varrho_2|_H, V_2)$ to be étale, then $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ is obviously prehomogenous and as

$$\dim G - \dim V_1 = \dim H = \dim V_2,$$

it is even étale. □

Corollary 4.2. *Let* (G, ϱ_1, V_1) *be a prehomogeneous module with reductive generic isotropy subgroup. Then* $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ *is étale if and only if* $(G, \varrho_1 \oplus \varrho_2^*, V_1 \oplus V_2^*)$ *is étale, and then their generic isotropy subgroups are isomorphic.*

Proof. For a reductive group G, (G, ϱ, V) is equivalent to (G, ϱ^*, V^*), see Remark 2.2. The corollary now follows from Proposition 4.1. □

4.1. **Regularity of étale modules.** First, we note that non-regular prehomogeneous modules are not étale modules for a reductive algebraic group.

Lemma 4.3. *Let* G *be a reductive algebraic group. If* (G, ϱ, V) *is an étale module, then it is a regular prehomogeneous module.*

Proof. The generic isotropy subgroup of an étale module is finite, hence reductive. By Theorem 2.1, the module is regular. □

This lemma does not imply that any irreducible component of an étale module must be regular. In fact, it will follow from Theorem B that for groups with one-dimensional center, an étale module that contains a regular irreducible component must be irreducible itself.

4.2. **Groups with trivial character group.** Let $\mathsf{X}(G)$ denote the character group of G (the group of rational homomorphisms $\chi : G \to \mathbb{C}^\times$). The following proposition is possibly well-known. Since we do not know a reference for it, we will give a proof here.

Proposition 4.4. *Let* G *be an algebraic group with* $\mathsf{X}(G) = \{1\}$. *Then* G *does not admit a rational linear étale representation.*

Proof. Assume that $\varrho : G \to V$ is a linear étale representation. Let $n = \dim G = \dim V > 0$. By Kimura [11, Proposition 2.20], the prehomogeneous module (G, ϱ, V) has a relative invariant f of degree n, so f is not constant. As $\mathsf{X}(G) = \{1\}$, the associated character χ of f must be $\chi = 1$, which means that f is an absolute invariant. But this is a contradiction to the fact that prehomogeneous modules do not admit non-constant absolute invariants. □

We conclude that unipotent groups and semisimple groups do not admit linear étale representations, since their respective groups of rational characters are trivial.

Corollary 4.5. *There is no rational linear étale representation for a semisimple algebraic group.*

Corollary 4.6. *There is no rational linear étale representation for a unipotent algebraic group.*

On the other hand, a unipotent algebraic group may admit an affine étale representation. This is not the case for a semisimple algebraic group S. An affine étale representation of S is automatically linear and vice versa ([2, Corollary 3.9]). It is already known that S does not admit an affine étale representation, because of the correspondence to LSA-structures on the semisimple Lie algebra $\mathfrak{s}$ of S. Indeed, a semisimple Lie algebra $\mathfrak{s}$ of characteristic zero does not admit an LSA-structure, because $H^1(\mathfrak{s}, M) = 0$ for all finite-dimensional $\mathfrak{s}$-modules M by the first Whitehead Lemma on Lie algebra cohomology. However, the argument with the character group here gives an independent proof.

Remark 4.7. The vanishing of the Lie algebra cohomology $H^n(\mathfrak{g}, \mathfrak{g})$ for all $n \geq 0$ with the adjoint module $\mathfrak{g}$ alone is not enough to ensure that G does not admit an affine étale representation. For example, the linear algebraic group $G = \mathrm{Aff}(V)$ for a vector space V has a cohomologically rigid Lie algebra $\mathfrak{g} = \mathfrak{aff}(V)$, which satisfies $H^n(\mathfrak{g}, \mathfrak{g}) = 0$ for all $n \geq 0$. But the coadjoint representation of G is étale.

5. Étale modules for groups with one or two simple factors

As stated in the introduction, certain classification results for étale modules are immediately obtained from the classification of prehomogeneous modules. These classifications have been collected in a convenient reference in [9].

Remark 5.1. In Kimura et al. [12], [13], the prehomogeneous modules are always stated with one scalar multiplication μ acting on each irreducible component, that is, $(\mathrm{GL}_1^k \times G, \varrho_1 \oplus \ldots \oplus \varrho_k)$, and in this case we do not explicitely state the scalar multiplications, as it is understood that each ϱ_i stands for $\mu \otimes \varrho_i$. But in some cases, we do not need an independent scalar multiplication on each component to achieve prehomogeneity. Consider for example the prehomogeneous module Ks A-2, $(\mathrm{GL}_1^n \times \mathrm{SL}_n, \omega_1^{\oplus n})$. For $\omega_1^{\oplus n}$ we need only the operation of SL_n and one scalar multiplication GL_1 acting on all components to obtain a prehomogeneous module, that is $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n})$.

Finding the étale modules of type SK, Ks and KI is rather easy, as the generic isotropy subgroup is known in each case. Thus we can just pick the modules with $G_v^\circ \cong \{1\}$ from the known classification tables. Finding étale modules in the class KII is significantly more complicated and will be done in a forthcoming article.

Proposition 5.2. *The following irreducible reduced prehomogeneous modules are all étale modules in the list SK:*

- *SK I-4:* $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\mathbb{C}^2)$.
- *SK I-8:* $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\mathbb{C}^3 \otimes \mathbb{C}^2)$.
- *SK I-11:* $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \mathbb{C}^5 \otimes \mathbb{C}^4)$.

Proposition 5.3. *The following non-irreducible one-simple prehomogeneous modules are all étale modules in the list Ks:*

- *Ks A-1 for $n = 2$: This is equivalent to Ks A-4 with $n = 2$.*
- *Ks A-2:* $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\mathbb{C}^n)^{\oplus n})$.
- *Ks A-3:* $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\mathbb{C}^n)^{\oplus n+1})$.
- *Ks A-4:* $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\mathbb{C}^n)^{\oplus n} \oplus \mathbb{C}^{n*})$.
- *Ks A-11 for $n = 2$:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_2, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\mathbb{C}^2 \otimes \mathbb{C}^2)$.

- *Ks A-12 for $n = 2$: Equivalent to Ks A-11 with $n = 2$.*
- *Ks A-20 for $n = 1$: Equivalent to Ks A-2 with $n = 2$.*

Corollary 5.4. *If $(\mathrm{GL}_1^k \times S, \varrho, V)$ for $k \geq 1$ and a simple group S is an étale module, then $S = \mathrm{SL}_n$ for some $n \geq 1$.*

Proposition 5.5. *The following two-simple prehomogeneous modules of type I are all étale modules in the list KI:*

- *KI I-1:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1), (\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2))$.
- *KI I-2:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1), (\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^4)$.
- *KI I-6:* $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1), (\bigwedge^2 \mathbb{C}^5 \otimes \mathbb{C}^2) \oplus \mathbb{C}^{5*} \oplus \mathbb{C}^{5(*)})$.
- *KI I-16:* $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\mathbb{C}^4 \otimes \mathbb{C}^3) \oplus V^5 \oplus \mathbb{C}^3)$.
- *KI I-18:* $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (V^5 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^2)$.
- *KI I-19:* $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (V^5 \otimes \mathbb{C}^4) \oplus \mathbb{C}^4 \oplus \mathbb{C}^4)$.

6. Étale modules for groups with one-dimensional center

In this section we study étale modules for algebraic groups $G = \mathrm{GL}_1 \times S$, where S is semisimple.

6.1. **Non-regularity of submodules.** Below, let $\mathbb{C}[V]^G$ denote the ring of G-invariant polynomial functions on V, and let $V /\!\!/ G$ denote the *algebraic quotient* of the G-action on V, that is, the algebraic variety with coordinate ring $\mathbb{C}[V]^G$ which parameterizes the closed orbits of G.

Theorem B. *Let $G = \mathrm{GL}_1 \times S$ with S semisimple and let (G, ϱ, V) be an étale module. Suppose (G, ϱ, W) is a proper submodule of (G, ϱ, V). Then (G, ϱ, W) is a non-regular prehomogeneous module.*

In the terminology of Rubenthaler [18], this theorem states that (G, ϱ, V) is *quasi-irreducible*.

Proof. Denote the étale module by $(\mathrm{GL}_1 \times S, \varrho, V)$ and let W be a non-trivial S-submodule and U an invariant complement in V.

Assume $\dim W = \dim S$. Then $\dim U = 1$, so the action of S on U is trivial by the semisimplicity of S. It follows that S has an open orbit on W, which contradicts the fact that semisimple groups do not admit étale modules. So $\dim W < \dim S$.

By Baues [2, Proposition 3.3], the submodule W must be contained in the fiber over the orbit $\{0\}$ of the algebraic quotient map $\pi : V \to V /\!\!/ S$, and by Baues [2, Proposition 3.2], $V /\!\!/ S$ is isomorphic to the affine line $\mathbb{C}$, and $\mathbb{C}[V]^S$ is generated by an irreducible non-constant homogeneous polynomial f.

Consider $h \in \mathbb{C}[W]^S$. Then h is also an element of $\mathbb{C}[V]^S$ and $h = f_0 + c$ with $c \in \mathbb{C}$ and $f_0 \in \langle f \rangle$. As any $w \in W$ is contained in the fiber over $\{0\}$, we have $h(w) = f_0(w) + c = c$, that is, $h = c$ and thus $\mathbb{C}[W]^S = \mathbb{C}$. So $\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}[W]^S = 0$, and now a formula by Rosenlicht [17],

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}[W]^S = \dim W - \max\{\dim \varrho(S)w \mid w \in W\},$$

implies

$$\dim W = \max\{\dim \varrho(S)w \mid w \in W\}.$$

This means W is a prehomogeneous module for S.

As S is semisimple, any non-constant relative invariant of $\mathrm{GL}_1 \times S$ on W is an absolute invariant for S on W. Then $\mathbb{C}[W]^S = \mathbb{C}$ implies that there are no non-constant relative invariants for $\mathrm{GL}_1 \times S$ on W, so W is a non-regular prehomogeneous module. ∎

A simple example illustrates the statement of Theorem B:

Example 6.1. The module Ks A-2, $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\mathbb{C}^n)^{\oplus n})$ is an étale module. We identify $(\mathbb{C}^n)^{\oplus n} = \mathrm{Mat}_n$, and then a relative invariant is given by the determinant of $n \times n$-matrices. This module decomposes into n irreducible and non-regular summands of type SK III-2, $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1, \mathbb{C}^n)$ corresponding to action by matrix-vector multiplication on each column of matrices in Mat_n.

Theorem B does not hold if the center of G has dimension ≥ 2:

Example 6.2. Consider the module KI I-2,

$$(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1), (\textstyle\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^4).$$

The first irreducible component of this module, $\omega_2 \otimes \omega_1$, corresponds to the regular irreducible module SK I-15 with parameters $n = 6$, $m = 2$ (recall that over the complex numbers, SO_6 and SL_4 are locally isomorphic).

6.2. **Groups with copies of one simple factor only.** We now want to study reductive groups of the form

$$G = \mathrm{GL}_1 \times S \times \overset{k}{\cdots} \times S, \tag{6.1}$$

where S is simple and $k \geq 2$ (for $k = 1$ see Section 5). If there are any irreducible étale modules for such a group, each of them must be castling-equivalent to one of the modules in Proposition 5.2,

- SK I-4: $(\mathrm{GL}_2 \times \mathrm{SL}_1, 3\omega_1 \otimes \omega_1, \mathrm{Sym}^3 \mathbb{C}^2)$.
- SK I-8: $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2 \mathbb{C}^3 \otimes \mathbb{C}^2)$.
- SK I-11: $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \mathbb{C}^5 \otimes \mathbb{C}^4)$.

Remark 6.3. As castling only adds additional factors SL_m, this list shows that no irreducible étale representation for (6.1) with $S \neq \mathrm{SL}_m$ can exist.

By Theorem B, any reducible étale module decomposes into irreducible components, each of which is a non-regular prehomogeneous module for G. Therefore, by the Sato-Kimura classification [19, III on p. 147] (or [9, Section 1]), each irreducible component is castling-equivalent to one of the following:

- SK III-1: $(L \times \mathrm{GL}_m, \varrho \otimes \omega_1, V^n \otimes \mathbb{C}^m)$, where $\varrho : L \to \mathrm{GL}(V^n)$ is an n-dimensional irreducible representation of a semisimple algebraic group L $(\neq \mathrm{SL}_n)$ with $m > n \geq 3$.
- SK III-2: $(\mathrm{SL}_n \times \mathrm{GL}_m, \omega_1 \otimes \omega_1, \mathbb{C}^n \otimes \mathbb{C}^m)$ for $\frac{1}{2}m \geq n \geq 1$.
- SK III-3: $(\mathrm{GL}_{2n+1}, \omega_2, \bigwedge^2 \mathbb{C}^{2n+1})$ for $n \geq 2$.
- SK III-4: $(\mathrm{GL}_2 \times \mathrm{SL}_{2n+1}, \omega_1 \otimes \omega_2, \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^{2n+1})$ for $n \geq 2$.
- SK III-5: $(\mathrm{Sp}_n \times \mathrm{GL}_{2m+1}, \omega_1 \otimes \omega_1, \mathbb{C}^{2n} \otimes \mathbb{C}^{2m+1})$ for $n > 2m + 1 \geq 1$.
- SK III-6: $(\mathrm{GL}_1 \times \mathrm{Spin}_{10}, \mu \otimes \text{halfspinrep}, \mathbb{C} \otimes V^{16})$.

Now it is obvious that any castling transform of one of these modules will have a group which has at least one factor SL_m with $m \geq 2$. So among these there is not even a prehomogeneous module for a group (6.1) with $S \neq \mathrm{SL}_m$. Combined with Remark 6.3, we have:

Corollary 6.4. *Let $G = \mathrm{GL}_1 \times S \times \cdots \times S$, where S is a simple algebraic group other than SL_m for any $m \geq 2$. Then there exist no étale modules for G.*

Lemma 6.5. $G = \mathrm{GL}_1 \times \mathrm{SL}_m \times \overset{k}{\cdots} \times \mathrm{SL}_m$ *has no irreducible étale representations.*

Proof. If G has an irreducible étale module (G, ϱ, V), then it is castling-equivalent to one of the modules SK I-4, SK I-8 or SK I-11 above. But in each of these cases, we have two factors SL_{m_1}, SL_{m_2} with $m_1 \neq m_2$. Any non-trivial castling transform of these modules would have at least three simple factors. By Theorem A, this means there are at least two simple factors SL_{m_1}, SL_{m_2} with $\gcd(m_1, m_2) = 1$, which again means $m_1 \neq m_2$. ☐

Theorem C. *Let $G = \mathrm{GL}_1 \times S \times \overset{k}{\cdots} \times S$, where S is a simple algebraic group and $k \geq 2$. Then G has no étale representations.*

Proof. Consider $G = \mathrm{GL}_1 \times S_1 \times \cdots \times S_k$. As we are interested in the case where all simple factors are identical, by Corollary 6.4, we only need to consider the case where all $S_i = \mathrm{SL}_{m_i}$ for $m_i \geq 2$. Let (G, ϱ, V) be an étale module. First, observe that the étale representation has at least one irreducible factor on which at least two of the factors, say S_1 and S_2, act non-trivially. In fact, otherwise (G, ϱ, V) would be a direct sum of étale modules $(\mathrm{GL}_1 \times S_i, \varrho_i, V_i)$, which is regular for $\mathrm{GL}_1 \times S_i$ by Lemma 4.3, hence for G, as the stabilizer on V_i is the product of the S_j with $j \neq i$. But the center of G is one-dimensional, so by Theorem B, an (G, ϱ, V) does not have proper regular submodules. This would imply $k = 1$, contradicting our assumption that $k \geq 2$.

Let (G, ϱ_1, V_1) be an irreducible factor on which at least two simple factors of G act non-trivially. By Lemma 6.5, there are no irreducible étale representations of G if all simple factors are identical, so we may assume the étale representation is reducible, and by Theorem B, (G, ϱ_1, V_1) must be a non-regular irreducible prehomogeneous module for G. After removing simple factors contained in the generic stabilizer of (ϱ_1, V_1) from G, we can assume that (G, ϱ_1, V_1) is castling equivalent to one of the reduced irreducible modules SK III-1 (with $(L, \varrho) \neq (\mathrm{SL}_n, \omega_1)$), SK III-2, SK III-3, and SK III-4. In each of these cases the group is of the form $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$ with $m \neq n$. By Theorem A, any of its castling transforms has at least two factors SL_{m_1} and SL_{m_2} with $m_1, m_2 > 1$ and $\gcd(m_1, m_2) = 1$. In particular, it is not possible that all simple factors of the group G are identical. ☐

Remark 6.6. If we admit a center GL_1^k, then we trivially obtain étale modules with semisimple part $\mathrm{SL}_n \times \cdots \times \mathrm{SL}_n$ by taking direct sums of étale modules for $\mathrm{GL}_1 \times \mathrm{SL}_n$.

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Dietrich Burde, Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

E-mail address: dietrich.burde@univie.ac.at

Wolfgang Globke, School of Mathematical Sciences, The University of Adelaide, SA 5005, Australia

E-mail address: wolfgang.globke@adelaide.edu.au