DERIVATION DOUBLE LIE ALGEBRAS

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Abstract. We study classical R-matrices D for Lie algebras $\mathfrak{g}$ such that D is also a derivation of $\mathfrak{g}$. This yields derivation double Lie algebras $(\mathfrak{g}, D)$. The motivation comes from recent work on post-Lie algebra structures on pairs of Lie algebras arising in the study of nil-affine actions of Lie groups. We prove that there are no nontrivial simple derivation double Lie algebras, and study related Lie algebra identities for arbitrary Lie algebras.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic zero. Motivated by studies in post-Lie algebras [9], [10] we are interested in the following question.

Question 1. *Let $\mathfrak{g}$ be a Lie algebra. For which derivations D of $\mathfrak{g}$ does the skew-symmetric bilinear map*

$$[x,y]_D = D([x,y])$$

satisfy the Jacobi identity ?

In other words, for which derivations D defines $[x,y]_D$ another Lie algebra, denoted by $\mathfrak{g}_D$? If $[x,y]_D$ is a Lie bracket, then the linear map D is also an example of a *classical R-matrix* for $\mathfrak{g}$, i.e., a linear transformations $R\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$[x,y]_R = [R(x),y] + [x,R(y)]$$

defines a Lie bracket. Classical R-matrices [19] have been studied by many authors. Our main result here is that for simple Lie algebras $\mathfrak{g}$ of rank $r \geq 2$ over an algebraically closed field of characteristic zero, $[x,y]_D = D([x,y])$ is a Lie bracket only for the trivial derivation $D = 0$, see Theorem 3.2. On the other hand, for $\mathfrak{sl}_2(K)$ this is a Lie bracket for all derivations $D \in \mathrm{Der}(\mathfrak{sl}_2(K))$.

Post-Lie algebra structures have been introduced in the context of nil-affine actions of Lie groups in [9], and also in connection with homology of partition posets and the study of Koszul operads in [15], [20]. Such structures are important in many areas of algebra, geometry and physics. They generalize both LR-structures and pre-Lie algebra structures, which we have studied in [4], [5], [6], [7], [8]. Related topics are Poisson structures and Lie bialgebra structures, which have been studied as well in connection with classical R-matrices and double Lie algebras, see [1].

Let us explain the motivation for question 1 in terms of post-Lie algebra structures. In [10],

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Theorem 6.4 we have determined all complex semisimple Lie algebras $\mathfrak{g}$ with Lie bracket $[\,,\,]$, and all $\lambda \in \mathbb{C}$, $z \in \mathfrak{g}$ such that

$$[x,y]_R = [z,[x,y]] + (2\lambda+1)[x,y]$$

defines a Lie bracket, and at the same time the Lie bracket $[\,,\,]$ satisfies

$$[[z,x],[z,y]] = [z,[z,[x,y]]] + (2\lambda+1)[z,[x,y]] + (\lambda^2+\lambda)[x,y]$$

for all $x,y \in \mathfrak{g}$. It turned out that this always implies $z=0$, except for the case where $\lambda = -\frac{1}{2}$, and the Lie algebra is isomorphic to a direct sum of $\mathfrak{sl}_2(\mathbb{C})$'s. For $\lambda = -\frac{1}{2}$ the second Lie bracket is given by $[x,y]_D = [z,[x,y]] = D([x,y])$ with the inner derivation $D = \mathrm{ad}(z)$. This motivated the question for which $z \in \mathfrak{g}$ the endomorphism $D = \mathrm{ad}(z)$ is a classical R-matrix.
Instead of $D = \mathrm{ad}(z)$ we consider here an arbitrary derivation D of any Lie algebra $\mathfrak{g}$, and study the Jacobi identity for the skew-symmetric bilinear map $[x,y]_D = D([x,y])$.

Question 1 naturally leads to another question, which is in particular also interesting for solvable and nilpotent Lie algebras:

Question 2. *Which Lie algebras $\mathfrak{g}$ have the property that $[x,y]_D = D([x,y])$ satisfies the Jacobi identity for all derivations $D \in \mathrm{Der}(\mathfrak{g})$?*

This is related to the theory of Lie algebra identities, and in particular to the variety $\mathrm{var}(\mathfrak{sl}_2(K))$, see [12], [13], [14]. We study this question together with the related identities (1), (2), (3) and (4) for some general classes of Lie algebras, and in particular for all complex nilpotent Lie algebras of dimension $n \leq 7$. A Lie algebra has the property given in question 2 if and only if identity (1) holds for it, for all derivations. We show that every almost abelian Lie algebra satisfies the Hom-Jacobi identity (2), and hence also (1), see Proposition 4.6. Finally we prove that every complex CNLA (characteristically nilpotent Lie algebra) of dimension 7 is a derivation double Lie algebra for all derivations, see Proposition 4.12.

2. Preliminaries

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic zero. Let $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$ for all $i \geq 1$. We say that $\mathfrak{g}$ is nilpotent if there exists an index $c \geq 1$ such that $\mathfrak{g}^c = 0$. In that case, the smallest such index is called the nilpotency class of $\mathfrak{g}$ and is denoted by $c(\mathfrak{g})$. Let $\mathfrak{g}^{(0)} = \mathfrak{g}$, and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, [\mathfrak{g}^{(i-1)}]$ for all $i \geq 1$. We say that $\mathfrak{g}$ is solvable if there exists an index $d \geq 1$ such that $\mathfrak{g}^{(d)} = 0$. In that case, the smallest such integer is called the solvability class and is denoted by $d(\mathfrak{g})$.

Classical R-matrices and double Lie algebras have been defined in [19] as follows.

Definition 2.1. Let V be a vector space over a field K, and $\mathfrak{g} = (V, [\,,])$ be a Lie bracket on V. A linear transformation $R\colon \mathfrak{g} \to \mathfrak{g}$ is called a *classical R-matrix*, if

$$[x,y]_R = [R(x),y] + [x,R(y)]$$

defines a Lie bracket, i.e., satisfies the Jacobi identity. In this case, the pair $(\mathfrak{g}, R)$ is called a *double Lie algebra*.

It is useful to set

$$B_R(x,y) = [R(x),R(y)] - R([R(x),y] + [x,R(y)]).$$

Then the Jacobi identity for $[x,y]_R$ can be formulated as follows [19]:

Proposition 2.2. *Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\,,\,]$. The bracket $[x,y]_R = [R(x),y] + [x,R(y)]$ satisfies the Jacobi identity if and only if*

$$[B_R(x,y),w] + [B_R(y,w),x] + [B_R(w,x),y] = 0$$

for all $x,y,w \in \mathfrak{g}$.

Definition 2.3. Let $\lambda \in K$. The identity $B_R(x,y) + \lambda[x,y] = 0$ for all $x,y \in \mathfrak{g}$ is called MYBE, the *modified Yang-Baxter equation*.

It is obvious that every solution R of MYBE is a classical R-matrix. The converse, however, need not be true in general.

Concerning question 1 we have the following result.

Proposition 2.4. *Let $\mathfrak{g}$ be a Lie algebra and $D \in \mathrm{Der}(\mathfrak{g})$ be a derivation. Then $[x,y]_D = D([x,y]) = [D(x),y] + [x,D(y)]$ satisfies the Jacobi identity if and only if*

$$D\Big([D(x),[y,w]] + [D(y),[w,x]] + [D(w),[x,y]]\Big) = 0 \tag{1}$$

for all $x,y,w \in \mathfrak{g}$.

Proof. We have

$$\begin{aligned}
[[x,y]_D,w]_D &= [[D(x),y] + [x,D(y)],w]_D \\
&= D([[D(x),y],w]) + D([[x,D(y)],w])
\end{aligned}$$

This yields

$$\begin{aligned}
[[x,y]_D,w]_D + [[y,w]_D,x]_D + [[w,x]_D,y]_D &= D([[D(x),y],w]) + D([[x,D(y)],w]) \\
&\quad + D([[D(y),w],x]) + D([[y,D(w)],x]) \\
&\quad + D([[D(w),x],y]) + D([[w,D(x)],y]) \\
&= -D([[y,w],D(x)] + [[w,x],D(y)] + [[x,y],D(w)]).
\end{aligned}$$

In the last step we have used the Jacobi identity three times, i.e.,

$$[[D(x),y],w] + [[y,w],D(x)] + [[w,D(x)],y] = 0,$$

and similarly for the terms with $D(y)$ and $D(w)$. $\square$

Note that identity (1) can also be stated as follows: for all $x,y,w \in \mathfrak{g}$ we have

$$\begin{aligned}
0 &= [x,[D(y),D(w)]] + [y,[D(w),D(x)]] + [w,[D(x),D(y)]] \\
&\quad + [D^2(y),[x,w]] + [D^2(w),[y,x]] + [D^2(x),[w,y]].
\end{aligned}$$

Definition 2.5. Let $\mathfrak{g}$ be a Lie algebra and $D \in \mathrm{Der}(\mathfrak{g})$ be a derivation, such that $[x,y]_D = D([x,y])$ defines another Lie bracket $\mathfrak{g}_D$. Then the pair $(\mathfrak{g},D)$ is called a *derivation double Lie algebra*.

The identity within the brackets of (1) for a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ is called the *Hom-Jacobi identity*, see [17] for further references. It says that

$$[D(x),[y,z]] + [D(y),[z,x]] + [D(z),[x,y]] = 0 \tag{2}$$

for all $x,y,z \in \mathfrak{g}$.

Corollary 2.6. *Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\,,\,]$ and $z \in \mathfrak{g}$. Then the bracket $[x,y]_D = [z,[x,y]]$ satisfies the Jacobi identity if and only if*

$$[z,[[z,x],[y,w]]] + [z,[[z,y],[w,x]]] + [z,[[z,w],[x,y]]] = 0 \tag{3}$$

for all $x, y, w \in \mathfrak{g}$.

Proof. This follows from Proposition 2.4 with $D = \operatorname{ad}(z)$. □

For $w = z$ the identity (3) implies, for $2 \neq 0$

$$[z,[[z,x],[z,y]]] = 0 \tag{4}$$

for all $x, y \in \mathfrak{g}$.

Lemma 2.7. *Let $\mathfrak{g}$ be a Lie algebra and suppose that $D = \operatorname{ad}(z)$ is a classical R-matrix, so that $[x,y]_D = [z,[x,y]]$ defines a second Lie bracket. Then $\operatorname{ad}(z)^3$ is a derivation of $\mathfrak{g}$.*

Proof. For $D = \operatorname{ad}(z)$ identity (4) gives

$$\begin{aligned} 0 &= D([D(x), D(y)]) \\ &= [D^2(x), D(y)] + [D(x), D^2(y)]. \end{aligned}$$

This yields

$$\begin{aligned} D^3([x,y]) &= [D^3(x), y] + 3[D^2(x), D(y)] + 3[D(x), D^2(y)] + [x, D^3(y)] \\ &= [D^3(x), y] + [x, D^3(y)]. \end{aligned}$$

Hence $D^3 = \operatorname{ad}(z)^3$ is a derivation. □

Conversely, if $\operatorname{ad}(z)^3$ is a derivation of $\mathfrak{g}$, and $3 \neq 0$, then identity (4) holds for z.

Remark 2.8. An element z of a Lie algebra $\mathfrak{g}$ is called *extremal*, if there is a linear map $f_z \colon \mathfrak{g} \to K$ such that

$$[z,[z,x]] = f_z(x)z$$

for all $x \in \mathfrak{g}$. For the study of extremal elements see [11] and the references therein. It is a well known result of Premet, that every simple Lie algebra over an algebraically closed field of characteristic different from 2 and 3 has a nontrivial extremal element. Note that for every extremal element $z \in \mathfrak{g}$ we have $\operatorname{ad}(z)^3 = 0$, so that identity (4) holds for all extremal elements $z \in \mathfrak{g}$ by the above Lemma.

3. Simple derivation double Lie algebras

We will give here an answer to question 1 for simple Lie algebras $\mathfrak{g}$ over an algebraically closed field of characteristic zero. In terms of classical R-matrices, the question is, for which $z \in \mathfrak{g}$ the linear map $R = \operatorname{ad}(z)$ is a classical R-matrix. We will show that for rank one every $\operatorname{ad}(z)$ is a classical R-matrix, and that for rank $r \geq 2$ only the zero transformation is a classical R-matrix. In other words, there are no nontrivial simple derivation double Lie algebras of rank $r \geq 2$.

One should remark that simple Lie algebras always admit nontrivial classical R-matrices, but not of the form $\operatorname{ad}(z)$. An easy example is given by $R = \lambda I_n$ with the identity matrix I_n. In general there are much more possibilities, see [2].

Denote by $\mathfrak{r}_{3,1}(\mathbb{C})$ the 3-dimensional solvable Lie algebra given by $[e_1, e_2] = e_2$ and $[e_1, e_3] = e_3$, see table 1.

Proposition 3.1. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Then $R = \mathrm{ad}(z)$ is a classical R-matrix for all $z \in \mathfrak{g}$. The Lie algebra $\mathfrak{g}_R$ is isomorphic to $\mathfrak{r}_{3,1}(\mathbb{C})$ for all $z \neq 0$.*

Proof. The claim follows by a direct computation. Let (e_1, e_2, e_3) be the standard basis of $\mathfrak{sl}(2,\mathbb{C})$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = -2e_1$, $[e_2, e_3] = 2e_2$, and write $z = z_1e_1 + z_2e_2 + z_3e_3$. Using $[x, y]_R = [z, [x, y]]$ we have

$$\begin{aligned}
[e_1, e_2]_R &= [z, [e_1, e_2]] = [z, e_3] = -2z_1e_1 + 2z_2e_2,\\
[e_1, e_3]_R &= [z, [e_1, e_3]] = [z, -2e_1] = -2z_3e_1 + 2z_2e_3,\\
[e_2, e_3]_R &= [z, [e_2, e_3]] = [z, 2e_2] = -2z_3e_2 + 2z_1e_3,
\end{aligned}$$

and

$$\begin{aligned}
[e_1, [e_2, e_3]_R]_R &= [e_1, -2z_3e_2 + 2z_1e_3]_R = -4z_2z_3e_2 + 4z_1z_2e_3,\\
[e_2, [e_3, e_1]_R]_R &= [e_2, 2z_3e_1 - 2z_2e_3]_R = 4z_1z_3e_1 - 4z_1z_2e_3,\\
[e_3, [e_1, e_2]_R]_R &= [e_3, -2z_1e_1 + 2z_2e_2]_R = -4z_1z_3e_1 + 4z_2z_3e_2.
\end{aligned}$$

Hence

$$[e_1, [e_2, e_3]_R]_R + [e_2, [e_3, e_1]_R]_R + [e_3, [e_1, e_2]_R]_R = 0.$$

It is easy to see that the resulting Lie algebra $\mathfrak{g}_R$ is isomorphic to $\mathfrak{r}_{3,1}(\mathbb{C})$, except for $z = 0$. □

Theorem 3.2. *Let $\mathfrak{g}$ be a simple Lie algebra of rank $r \geq 2$ over an algebraically closed field K of characteristic zero, and $z \in \mathfrak{g}$. Suppose that $R = \mathrm{ad}(z)$ is a classical R-matrix. Then $z = 0$ and $R = 0$.*

Proof. Let G be the identity component of the algebraic group $\mathrm{Aut}(\mathfrak{g})$, i.e., $G = \mathrm{Aut}(\mathfrak{g})^\circ$. Then G acts on $\mathfrak{g}$ and the set of $z \in \mathfrak{g}$ satisfying the identity (3) is a G-invariant closed set. Denote by $z = s + n$ the Jordan-Chevalley decomposition of z, with semisimple part s and nilpotent part n. We have $\mathrm{ad}(z) = \mathrm{ad}(z)_s + \mathrm{ad}(z)_n = \mathrm{ad}(s) + \mathrm{ad}(n)$, since $\mathfrak{g}$ is simple. Since the identity (3) holds also for the orbit closure and $Gs \subseteq \overline{Gz}$ is closed, we may apply a standard limit argument and pass to the semisimple part s of z. But for semisimple elements s we will show that the identity forces $s = 0$. Hence we obtain that $z = n$ must be nilpotent.
Let $z = s \neq 0$ be semisimple. Let α, β be roots such that $\alpha + \beta$ is again a root. We may assume that $(\alpha + \beta)(z) \neq 0$, since $\mathfrak{g}$ has rank at least 2. Now we take $(x, y, w) = (h, e_\alpha, e_\beta)$ for identity (1), with $h \in \mathfrak{h}$, the Cartan subalgebra of $\mathfrak{g}$. We have $[e_\alpha, e_\beta] = n_{\alpha\beta}e_{\alpha+\beta}$ with $n_{\alpha\beta} \neq 0$, since $\alpha + \beta$ is a root. Furthermore $[h, e_\alpha] = \alpha(h)e_\alpha$ and $[h, e_\beta] = \beta(h)e_\beta$. We have

$$\begin{aligned}
[[z, x], [y, w]] + [[z, y], [w, x]] + [[z, w], [x, y]] &= [[z, h], [e_\alpha, e_\beta]] + [[z, e_\alpha], [e_\beta, h]] + [[z, e_\beta], [h, e_\alpha]]\\
&= -(\alpha(z)\beta(h) - \beta(z)\alpha(h))[e_\alpha, e_\beta]\\
&= -n_{\alpha\beta}(\alpha(z)\beta(h) - \beta(z)\alpha(h))e_{\alpha+\beta}.
\end{aligned}$$

Applying $\mathrm{ad}(z)$ on the left-hand side we obtain by (3),

$$0 = n_{\alpha\beta}(\alpha + \beta)(z)(\alpha(z)\beta(h) - \beta(z)\alpha(h))$$

for all $h \in \mathfrak{h}$. This means $(\alpha(z)\beta(h) - \beta(z)\alpha(h)) = 0$ for all $h \in \mathfrak{h}$. This implies $\alpha(z) = \beta(z) = 0$, so that $(\alpha + \beta)(z) = 0$, a contradiction.
So we may assume that z is nilpotent. By Morozov's theorem $[z, x] = z$ for some $x \in \mathfrak{g}$, so that identity (4) implies $\operatorname{ad}(z)^3 = 0$. Again by the limit argument we can assume that z lies in the minimal nilpotent orbit, i.e., $z = e_\theta$, where θ is the maximal root, or just a long root. This implies that we may already assume that $\mathfrak{g}$ is of type A_2, B_2 or G_2. But now a direct computation with a computer algebra system shows that in all three cases the identity forces $z = 0$ and we are done. □

Corollary 3.3. *Let $\mathfrak{g}$ be a simple Lie algebra of rank $r \geq 2$ over an algebraically closed field K of characteristic zero, and suppose that $R = \operatorname{ad}(z)$ satisfies the modified Yang-Baxter equation, that is*

$$[z, [z, [x, y]]] = [[z, x], [z, y]] + \lambda[x, y] \tag{5}$$

for all $x, y \in \mathfrak{g}$. Then $z = 0$ and $\lambda = 0$.

Note that the operator form of identity (5) is given by

$$\operatorname{ad}(z)^2 \operatorname{ad}(x) - \operatorname{ad}(z) \operatorname{ad}(x) \operatorname{ad}(z) + \operatorname{ad}(x) \operatorname{ad}(z)^2 = \lambda \operatorname{ad}(x)$$

for all x. For the rank one case the following result can be shown, for all fields K of characteristic zero, again by a direct computation.

Proposition 3.4. *Let $\mathfrak{g} = \mathfrak{sl}(2, K)$ with standard basis (e_1, e_2, e_3), and $z = z_1e_1 + z_2e_2 + z_3e_3$. Then $R = ad(z)$ solves MYBE for $z \in \mathfrak{g}$ and $\lambda \in K$ if and only if $\lambda = 4(z_1z_2 + z_3^2)$.*

Concerning identity (2) we obtain a result analogous to Theorem 3.2, but for all simple Lie algebras.

Proposition 3.5. *Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field K of characteristic zero, and D be a derivation of $\mathfrak{g}$ satisfying the identity (2). Then $D = 0$.*

Proof. The result follows from Theorem 3.2 if $\mathfrak{g}$ has rank $r \geq 2$, and for the rank 1 case by a direct computation. On the other hand, there is also a direct proof without a computation. Since every derivation of $\mathfrak{g}$ is inner, there is an element $z \in \mathfrak{g}$ with $D = \operatorname{ad}(z)$. Then identity (2) gives, with $w = z$,

$$[[z, x], [z, y]] = 0$$

for all $x, y \in \mathfrak{g}$. As before in Theorem 3.2 we may assume that z is nilpotent. For $z = 0$ we are done. Otherwise, since z is nilpotent, there exists an element $x \in \mathfrak{g}$ with $[z, x] = z$ by Jacobson-Morozov. This implies $[z, [z, y]] = 0$ for all $y \in \mathfrak{g}$, so that z is a sandwich element, i.e., with $\operatorname{ad}(z)^2 = 0$. It is well-known that there are no nontrivial sandwich elements in simple Lie algebras. Hence $z = 0$ and $D = 0$. □

We note that for nilpotent Lie algebras we have a quite different behavior. There we always have a nontrivial solution for (1) and (2), different from the trivial derivation $D = 0$.

Proposition 3.6. *Let $\mathfrak{g}$ be nilpotent. Then there exists a nontrivial derivation $D \in \operatorname{Der}(\mathfrak{g})$ satisfying identities (1) and (2).*

Proof. If $\mathfrak{g}$ is nilpotent of class $c(\mathfrak{g}) \leq 2$, then obviously identity (2), and hence also (1), holds for every derivation. Consider the lower central series of $\mathfrak{g}$. Suppose that $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$, with $k \geq 3$, i.e., $c(\mathfrak{g}) \geq 3$. Choose an element w in $\mathfrak{g}^{k-2}$ which is not in the center of $\mathfrak{g}$. This is possible, because otherwise $\mathfrak{g}^{k-2} \subseteq Z(\mathfrak{g})$, and hence $\mathfrak{g}^{k-1} = [\mathfrak{g}, \mathfrak{g}^{k-2}] = 0$, a contradiction. It follows that $D = \mathrm{ad}(w)$ is a nontrivial derivation satisfying identity (2), because each term is zero. Indeed, $[[w,x],[y,z]] \in [\mathfrak{g}^{k-1}, \mathfrak{g}^1] \subseteq \mathfrak{g}^k = 0$. $\square$

4. Lie algebra identities

In this section we study Lie algebras $\mathfrak{g}$ satisfying one of the identities (1), (2), (3), (4), i.e.,

$$
\begin{aligned}
D\big([D(x),[y,z]] + [D(y),[z,x]] + [D(z),[x,y]]\big) &= 0,\\
[D(x),[y,z]] + [D(y),[z,x]] + [D(z),[x,y]] &= 0,\\
[z,[[z,x],[y,w]]] + [z,[[z,y],[w,x]]] + [z,[[z,w],[x,y]]] &= 0,\\
[z,[[z,x],[z,y]]] &= 0,
\end{aligned}
$$

for all $x,y,z,w \in \mathfrak{g}$ and not just for a given derivation, but *for all* derivations $D \in \mathrm{Der}(\mathfrak{g})$. This leads us to the theory of Lie algebra identities, which has a large literature. Clearly we have the implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4). Concerning question 2 , identity (1) holds if and only if $[x,y]_D = D([x,y])$ is a Lie bracket for all derivations $D \in \mathrm{Der}(\mathfrak{g})$.
We start with the identity $[[z,x],[z,y]] = 0$, which is a consequence of (2) by taking $D = \mathrm{ad}(z)$. A Lie algebra satisfying this identity is metabelian, i.e., satisfies $\mathfrak{g}^{(2)} = 0$. This is known but rarely mentioned. Therefore it seems useful to give a proof here. We always assume that the field K has characteristic zero, if not said otherwise.

Lemma 4.1. *Let $\mathfrak{g}$ be a Lie algebra over a field K of characteristic not 2. Then $\mathfrak{g}$ is metabelian if and only if it satisfies the identity*

$$[[z,x],[z,y]] = 0$$

for all $x,y,z \in \mathfrak{g}$.

Proof. Suppose that $\mathfrak{g}$ is metabelian, i.e., satisfies the identity $[[z,x],[w,y]] = 0$. Setting $w = z$ we obtain the required identity. Conversely, if we assume $[[z,x],[z,y]] = 0$ and formally replace z by $u+v$ we obtain

$$[[u,x],[v,y]] = [[u,y],[v,x]]$$

for all x,y,u,v. Now we use this identity and skew-symmetry twice to obtain

$$
\begin{aligned}
[[z,x],[w,y]] &= [[w,y],[x,z]]\\
&= [[w,z],[x,y]]\\
&= [[z,w],[y,x]]\\
&= [[z,x],[y,w]].
\end{aligned}
$$

This implies $2[[z,x],[w,y]] = 0$. $\square$

For Lie algebras satisfying the strongest identity, namely (2), we obtain the following necessary condition.

Proposition 4.2. *Let $\mathfrak{g}$ be a Lie algebra satisfying identity (2). Then $\mathfrak{g}$ is metabelian.*

Proof. Applying identity (2) for $D = \mathrm{ad}(w)$ we obtain

$$[[w,x],[y,z]] + [[w,y],[z,x]] + [[w,z],[x,y]] = 0$$

for all $x, y, z, w \in \mathfrak{g}$. Setting $w = z$ this implies

$$[[z, x], [z, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$. By Lemma 4.1, $\mathfrak{g}$ is metabelian. $\square$

The converse is not true in general, but of course true for all inner derivations.

Corollary 4.3. *A Lie algebra $\mathfrak{g}$ satisfies identity* (2) *for all inner derivations if and only if it is metabelian.*

The problem for the converse in general are the outer derivations of a metabelian Lie algebra. They need not satisfy identity (2). The following example demonstrates this, and is of minimal dimension with this property.

Example 4.4. *Let $\mathfrak{g}$ be the non-nilpotent metabelian Lie algebra of dimension* 4 *with basis $\{e_1, \ldots, e_4\}$, defined by the brackets*

$$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2,\ [e_1, e_4] = e_4,\ [e_2, e_3] = e_4.$$

Then the outer derivations $D = \operatorname{diag}(0, \lambda, \lambda, 2\lambda)$ satisfy (2) *if and only if $\lambda = 0$. They also satisfy* (1) *if and only if $\lambda = 0$.*

Indeed, we have

$$[D(e_1), [e_2, e_3]] + [D(e_2), [e_3, e_1]] + [D(e_3), [e_1, e_2]] = -2\lambda e_4.$$

There are also sufficient conditions for a Lie algebra $\mathfrak{g}$ to satisfy identity (2), such as $\mathfrak{g}^2 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. One would like to find more interesting conditions, of course. A view on low-dimensional Lie algebras already shows that it is not so easy.

Proposition 4.5. *All complex Lie algebras of dimension $n \leq 4$ satisfy identity* (2) *with the exception of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{gl}_2(\mathbb{C})$, $\mathfrak{g}_3$ and $\mathfrak{g}_5(\alpha)$, which are listed in table* 1.

Note that the algebra given in Example 4.4 is $\mathfrak{g}_5(0)$.

A finite-dimensional Lie algebra $\mathfrak{g}$ is called *almost abelian*, if it has an abelian ideal $\mathfrak{a}$ of codimension 1. We may choose a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ such that $\mathfrak{a} = \langle e_2, \ldots, e_n \rangle$ and $\mathfrak{g} \simeq \mathfrak{a} \rtimes \langle e_1 \rangle$.

Proposition 4.6. *Any almost abelian Lie algebra satisfies identity* (2).

Proof. Let $\mathfrak{g} = \mathfrak{a} \rtimes \langle e_1 \rangle$ be an almost abelian Lie algebra of dimension n. Since $\mathfrak{a}$ is an ideal of codimension 1 we know that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$. Consider the annihilator of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$,

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid [x, [\mathfrak{g}, \mathfrak{g}]] = 0\}.$$

Since $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal of $\mathfrak{g}$, i.e., $D([\mathfrak{g}, \mathfrak{g}]) \subseteq [\mathfrak{g}, \mathfrak{g}]$ for all derivations D of $\mathfrak{g}$, we have

$$\begin{aligned} [D(\mathfrak{h}), [\mathfrak{g}, \mathfrak{g}]] &= D([\mathfrak{h}, [\mathfrak{g}, \mathfrak{g}]]) + [\mathfrak{h}, D([\mathfrak{g}, \mathfrak{g}])] \\ &\subseteq 0 + [\mathfrak{h}, [\mathfrak{g}, \mathfrak{g}]]. \end{aligned}$$

This shows $D(\mathfrak{h}) \subseteq \mathfrak{h}$ for all $D \in \operatorname{Der}(\mathfrak{g})$. Taking $D = \operatorname{ad}(x)$ we see that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. So $\mathfrak{h}$ is a characteristic ideal of $\mathfrak{g}$ with $\mathfrak{a} \subseteq \mathfrak{h}$. We have $\mathfrak{h} = \mathfrak{g}$ if and only if $c(\mathfrak{g}) \leq 2$. However, for Lie algebras of nilpotency class at most 2 we are done. Otherwise we have $\mathfrak{h} = \mathfrak{a}$, i.e., $D(\mathfrak{a}) \subseteq \mathfrak{a}$ for all $D \in \operatorname{Der}(\mathfrak{g})$. Now it is easy to see that the identity

$$[D(e_i), [e_j, e_k]] + [D(e_j), [e_k, e_i]] + [D(e_k), [e_i, e_j]] = 0$$

holds for all i,j,k. If all $i,j,k \geq 2$, then all $[e_r,e_s] \in [\mathfrak{a},\mathfrak{a}] = 0$ for $r,s \in \{i,j,k\}$. Hence we may assume that $i=1$. If both $j,k \geq 2$, then

$$[D(e_1),[e_j,e_k]] = [D(e_j),[e_k,e_1]] = [D(e_k),[e_1,e_j]] = 0,$$

since $[e_j,e_k] = 0$ and $D(e_j), D(e_k) \in \mathfrak{a}$ because of $D(\mathfrak{a}) \subseteq \mathfrak{a}$. Hence we may assume that $i=j=1$. But then we obtain

$$[D(e_1),[e_1,e_k]] + [D(e_1),[e_k,e_1]] + [D(e_k),[e_1,e_1]] = 0,$$

so that identity (2) is satisfied.

□

An example for an almost abelian Lie algebra is the standard graded filiform Lie algebra $\mathfrak{f}_n$ of dimension $n \geq 3$, with basis $\{e_1,\ldots,e_n\}$ and Lie brackets $[e_1,e_i] = e_{i+1}$ for $i = 2,\ldots,n-1$.

Corollary 4.7. *The filiform nilpotent Lie algebra $\mathfrak{f}_n$ satisfies identity* (2) *for every $n \geq 3$.*

Remark 4.8. The above result shows that the nilpotency class of a Lie algebra satisfying identity (2) can be arbitrarily large, whereas the solvability class is bounded by 2.

It is easy to verify the following result for low-dimensional nilpotent Lie algebras. The notation is taken from Magnin [16].

Proposition 4.9. *Every complex nilpotent Lie algebra of dimension $n \leq 5$ satisfies identity* (2)*. In dimension* 6 *all nilpotent algebras satisfy* (2) *with the exception of $\mathfrak{g}_{6,9}$, $\mathfrak{g}_{6,13}$, $\mathfrak{g}_{6,15}$, $\mathfrak{g}_{6,18}$, $\mathfrak{g}_{6,19}$, and $\mathfrak{g}_{6,20}$.*

One can obtain a similar result for dimension 7 by using the classification list of Magnin - see table 2. We have shortened Magnin's notation there by omitting the dimension index 7. There is one interesting infinite family of Lie algebras, depending on a complex parameter λ, where identity (2) holds precisely for one singular value of λ. The family is $\mathfrak{g}_{7,1.2(i_\lambda)}$ in Magnin's notation:

Example 4.10. *For $\lambda \in \mathbb{C}$ let $\mathfrak{g}_\lambda$ denote the following complex* 7*-dimensional nilpotent Lie algebra given by the Lie brackets*

$$\begin{aligned}
&[x_1,x_2] = x_4,\ [x_1,x_3] = x_6,\ [x_1,x_4] = x_5,\ [x_1,x_5] = x_7,\\
&[x_2,x_3] = \lambda x_5,\ [x_2,x_4] = x_6,\ [x_2,x_6] = x_7,\ [x_3,x_4] = (1-\lambda)x_7.
\end{aligned}$$

Then (1) *holds* $\Leftrightarrow$ (2) *holds* $\Leftrightarrow \lambda = 1$.

Note that we have $c(\mathfrak{g}_\lambda) = 4$, $d(\mathfrak{g}_\lambda) = 2$ for all $\lambda \in \mathbb{C}$, and

$$\dim \operatorname{Der}(\mathfrak{g}_\lambda) = \begin{cases} 13, & \text{for } \lambda = -1 \\ 12, & \text{for } \lambda \neq -1 \end{cases}.$$

One might ask for invariants which differ exactly for $\lambda = 1$ and $\lambda \neq 1$. For the $(t,1,1)$-space of generalized derivations with $t \neq 0,1,-1,2$ we have

$$\dim \operatorname{Der}_{(t,1,1)}(\mathfrak{g}_\lambda) = \begin{cases} 12, & \text{for } \lambda = 1 \\ 11, & \text{for } \lambda \neq 1 \end{cases},$$

see [10], [21] for more on generalized derivations. However, there seems to be no relation in general between these spaces and identity (1) or (2).

Identity (1) is weaker than identity (2) in general. Indeed, a Lie algebra satisfying identity (1) need not be metabelian as we have already seen in the cases of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{gl}_2(\mathbb{C})$, see table 1. However, for nilpotent Lie algebras of dimension $n \leq 6$ they are equivalent, and also for nilpotent Lie algebras of dimension 7 which are not CNLAs. This follows from an easy but lengthy computation, see table 2.

Proposition 4.11. *A complex nilpotent Lie algebra of dimension $n \leq 6$ satisfies identity* (1) *if and only it satisfies identity* (2)*. In dimension* 7 *this holds true for all complex nilpotent Lie algebras which are not a CNLA.*

For CNLAs of dimension 7 identity (1) always holds, but identity (2) does not.

Proposition 4.12. *Every complex CNLA of dimension* 7 *satisfies identity* (1)*.*

In general, the last result is not true in higher dimension.

Example 4.13. *Let $\mathfrak{g}$ be the filiform nilpotent Lie algebra of dimension* 8 *defined by the brackets*

$$\begin{aligned}
&[x_1, x_i] = x_{i+1},\ i = 2, \ldots, 7,\\
&[x_2, x_3] = x_5 + x_6,\ [x_2, x_4] = x_6 + x_7,\ [x_2, x_5] = 2x_7 + x_8,\\
&[x_2, x_6] = 3x_8,\ [x_3, x_4] = -x_7,\ [x_3, x_5] = -x_8.
\end{aligned}$$

Then $\mathfrak{g}$ is a CNLA which does not satisfy identity (1)*.*

In fact, $\mathfrak{g}$ does not even satisfy identity (4), since we have

$$[x_1, [[x_1, x_2], [x_1, x_3]]] = -x_8.$$

We have $c(\mathfrak{g}) = 7$ and $d(\mathfrak{g}) = 3$.

Let us finally discuss the identities (3) and (4). They have been studied by many authors in connection with identities in $\mathfrak{sl}_2(K)$. Identity (4) appears in the basis for identities of $\mathfrak{sl}_2(K)$ found by Razmyslov [18]:

Theorem 4.14. *Let K be a field of characteristic zero. A finite basis of identities for the Lie algebra $\mathfrak{sl}_2(K)$ is given by identity* (4) *and the standard identity of degree* 5*,*

$$\sum_{\pi \in S_4} (-1)^{\pi} [x_{\pi(1)}, [x_{\pi(2)}, [x_{\pi(3)}, [x_{\pi(4)}, x_0]]]] = 0$$

for all $x_i \in \mathfrak{sl}_2(K)$.

This theorem was generalized by Fillipov [13] to arbitrary fields K of characteristic not 2. Moreover he showed that all such identities for $\mathfrak{sl}_2(K)$ are a consequence of one single identity, namely

$$[z, [[w, x], [w, y]]] = [w, [[z, w], [x, y]]]. \tag{6}$$

This is related to our identity (3) as follows.

Proposition 4.15. *Identity* (6) *is a consequence of identity* (3)*.*

Proof. Formally replacing z by $z+v$ in (3) gives

$$\begin{aligned}0 &= [z+v,[[z+v,x],[y,w]]]+[z+v,[[z+v,y],[w,x]]]+[z+v,[[z+v,w],[x,y]]]\\ &= [z,[[z,x],[y,w]]]+[z,[[v,x],[y,w]]]+[v,[[z,x],[y,w]]]+[v,[[v,x],[y,w]]]\\ &+ [z,[[z,y],[w,x]]]+[z,[[v,y],[w,x]]]+[v,[[z,y],[w,x]]]+[v,[[v,y],[w,x]]]\\ &+ [z,[[z,w],[x,y]]]+[z,[[v,w],[x,y]]]+[v,[[z,w],[x,y]]]+[v,[[v,w],[x,y]]].\end{aligned}$$

Applying (3) for the first and last column of terms we obtain

$$\begin{aligned}0 &= [z,[[v,x],[y,w]]]+[z,[[v,y],[w,x]]]+[z,[[v,w],[x,y]]]\\ &+ [v,[[z,x],[y,w]]]+[v,[[z,y],[w,x]]]+[v,[[z,w],[x,y]]].\end{aligned}$$

Setting $v=w$ and applying (3) with z and w interchanged, i.e., in the form

$$[w,[[w,x],[y,z]]]+[w,[[w,y],[z,x]]=[w,[z,w],[x,y]]],$$

we obtain

$$\begin{aligned}0 &= 2[z,[[w,x],[y,w]]]+[w,[[z,x],[y,w]]]\\ &+ [w,[[z,y],[w,x]]]+[w,[[z,w],[x,y]]\\ &= 2[z,[[w,x],[y,w]]]+2[w,[[z,w],[x,y]]].\end{aligned}$$

This is identity (6). ☐

Identity (4) has been studied further, but mostly for simple and semisimple algebras. Filippov [14] termed algebras satisfying identity (4) also h_0-*algebras*, and algebras satisfying identity (6) also h-*algebras*. Another term for the variety of h-algebras is given by $\mathrm{var}(\mathfrak{sl}_2(K))$. This variety and its subvarieties have also been studied by several authors, see [12] and the references therein. A study of identities (3) and (4) for solvable and nilpotent Lie algebras seems to be less known. Table 1 shows the result for complex Lie algebras of dimension $n \le 4$.

Proposition 4.16. *All complex Lie algebras of dimension $n \le 4$ satisfy identity* (3)*, and hence also* (4)*, except for $\mathfrak{g}_3$ and $\mathfrak{g}_5(\alpha)$ with $\alpha \neq 0,-1$.*

There is a trivial reason in low dimensions, why these identities are often satisfied. Every center-by-metabelian Lie algebra $\mathfrak{g}$ satisfies identity (3) and (4). By definition, center-by-metabelian means that $\mathfrak{g}^{(2)} \subseteq Z(\mathfrak{g})$. This immediately implies that every term in (3) is zero. Indeed, all low-dimensional nilpotent Lie algebras are center-by-metabelian. More precisely, we have the following result:

Proposition 4.17. *Every nilpotent Lie algebra $\mathfrak{g}$ of dimension $n \le 7$ over a field of characteristic zero is center-by-metabelian, and hence satisfies identity* (3) *and* (4)*.*

Proof. The claim follows from results in [3]. We have $\mathfrak{g}^{(3)}=0$ because of $n \le 7$. Suppose that $\mathfrak{g}^{(2)} \neq 0$. Then $\dim \mathfrak{g}/\dim \mathfrak{g}^{(1)} \ge 2$, $\dim \mathfrak{g}^{(1)}/\dim \mathfrak{g}^{(2)} \ge 3$, and $n \ge 6$. Moreover, if $\dim \mathfrak{g}^{(1)}/\dim \mathfrak{g}^{(2)} = 3$, then $\dim \mathfrak{g}^{(2)} \le 1$, see [3]. Otherwise we have $\dim \mathfrak{g}^{(1)}/\dim \mathfrak{g}^{(2)} \ge 4$ and

$$\begin{aligned}n &= \dim \mathfrak{g}/\dim \mathfrak{g}^{(1)} + \dim \mathfrak{g}^{(1)}/\dim \mathfrak{g}^{(2)} + \dim \mathfrak{g}^{(2)}\\ &\ge 6 + \dim \mathfrak{g}^{(2)}\end{aligned}$$

This gives again $\dim \mathfrak{g}^{(2)} \le 1$. Because $\mathfrak{g}$ is nilpotent, $\mathfrak{g}^{(2)} \cap Z(\mathfrak{g}) \neq 0$, and hence $\mathfrak{g}^{(2)} \subseteq Z(\mathfrak{g})$. ☐

The result does not hold in higher dimensions. Indeed, the 8-dimensional nilpotent Lie algebra of Example 4.13 is not center-by-metabelian. It does not satisfy identity (3) or (4). Recall that a Lie algebra $\mathfrak{g}$ satisfying identity (3), or (4) need not be center-by-metabelian, e.g., consider $\mathfrak{sl}_2(K)$.

5. Tables

1.) Complex Lie algebras of dimension $n \leq 4$:

$\mathfrak{g}$	Lie brackets	(1)	(2)	(3)	(4)
$\mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_2$	✓	✓	✓	✓
$\mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$	✓	✓	✓	✓
$\mathfrak{r}_{3,\lambda}(\mathbb{C})$	$[e_1, e_2] = e_2,\ [e_1, e_3] = \lambda e_3$	✓	✓	✓	✓
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3,\ [e_1, e_3] = -2e_1,\ [e_2, e_3] = 2e_2$	✓	–	✓	✓
$\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3$	✓	✓	✓	✓
$\mathfrak{n}_4(\mathbb{C})$	$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4$	✓	✓	✓	✓
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$	$[e_1, e_2] = e_2$	✓	✓	✓	✓
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(C)$	$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4$	✓	✓	✓	✓
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3,\ [e_1, e_3] = -2e_1,\ [e_2, e_3] = 2e_2$	✓	–	✓	✓
$\mathfrak{g}_1$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = e_4$	✓	✓	✓	✓
$\mathfrak{g}_2(\alpha)$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = e_3 + \alpha e_4$	✓	✓	✓	✓
$\mathfrak{g}_3$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = 2e_4,\ [e_2, e_3] = e_4$	–	–	–	–
$\mathfrak{g}_4(\alpha, \beta)$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + \alpha e_3,\ [e_1, e_4] = e_3 + \beta e_4$	✓	✓	✓	✓
$\mathfrak{g}_5(\alpha),\ \alpha \neq 0, -1$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + \alpha e_3,$	–	–	–	–
	$[e_1, e_4] = (\alpha + 1)e_4,\ [e_2, e_3] = e_4$				
$\alpha = 0, -1$		–	–	✓	✓

2.) Indecomposable complex nilpotent Lie algebras of dimension 7, see [16]:

Identity	$\mathfrak{g}_{0.1}$	$\mathfrak{g}_{0.2}$	$\mathfrak{g}_{0.3}$	$\mathfrak{g}_{0.4(\lambda)}$	$\mathfrak{g}_{0.5}$	$\mathfrak{g}_{0.6}$	$\mathfrak{g}_{0.7}$	$\mathfrak{g}_{0.8}$	$\mathfrak{g}_{1.01(i)}$	$\mathfrak{g}_{1.01(ii)}$	$\mathfrak{g}_{1.02}$	$\mathfrak{g}_{1.03}$
(1)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	–	–
(2)	–	✓	✓	–	–	–	–	–	✓	✓	–	–

Identity	$\mathfrak{g}_{1.1(i_\lambda)}$	$\mathfrak{g}_{1.1(ii)}$	$\mathfrak{g}_{1.1(iii)}$	$\mathfrak{g}_{1.1(iv)}$	$\mathfrak{g}_{1.1(v)}$	$\mathfrak{g}_{1.1(vi)}$	$\mathfrak{g}_{1.2(i_{\lambda\neq 1})}$	$\mathfrak{g}_{1.2(i_{\lambda=1})}$	$\mathfrak{g}_{1.2(ii)}$
(1)	–	–	–	–	–	–	–	✓	–
(2)	–	–	–	–	–	–	–	✓	–

Identity	$\mathfrak{g}_{1.2(iii)}$	$\mathfrak{g}_{1.2(iv)}$	$\mathfrak{g}_{1.3(i_\lambda)}$	$\mathfrak{g}_{1.3(ii)}$	$\mathfrak{g}_{1.3(iii)}$	$\mathfrak{g}_{1.3(iv)}$	$\mathfrak{g}_{1.3(v)}$	$\mathfrak{g}_{1.4}$	$\mathfrak{g}_{1.5}$	$\mathfrak{g}_{1.6}$	$\mathfrak{g}_{1.7}$
(1)	–	–	–	–	✓	✓	–	–	–	✓	✓
(2)	–	–	–	–	✓	✓	–	–	–	✓	✓

Identity	$\mathfrak{g}_{1.8}$	$\mathfrak{g}_{1.9}$	$\mathfrak{g}_{1.10}$	$\mathfrak{g}_{1.11}$	$\mathfrak{g}_{1.12}$	$\mathfrak{g}_{1.13}$	$\mathfrak{g}_{1.14}$	$\mathfrak{g}_{1.15}$	$\mathfrak{g}_{1.16}$	$\mathfrak{g}_{1.17}$	$\mathfrak{g}_{1.18}$	$\mathfrak{g}_{1.19}$
(1)	✓	✓	✓	–	–	–	–	✓	✓	–	✓	✓
(2)	✓	✓	✓	–	–	–	–	✓	✓	–	✓	✓

Identity	$\mathfrak{g}_{1.20}$	$\mathfrak{g}_{1.21}$	$\mathfrak{g}_{2.1(i_\lambda)}$	$\mathfrak{g}_{2.1(ii)}$	$\mathfrak{g}_{2.1(iii)}$	$\mathfrak{g}_{2.1(iv)}$	$\mathfrak{g}_{2.1(v)}$	$\mathfrak{g}_{2.2}$	$\mathfrak{g}_{2.3}$	$\mathfrak{g}_{2.4}$	$\mathfrak{g}_{2.5}$	$\mathfrak{g}_{2.6}$
(1)	–	–	–	–	–	–	–	–	✓	–	–	–
(2)	–	–	–	–	–	–	–	–	✓	–	–	–

Identity	$\mathfrak{g}_{2.7}$	$\mathfrak{g}_{2.8}$	$\mathfrak{g}_{2.9}$	$\mathfrak{g}_{2.10}$	$\mathfrak{g}_{2.11}$	$\mathfrak{g}_{2.12}$	$\mathfrak{g}_{2.13}$	$\mathfrak{g}_{2.14}$	$\mathfrak{g}_{2.15}$	$\mathfrak{g}_{2.16}$	$\mathfrak{g}_{2.17}$	$\mathfrak{g}_{2.18}$
(1)	✓	–	✓	–	✓	✓	–	–	–	✓	–	✓
(2)	✓	–	✓	–	✓	✓	–	–	–	✓	–	✓

Identity	$\mathfrak{g}_{2.19}$	$\mathfrak{g}_{2.20}$	$\mathfrak{g}_{2.21}$	$\mathfrak{g}_{2.22}$	$\mathfrak{g}_{2.23}$	$\mathfrak{g}_{2.24}$	$\mathfrak{g}_{2.25}$	$\mathfrak{g}_{2.26}$	$\mathfrak{g}_{2.27}$	$\mathfrak{g}_{2.28}$	$\mathfrak{g}_{2.29}$	$\mathfrak{g}_{2.30}$
(1)	✓	✓	✓	✓	✓	–	–	–	✓	✓	✓	✓
(2)	✓	✓	✓	✓	✓	–	–	–	✓	✓	✓	✓

Identity	$\mathfrak{g}_{2.31}$	$\mathfrak{g}_{2.32}$	$\mathfrak{g}_{2.33}$	$\mathfrak{g}_{2.34}$	$\mathfrak{g}_{2.35}$	$\mathfrak{g}_{2.36}$	$\mathfrak{g}_{2.37}$	$\mathfrak{g}_{2.38}$	$\mathfrak{g}_{2.39}$	$\mathfrak{g}_{2.40}$	$\mathfrak{g}_{2.41}$	$\mathfrak{g}_{2.42}$
(1)	✓	✓	✓	✓	–	✓	–	✓	✓	✓	–	✓
(2)	✓	✓	✓	✓	–	✓	–	✓	✓	✓	–	✓

Identity	$\mathfrak{g}_{2.43}$	$\mathfrak{g}_{2.44}$	$\mathfrak{g}_{2.45}$	$\mathfrak{g}_{3.1(i_\lambda)}$	$\mathfrak{g}_{3.1(iii)}$	$\mathfrak{g}_{3.2}$	$\mathfrak{g}_{3.3}$	$\mathfrak{g}_{3.4}$	$\mathfrak{g}_{3.5}$	$\mathfrak{g}_{3.6}$	$\mathfrak{g}_{3.7}$	$\mathfrak{g}_{3.8}$
(1)	✓	✓	✓	–	–	✓	✓	✓	–	✓	✓	✓
(2)	✓	✓	✓	–	–	✓	✓	✓	–	✓	✓	✓

Identity	$\mathfrak{g}_{3.9}$	$\mathfrak{g}_{3.10}$	$\mathfrak{g}_{3.11}$	$\mathfrak{g}_{3.12}$	$\mathfrak{g}_{3.13}$	$\mathfrak{g}_{3.14}$	$\mathfrak{g}_{3.15}$	$\mathfrak{g}_{3.16}$	$\mathfrak{g}_{3.17}$	$\mathfrak{g}_{3.18}$	$\mathfrak{g}_{3.19}$	$\mathfrak{g}_{3.20}$
(1)	✓	–	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(2)	✓	–	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Identity	$\mathfrak{g}_{3.21}$	$\mathfrak{g}_{3.22}$	$\mathfrak{g}_{3.23}$	$\mathfrak{g}_{3.24}$	$\mathfrak{g}_{4.1}$	$\mathfrak{g}_{4.2}$	$\mathfrak{g}_{4.3}$	$\mathfrak{g}_{4.4}$
(1)	✓	–	✓	✓	✓	✓	✓	✓
(2)	✓	–	✓	✓	✓	✓	✓	✓

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