CLASSIFICATION OF NOVIKOV ALGEBRAS

DIETRICH BURDE AND WILLEM DE GRAAF

Abstract. We describe a method for classifying the Novikov algebras with a given associated Lie algebra. Subsequently we give the classification of the Novikov algebras of dimension 3 over $\mathbb{R}$ and $\mathbb{C}$, as well as the classification of the 4-dimensional Novikov algebras over $\mathbb{C}$ whose associated Lie algebra is nilpotent. In particular this includes a list of all 4-dimensional commutative associative algebras over $\mathbb{C}$.

1. Introduction

Novikov algebras arise in many areas of mathematics and physics. They form a special class of pre-Lie algebras which arise among other things in the study of rooted trees (Cayley), convex homogeneous cones (Vinberg), affinely flat manifolds and their fundamental groups (Auslander, Milnor) and renormalization theory (Connes, Kreimer, Kontsevich). Novikov algebras in particular were considered in the study of Hamiltonian operators, Poisson brackets of hydrodynamic type (Balinskii, Novikov), operator Yang-Baxter equations and left-invariant affine structures on Lie groups [5]. We refer to [4], and the references therein, for more information on these topics.

The theory of Novikov algebras and its classification has been started by Zelmanov [15]. Further structure theory has been developed in [6]. There have been several efforts to classify complex Novikov algebras in low dimensions: Bai and Meng classify the Novikov algebras over $\mathbb{C}$ of dimension up to 3 in [1], and complete Novikov algebras over $\mathbb{C}$ that have a nilpotent associated Lie algebra in [2]. We recall that a Novikov algebra is said to be complete, or transitive, if the right multiplication by any element is a nilpotent linear map. Also we note that Novikov algebras are Lie admissible algebras, i.e., the commutator defines the structure of a Lie algebra on them. In this paper we outline a systematic method to classify the Novikov algebras with a given associated Lie algebra. Using it we obtain classifications that extend the ones by Bai and Meng in several ways. Firstly, we get the classification of the 3-dimensional Novikov algebras over $\mathbb{R}$ and $\mathbb{C}$ (Section 4). Secondly (Section 5) we have classified all 4-dimensional Novikov algebras over $\mathbb{C}$ that have a nilpotent associated Lie algebra (not just the complete ones). Furthermore, using our method we obtain a list of Novikov algebras that is ordered with respect to the associated Lie algebra, i.e., all Novikov algebras with the same Lie algebra are grouped together.

If the associated Lie algebra of a Novikov algebra is abelian, then the Novikov algebra is nothing other than a commutative and associative algebra. This is an interesting class of algebras in its own. It also appears as a subclass of other classes of algebras being of interest in geometry; as an example we name LR-algebras [7]. Hence an explicit classification of associative, commutative algebras in low dimension is also very desirable. Over the past century

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many classifications of commutative associative algebras have appeared in the literature. Here we mention [3], [12], [13], [14]. The last three of these references have classifications of nilpotent associative algebras. In Section 3 we describe an elementary way to obtain the classification of all commutative associative algebras from the classification of the nilpotent commutative associative algebras. We apply this method to obtain a list of commutative associative algebras of dimensions 3 (over $\mathbb{R}$ and $\mathbb{C}$) and 4 (over $\mathbb{C}$; the latter is contained in Section 5). The list of commutative associative algebras of dimension 3 over $\mathbb{C}$ is also contained in [3]. For completeness we also include it in this paper. Although the classification of the commutative associative algebras can be deduced from the classification of the nilpotent commutative associative algebras, no such classification in dimension 4 has explicitly been done, to the best of our knowledge.

We start, in the next section, by describing the method that we use for classifying Novikov algebras. In order to compute isomorphisms between these algebras, we rely on computer calculations, using the technique of Gröbner bases. Using this technique we have also established the correspondence between our list of 3-dimensional Novikov algebras over $\mathbb{C}$ and the list in [1]; therefore, the two classifications are equivalent.

Finally we remark that it turned out that the problem of classifying all 4-dimensional Novikov algebras is far too complex to be undertaken. Therefore, we have restricted ourselves to the Novikov algebras with a nilpotent associated Lie algebra. We remark that many of the Novikov algebras that we get are not complete (for example $N_{17}^{\mathfrak{h}_1}$ in Table 10). Therefore our classification substantially extends the one in [2].

2. Classifying Novikov algebras with given Lie algebra

Let F be a field. An algebra A over F is called Novikov if

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \tag{1}$$

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y, \tag{2}$$

for all $x, y, z \in A$. Let A be a Novikov algebra, and define a bracket on A by

$$[x, y] = x \cdot y - y \cdot x.$$

Then it is straightforward to see that $(A, [\ ,\ ])$ is a Lie algebra. The problem considered here is to find, up-to isomorphism, all Novikov algebras that have a given Lie algebra as associated Lie algebra.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Let $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a bilinear map. Then we define an algebra $A(\mathfrak{g}, \theta)$ as follows. The underlying vector space of $A(\mathfrak{g}, \theta)$ is $\mathfrak{g}$. Furthermore, for $x, y \in \mathfrak{g}$ we set $x \cdot y = \theta(x, y)$. We define $T(\mathfrak{g})$ to be the set of all bilinear θ such that $[x, y] = \theta(x, y) - \theta(y, x)$ for all $x, y \in \mathfrak{g}$ and such that $A(\mathfrak{g}, \theta)$ is a Novikov algebra.

The automorphism group, $\mathrm{Aut}(\mathfrak{g})$, of $\mathfrak{g}$ acts on $T(\mathfrak{g})$ by $\phi\theta(x, y) = \phi^{-1}(\theta(\phi(x), \phi(y)))$, for $\phi \in \mathrm{Aut}(\mathfrak{g})$, and $\theta \in T(\mathfrak{g})$.

Lemma 2.1. *Let $\theta_1, \theta_2 \in T(\mathfrak{g})$. Then $A(\mathfrak{g}, \theta_1)$ and $A(\mathfrak{g}, \theta_2)$ are isomorphic if and only if there is a $\phi \in \mathrm{Aut}(\mathfrak{g})$ with $\phi\theta_1 = \theta_2$.*

Proof. If $\phi\theta_1 = \theta_2$, then $\phi(\theta_2(x, y)) = \theta_1(\phi(x), \phi(y))$. Hence $\phi : A(\mathfrak{g}, \theta_2) \to A(\mathfrak{g}, \theta_1)$ is an isomorphism.

On the other hand, if $\phi : A(\mathfrak{g}, \theta_2) \to A(\mathfrak{g}, \theta_1)$ is an isomorphism of Novikov algebras, then $\phi\theta_2(x, y) = \theta_1(\phi(x), \phi(y))$ for all $x, y \in \mathfrak{g}$. Hence $\phi\theta_1 = \theta_2$. This also implies that ϕ is an automorphism of $\mathfrak{g}$. □

Set $n = \dim \mathfrak{g}$, and let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$. Then an element in $T(\mathfrak{g})$ is given by a sequence of $n \times n$-matrices $L_1, \ldots, L_n$, where L_i describes the left multiplication of e_i. More precisely, if $\theta \in T(\mathfrak{g})$ satisfies $\theta(e_i, e_j) = \sum_{k=1}^{n} c_{ij}^k e_k$, then for the L_i we have $L_i(k, j) = c_{ij}^k$.

This way we view $T(\mathfrak{g})$ is an affine variety in F^{n^3}. We get equations for this variety by plugging $x = e_i$, $y = e_j$, $z = e_k$ in (1) and (2), for $1 \leq i, j, k \leq n$, and by requiring that $e_i \cdot e_j - e_j \cdot e_i = [e_i, e_j]$ for $i < j$. The equations corresponding to this last requirement will be linear. The equations corresponding to (1) and (2) will be polynomial; however we can reformulate (2) such that it leads to linear equations as well. This works as follows. Let A be an algebra. For $x \in A$ let $L(x) : A \to A$ be the linear map given by $L(x)(y) = x \cdot y$. Similarly we define $R(x)(y) = y \cdot x$. The adjoint map is defined by $\mathrm{ad}(x) = L(x) - R(x)$. Now assume that A satisfies (1). Then (2) holds as well if and only if $[R(x), R(y)] = 0$ for all $x, y \in A$. Furthermore, (1) implies that $[L(x), L(y)] = L([x, y])$. Now, using $R(x) = L(x) - \mathrm{ad}(x)$ we get that $[R(x), R(y)] = 0$ is equivalent to

$$L([x, y]) + \mathrm{ad}([x, y]) - [L(x), \mathrm{ad}(y)] - [\mathrm{ad}(x), L(y)] = 0. \tag{3}$$

If in this equation we set $x = e_i$, $y = e_j$, and noting that the maps $\mathrm{ad}e_k$ are given by the Lie multiplication on $\mathfrak{g}$, we get that (3) is equivalent to a set of linear equations.

Example 2.2. *Consider the 2-dimensional Lie algebra $\mathfrak{g}$ over $F = \mathbb{C}$ with basis e_1, e_2 and Lie bracket $[e_1, e_2] = e_1$. Write $L_i = L(e_i)$ and*

$$L_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; L_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then (3) with $x = e_1$, $y = e_2$ is equivalent to the equations $a_{21} = 0$, $a_{11} = b_{21}$, $a_{22} = -b_{21}$, $b_{22} = b_{11} + 1$. Furthermore, $L(e_1)(e_2) - L(e_2)(e_1) = [e_1, e_2]$ is tantamount to $a_{22} = b_{21}$, $a_{12} = b_{11} + 1$. So the linear equations that we get imply that

$$L_1 = \begin{pmatrix} 0 & b_{22} \\ 0 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} b_{22} - 1 & b_{12} \\ 0 & b_{22} \end{pmatrix}. \tag{4}$$

These matrices already define a Novikov structure, i.e., (1) is automatically satisfied.

By Lemma 2.1, classifying the Novikov algebras with associated Lie algebra equal to $\mathfrak{g}$ is the same as listing the $\mathrm{Aut}(\mathfrak{g})$-orbits on $T(\mathfrak{g})$. In order to carry this out we write an element of $\mathrm{Aut}(\mathfrak{g})$ as an $n \times n$-matrix, with indeterminates as entries that satisfy some polynomial equations. Then we try and work out what the orbits are. We illustrate this by example.

Example 2.3. *We consider the situation of Example 2.2. An element of* $\mathrm{Aut}(\mathfrak{g})$ *is given by*

$$\phi = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

where $a \neq 0$. (Here we use the column convention, so $\phi(e_1) = ae_1$, $\phi(e_2) = be_1 + e_2$.) Let a $\theta \in T(\mathfrak{g})$ be given by the matrices (4). Then a short calculation shows that $\phi\theta$ corresponds to two matrices of the same shape, where b_{22} is unchanged, but where b_{12} is changed into

$$a^{-1}(b(b_{22} - 1) + b_{12}).$$

We now distinguish two cases. In the first case $b_{22} \neq 1$. Then we can choose b so that b_{12} is mapped to 0. So we get a 1-parameter family of Novikov algebras given by

$$e_1 \cdot e_2 = b_{22}e_1, \; e_2 \cdot e_1 = (b_{22} - 1)e_1, \; e_2 \cdot e_2 = b_{22}e_2.$$

Algebras corresponding to different values of the parameter b_{22} are not isomorphic.

In the second case we have $b_{22} = 1$. Here we have two subcases. If $b_{12} = 0$ then we get an algebra which is included in the above parametric family. On the other hand, if $b_{12} \neq 0$ then we can choose $a = b_{12}$ and we see that it is changed to 1. So we get one more algebra, given by

$$e_1 \cdot e_2 = e_1, \ e_2 \cdot e_2 = e_1 + e_2.$$

Summarising, our procedure to classify the Novikov algebras with given associated Lie algebra $\mathfrak{g}$ consists of two steps:

(1) Find the equations for $T(\mathfrak{g})$.
(2) List the orbits of $\mathrm{Aut}(\mathfrak{g})$ on $T(\mathfrak{g})$.

The last step is by far the most difficult one. And we are not always able to carry it out in full; in other words, sometimes we obtain two Novikov algebras that are isomorphic without us being able to show that the corresponding bilinear maps in $T(\mathfrak{g})$ lie in the same $\mathrm{Aut}(\mathfrak{g})$-orbit. To deal with a situation of this kind we use a method based on the algorithmic technique of Gröbner bases (cf. [9]). For this again we write an element of $\mathrm{Aut}(\mathfrak{g})$ as a matrix with entries that are indeterminates, satisfying some polynomial equations. Let R denote the ring containing these indeterminates. Let $\theta_1, \theta_2 \in T(\mathfrak{g})$ be given. Then we compute the matrices L_i corresponding to $\phi\theta_1$; they have entries that are polynomials in the coefficients of ϕ (which are indeterminates). Then the requirement that $\phi\theta_1 = \theta_2$ leads to a set of polynomial equations $p = 0$ where $p \in P \subset R$. Let I be the ideal of R generated by P. Now, solving $p = 0$ for $p \in P$ is the same as solving $g = 0$ for any generating set G of I. For this a Gröbner basis is particularly convenient. In particular if the Gröbner basis is computed relative to a lexicographical order, then the resulting polynomial equations have a triangular structure, which makes them easier to solve. A second feature of this method is that if the algebras corresponding to the θ_i happen *not* to be isomorphic, then the reduced Gröbner basis is $\{1\}$. So in this case no hand calculations are necessary. For the computation of the Gröbner bases we used the computer algebra system Magma ([8]). We illustrate this with an example.

Example 2.4. *Let $\mathfrak{g}$ be the 3-dimensional Lie algebra with basis e_1, e_2, e_3 and nonzero bracket $[e_1, e_2] = e_3$. Consider a family of Novikov algebras N_1^α given by $e_1 \cdot e_2 = (\alpha+1)e_3$, $e_2 \cdot e_1 = \alpha e_3$. Denote the corresponding element of $T(\mathfrak{g})$ by θ_α. The elements of $\mathrm{Aut}(\mathfrak{g})$ are given by*

$$\phi = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & \delta \end{pmatrix},$$

with $\delta = x_{11}x_{22} - x_{12}x_{21}$. The Novikov algebra corresponding to $\phi\theta_\alpha$ has nonzero products

$$\begin{aligned} e_1 \cdot e_1 &= (2Dx_{11}x_{21}\alpha + Dx_{11}x_{21})e_3 \\ e_1 \cdot e_2 &= (2Dx_{12}x_{21}\alpha + Dx_{12}x_{21} + \alpha + 1)e_3 \\ e_2 \cdot e_1 &= (2Dx_{12}x_{21}\alpha + Dx_{12}x_{21} + \alpha)e_3 \\ e_2 \cdot e_2 &= (2Dx_{12}x_{22}\alpha + Dx_{12}x_{22})e_3, \end{aligned}$$

where $D = \delta^{-1}$.

Now let N_2^β be the family of Novikov algebras given by $e_1 \cdot e_1 = \beta e_3$, $e_1 \cdot e_2 = e_3$, $e_2 \cdot e_2 = e_3$. Then $N_1^\alpha \cong N_2^\beta$ if and only if the following polynomial equations are satisfied

$$\begin{aligned} 2Dx_{11}x_{21}\alpha + Dx_{11}x_{21} - \beta &= 0 \\ 2Dx_{12}x_{21}\alpha + Dx_{12}x_{21} + \alpha &= 0 \\ 2Dx_{12}x_{22}\alpha + Dx_{12}x_{22} - 1 &= 0. \end{aligned}$$

We let I be the ideal of the polynomial ring $\mathbb{C}[D, x_{11}, x_{12}, x_{21}, x_{22}, \alpha, \beta]$ generated by the left hand sides of these equations along with the polynomial $D(x_{11}x_{22} - x_{12}x_{21}) - 1$. A reduced Gröbner basis of I with respect to the lexicographical order with $D > x_{11} > x_{12} > \cdots > \alpha > \beta$ consists of the polynomials

$$\begin{aligned}
&Dx_{12}x_{22}\alpha + \tfrac{1}{2}Dx_{12}x_{22} - \tfrac{1}{2},\\
&Dx_{12}x_{22}\beta - \tfrac{1}{4}Dx_{12}x_{22} + \tfrac{1}{2}\alpha + \tfrac{1}{4},\\
&x_{11} - x_{12}\alpha - x_{12},\\
&x_{21} + x_{22}\alpha,\\
&\alpha^2 + \alpha + \beta.
\end{aligned}$$

It follows that $N_1^\alpha \cong N_2^\beta$ if and only if these polynomial equations have a solution. So from the last polynomial we see that $N_1^\alpha \cong N_2^\beta$ implies that $\beta = -\alpha^2 - \alpha$. Conversely, suppose that this is satisfied. From the third and fourth elements of the Gröbner basis we get that an isomorphism ϕ has to be of the form

$$\begin{pmatrix} (\alpha+1)u & u & 0\\ -\alpha v & v & 0\\ x_{31} & x_{32} & \delta \end{pmatrix}.$$

The determinant of the 2×2-block in the upper left corner is $(2\alpha+1)uv$. Now if $\alpha \neq -\frac{1}{2}$ then we choose $u = v = 1$, and $x_{31} = x_{32} = 0$ and get the linear map $\phi : N_2^{-\alpha^2-\alpha} \to N_1^\alpha$ given by

$$\begin{pmatrix} \alpha+1 & 1 & 0\\ -\alpha & 1 & 0\\ 0 & 0 & 2\alpha+1 \end{pmatrix}.$$

It is straightforward to check that in fact this is an isomorphism.

There remains the case where $\alpha = -\frac{1}{2}$. Adding the polynomial $\alpha + \frac{1}{2}$ to the generating set of the ideal we get that the Gröbner basis is $\{1\}$. Hence in this case the algebras are not isomorphic.

3. Novikov algebras with an abelian Lie algebra

If the associated Lie algebra of a Novikov algebra is abelian, then the Novikov algebra is a commutative associative algebra (CAA). Conversely, every CAA is a Novikov algebra with abelian Lie algebra. In this section we describe how to obtain a classification of the CAA's of dimension 3 over $\mathbb{R}$ and $\mathbb{C}$.

First we introduce some notation, and recall some facts on associative algebras. We refer to [10] for an in-depth account of these matters.

Let A be an associative algebra over a field F of characteristic 0. If A does not have a 1, then we write $\widetilde{A}$ for the algebra $A \oplus \langle 1\rangle$ (with $a\cdot 1 = 1\cdot a = a$ for all $a \in A$). If there are nontrivial proper ideals B, C of A such that $A = B \oplus C$ (direct sum of vector spaces), then A is said to be a direct sum. (Note that necessarily $BC = CB = 0$.) The algebra A is said to be nilpotent if there is an $m > 0$ such that $a_1 \cdots a_m = 0$ for all $a_1, \ldots, a_m \in A$. The radical R of A is its largest nilpotent ideal; and A is said to be semisimple if $R = 0$. More generally, if A has a one, then there exists a semisimple subalgebra S of A such that $A = S \oplus R$ (direct sum of vector spaces), and $S \cong A/R$. Furthermore, if S is a semisimple commutative algebra, then it is a direct sum: $S = K_1 \oplus \cdots \oplus K_m$, where each K_i is a field extension of F. This decomposition corresponds to a decomposition of the identity element: $1 = \epsilon_1 + \cdots + \epsilon_m$, where the ϵ_i are

orthogonal primitive idempotents and $K_i = \epsilon_i S$. In particular, if the base field is $\mathbb{R}$, then K_i is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$. We write $\mathbb{C}$ for the commutative associative algebra over $\mathbb{R}$ with basis e_1, e_2 and non-zero products $e_1^2 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e_2^2 = -e_1$.

Lemma 3.1. *Let A be a commutative associative algebra over $\mathbb{R}$. Suppose that A is not a direct sum of proper non-trivial ideals. Then A is either*

- *nilpotent, or*
- *equal to $\widetilde{B}$ for a nilpotent B, or*
- *equal to $\mathbb{C} \oplus R$, where R is the radical.*

Proof. If A has a 1 then $A = S \oplus R$, where S is semisimple, and R is the radical. As above $S = K_1 \oplus \cdots \oplus K_m$, where the K_i are field extensions of $\mathbb{R}$. Let $1 = \epsilon_1 + \cdots + \epsilon_m$ be the corresponding decomposition of 1, where $K_i = \epsilon_i S$. Then A is the direct sum of the ideals $\epsilon_i A$. If $m > 1$, they are all nontrivial and proper. Hence it follows that $m = 1$ and S is a field. If $S \cong \mathbb{R}$ then $A \cong \widetilde{R}$. If $S \cong \mathbb{C}$ then we are in the third case.

If A has no 1, then we consider $\widetilde{A}$. Again we get a decomposition $\widetilde{A} = S \oplus R$, and m orthogonal idempotents ϵ_i. Writing $\epsilon_i = \mu_i \cdot 1 + a_i$, with $\mu_i \in \mathbb{R}$ and $a_i \in A$ we see that the fact that the ϵ_i are orthogonal idempotents with sum 1 implies that $m-1$ of them (say $\epsilon_1, \ldots, \epsilon_{m-1}$) lie in A, and ϵ_m does not. Again we get that A is the direct sum of the ideals $\epsilon_i A$. If $\epsilon_i A = A$ for some $i \leq m-1$ then ϵ_i is an identity element in A. So this cannot happen. Again it follows that $m = 1$. If $S \cong \mathbb{R}$ then A is nilpotent. If $S \cong \mathbb{C}$ then a basis of S is $e_1 = 1$, e_2 with $e_2^2 = -e_1$. However, we can also write $e_2 = \mu \cdot 1 + a$ for some $\mu \in \mathbb{R}$ and $a \in A$. From this we get that $\mu^2 = -1$. We conclude that this case cannot occur. □

From [12] we get all real nilpotent CAA's of dimensions ≤ 3. They are

Table 1: Nilpotent CAA's of dimensions ≤ 3 over $\mathbb{R}$.

dim	name	multiplication table
0	A_0	
1	A_1	
2	$A_{2,1}$	
2	$A_{2,2}$	$e_1^2 = e_2$
3	$A_{3,1}$	
3	$A_{3,2}$	$e_1^2 = e_2$
3	$A_{3,3}$	$e_1^2 = e_2$, $e_1e_2 = e_3$
3	$A_{3,4}$	$e_1^2 = e_3$, $e_2^2 = e_3$
3	$A_{3,5}$	$e_1^2 = -e_3$, $e_2^2 = e_3$

Here, over $\mathbb{C}$ the algebras $A_{3,4}$ and $A_{3,5}$ are isomorphic.

Now, using Lemma 3.1, we get all CAA's of dimension 3 over $\mathbb{R}$. They are: $A_{3,i}$ for $1 \leq i \leq 5$, $\widetilde{A_{2,i}}$, $i = 1, 2$, $\widetilde{A_0} \oplus \widetilde{A_0} \oplus \widetilde{A_0}$, $\widetilde{A_0} \oplus \widetilde{A_0} \oplus A_1$, $\widetilde{A_0} \oplus A_1 \oplus A_1$, $\widetilde{A_1} \oplus A_1$, $\widetilde{A_1} \oplus \widetilde{A_0}$, $\mathbb{C} \oplus A_1$, $\mathbb{C} \oplus \widetilde{A_0}$, $A_{2,2} \oplus \widetilde{A_0}$.

So we get 15 algebras in total. Over $\mathbb{C}$ we get 12 of them, as the pairs $A_{3,4}$, $A_{3,5}$ and $\widetilde{A_0} \oplus \widetilde{A_0} \oplus \widetilde{A_0}$, $\mathbb{C} \oplus \widetilde{A_0}$ and $\widetilde{A_0} \oplus \widetilde{A_0} \oplus A_1$, $\mathbb{C} \oplus A_1$ become isomorphic and there are no other isomorphisms.

4. Novikov algebras of dimension three over $\mathbb{R}$ and $\mathbb{C}$

A simple Lie algebra of dimension 3 does not have Novikov structures. The Novikov algebras of dimension 3 with abelian Lie algebra were classified in the previous section. So this leaves the classification of the Novikov algebras of dimension 3, where the associated Lie algebra is solvable and non-abelian.

From [11] we get that over $\mathbb{R}$ and $\mathbb{C}$ there are the following solvable Lie algebras:

name	nonzero brackets
$\mathfrak{g}_1$	$[e_1,e_2]=e_2$, $[e_1,e_3]=e_3$
$\mathfrak{g}_2^\alpha$	$[e_1,e_2]=e_3$, $[e_1,e_3]=\alpha e_2+e_3$
$\mathfrak{g}_3$	$[e_1,e_2]=e_3$
$\mathfrak{g}_4$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_2$
$\mathfrak{g}_5$	$[e_1,e_2]=e_3$, $[e_1,e_3]=-e_2$

Among these algebras there are no isomorphisms, except that $\mathfrak{g}_4$ and $\mathfrak{g}_5$ are isomorphic over $\mathbb{C}$, but not over $\mathbb{R}$.

Next are the tables of Novikov algebras that we get. On some occasions we give a parametrised class of algebras. In those cases, if nothing is stated about isomorphisms, then different values of the parameter give non-isomorphic Novikov algebras. Moreover, the classification that we give is over $\mathbb{R}$. Over $\mathbb{C}$ some isomorphisms between elements of the list arise, and those are explicitly given. The Novikov algebras with associated Lie algebra $\mathfrak{g}$ will be denoted $N_i^{\mathfrak{g}}$, $i=1,2,\ldots$. For the Lie algebra $\mathfrak{g}_2^\alpha$ we have three tables with associated Novikov algebras. In Table 3 we list the Novikov algebras that we get for generic values of the parameter α. Table 4 contains the extra Novikov algebras that we get when $\alpha=-\frac{2}{9}$. And in Table 5 we give the extra algebras that arise when $\alpha=0$.

Table 2: Novikov algebras with Lie algebra $\mathfrak{g}_1$.

name	multiplication table
$N_1^{\mathfrak{g}_1}(a)$	$e_1e_1=ae_1$, $e_1e_2=(a+1)e_2$, $e_1e_3=(a+1)e_3$, $e_2e_1=ae_2$, $e_3e_1=ae_3$
$N_2^{\mathfrak{g}_1}$	$e_1e_1=-e_1+e_2$, $e_2e_1=-e_2$, $e_3e_1=-e_3$

Table 3: Novikov algebras with Lie algebra $\mathfrak{g}_2^\alpha$.

name	multiplication table
$N_1^{\mathfrak{g}_2^\alpha}(a)$	$e_1e_1=ae_1$, $e_1e_2=ae_2+e_3$, $e_1e_3=\alpha e_2+(a+1)e_3$, $e_2e_1=ae_2$, $e_3e_1=ae_3$
$N_2^{\mathfrak{g}_2^{a^2+a}}(a)$	$e_1e_1=ae_1+e_2$, $e_1e_2=ae_2+e_3$, $e_1e_3=(a^2+a)e_2+(a+1)e_3$, $e_2e_1=ae_2$, $e_3e_1=ae_3$

Table 4: Novikov algebras with Lie algebra $\mathfrak{g}_2^{-\frac{2}{9}}$.

name	multiplication table
$N_3^{\mathfrak{g}_2^{-\frac{2}{9}}}$	$e_1e_1=-\frac{1}{3}e_1$, $e_1e_2=\frac{8}{3}e_2-8e_3$, $e_1e_3=\frac{7}{9}e_2-\frac{7}{3}e_3$, $e_2e_1=\frac{8}{3}e_2-9e_3$, $e_3e_1=e_2-\frac{10}{3}e_3$

Novikov algebras corresponding to $\mathfrak{g}_2^{-\frac{2}{9}}$.	
$N_4^{\mathfrak{g}_2^{-\frac{2}{9}}}$	$e_1e_1 = -\frac{1}{3}e_1 + e_2$, $e_1e_2 = \frac{8}{3}e_2 - 8e_3$, $e_1e_3 = \frac{7}{9}e_2 - \frac{7}{3}e_3$, $e_2e_1 = \frac{8}{3}e_2 - 9e_3$, $e_3e_1 = e_2 - \frac{10}{3}e_3$
$N_5^{\mathfrak{g}_2^{-\frac{2}{9}}}(a)$	$e_1e_1 = 3ae_1 + (-3a^2 - \frac{1}{3}a)e_2$, $e_1e_2 = 6ae_2 + (-9a+1)e_3$, $e_1e_3 = (a - \frac{2}{9})e_2 + e_3$,$e_2e_1 = 6ae_2 - 9ae_3$, $e_2e_2 = -3e_2 + 9e_3$, $e_2e_3 = -e_2 + 3e_3$, $e_3e_1 = ae_2$, $e_3e_2 = -e_2 + 3e_3$, $e_3e_3 = -\frac{1}{3}e_2 + e_3$
$N_6^{\mathfrak{g}_2^{-\frac{2}{9}}}$	$e_1e_1 = -\frac{2}{3}e_1 - \frac{8}{27}e_2 + \frac{2}{3}e_3$, $e_1e_2 = -\frac{4}{3}e_2 + 3e_3$, $e_1e_3 = -\frac{4}{9}e_2 + e_3$, $e_2e_1 = -\frac{4}{3}e_2 + 2e_3$, $e_2e_2 = -3e_2 + 9e_3$, $e_2e_3 = -e_2 + 3e_3$, $e_3e_1 = -\frac{2}{9}e_2$, $e_3e_2 = -e_2 + 3e_3$, $e_3e_3 = -\frac{1}{3}e_2 + e_3$
$N_7^{\mathfrak{g}_2^{-\frac{2}{9}}}$	$e_1e_1 = -\frac{2}{3}e_1 - \frac{11}{27}e_2 + e_3$, $e_1e_2 = -\frac{4}{3}e_2 + 3e_3$, $e_1e_3 = -\frac{4}{9}e_2 + e_3$, $e_2e_1 = -\frac{4}{3}e_2 + 2e_3$, $e_2e_2 = -3e_2 + 9e_3$, $e_2e_3 = -e_2 + 3e_3$, $e_3e_1 = -\frac{2}{9}e_2$, $e_3e_2 = -e_2 + 3e_3$, $e_3e_3 = -\frac{1}{3}e_2 + e_3$

Remarks:

- The algebras from Table 3, $N_1^{\mathfrak{g}_2^{\alpha}}(a)$ for $\alpha = -\frac{2}{9}$ and $N_2^{\mathfrak{g}_2^{a^2+a}}(a)$ for $a^2 + a = -\frac{2}{9}$ are not in Table 4. The latter condition means $a = -\frac{1}{3}$ or $a = -\frac{2}{3}$. This yields two non-isomorphic algebras.
- Over $\mathbb{C}$ the algebra $N_7^{\mathfrak{g}_2^{-\frac{2}{9}}}$ is isomorphic to $N_5^{\mathfrak{g}_2^{-\frac{2}{9}}}(-\frac{2}{9})$. To describe the isomorphism, let e_i be the basis elements of $N_5^{\mathfrak{g}_2^{-\frac{2}{9}}}(-\frac{2}{9})$ and y_i those of $N_7^{\mathfrak{g}_2^{-\frac{2}{9}}}$. Then

$$e_1 \mapsto y_1 + \frac{-2-\sqrt{-2}}{9}y_2$$
$$e_2 \mapsto \frac{1-2\sqrt{-2}}{2}y_2 + \frac{-3+3\sqrt{-2}}{2}y_3$$
$$e_3 \mapsto \frac{1-\sqrt{-2}}{3}y_2 + \frac{-2+\sqrt{-2}}{2}y_3$$

  defines an isomorphism $N_5^{\mathfrak{g}_2^{-\frac{2}{9}}}(-\frac{2}{9}) \to N_7^{\mathfrak{g}_2^{-\frac{2}{9}}}$. Over $\mathbb{R}$ they are not isomorphic.

Table 5: Novikov algebras with Lie algebra $\mathfrak{g}_2^0$.

name	multiplication table
$N_8^{\mathfrak{g}_2^0}(a)$	$e_1e_1 = ae_1$, $e_1e_2 = (a+1)e_3$, $e_1e_3 = (a+1)e_3$, $e_2e_1 = ae_3$, $e_3e_1 = ae_3$
$N_9^{\mathfrak{g}_2^0}$	$e_1e_1 = -e_1 + e_3$, $e_2e_1 = -e_3$, $e_3e_1 = -e_3$
$N_{10}^{\mathfrak{g}_2^0}(a)$	$e_1e_1 = ae_1$, $e_1e_2 = ae_2 + e_3$, $e_1e_3 = (a+1)e_3$, $e_2e_1 = ae_2$, $e_2e_2 = -e_2 + e_3$, $e_3e_1 = ae_3$
$N_{11}^{\mathfrak{g}_2^0}$	$e_1e_1 = -e_1 + e_3$, $e_1e_2 = -e_2 + e_3$, $e_2e_1 = -e_2$, $e_2e_2 = -e_2 + e_3$, $e_3e_1 = -e_3$

Remark: $N_8^{\mathfrak{g}_2^0}(0)$ is $N_1^{\mathfrak{g}_2^0}(0)$; so for the first algebra we take $a \neq 0$.

Table 6: Novikov algebras with Lie algebra $\mathfrak{g}_3$.

name	multiplication table
$N_1^{\mathfrak{g}_3}(a)$	$e_1e_1 = e_2$, $e_1e_2 = (a+1)e_3$, $e_2e_1 = ae_3$
$N_2^{\mathfrak{g}_3}(a)$	$e_1e_1 = ae_3$, $e_1e_2 = e_3$, $e_2e_2 = e_3$
$N_3^{\mathfrak{g}_3}$	$e_1e_1 = e_3$, $e_1e_2 = e_1$, $e_2e_1 = e_1 - e_3$, $e_2e_2 = e_2$, $e_2e_3 = e_3$, $e_3e_2 = e_3$
$N_4^{\mathfrak{g}_3}$	$e_1e_2 = e_1$, $e_2e_1 = e_1 - e_3$, $e_2e_2 = e_2$, $e_2e_3 = e_3$, $e_3e_2 = e_3$
$N_5^{\mathfrak{g}_3}$	$e_1e_2 = \frac{1}{2}e_3$, $e_2e_1 = -\frac{1}{2}e_3$

Table 7: Novikov algebras with Lie algebra $\mathfrak{g}_4$.

name	multiplication table
$N_1^{\mathfrak{g}_4}(a)$	$e_1e_1 = ae_1$, $e_1e_2 = ae_2 + e_3$, $e_1e_3 = e_2 + ae_3$, $e_2e_1 = ae_2$, $e_3e_1 = ae_3$
$N_2^{\mathfrak{g}_4}$	$e_1e_1 = e_1 + e_3$, $e_1e_2 = e_2 + e_3$, $e_1e_3 = e_2 + e_3$, $e_2e_1 = e_2$, $e_3e_1 = e_3$

Remark: $N_1^{\mathfrak{g}_4}(a)$ is isomorphic to $N_1^{\mathfrak{g}_4}(b)$, if and only if $a = b$ or $a = -b$. (In the latter case, $\phi(e_1) = -x_1$, $\phi(e_2) = x_2$, $\phi(e_3) = -x_3$ defines an isomorphism. Here the x_i are the basis elements of $N_1^{\mathfrak{g}_4}(b)$.)

Table 8: Novikov algebras over $\mathbb{R}$ with Lie algebra $\mathfrak{g}_5$.

name	multiplication table
$N_1^{\mathfrak{g}_5}(a)$	$e_1e_1 = ae_1$, $e_1e_2 = ae_2 + e_3$, $e_1e_3 = -e_2 + ae_3$, $e_2e_1 = ae_2$, $e_3e_1 = ae_3$

Remarks:

- Let e_i denote the basis elements of $N_1^{\mathfrak{g}_5}(a)$, and let the base field be $\mathbb{C}$. Then setting $x_1 = ie_1$, $x_2 = e_2$, $x_3 = ie_3$, we see that the x_i satisfy the multiplication table of $N_1^{\mathfrak{g}_4}(ia)$. Hence, over $\mathbb{C}$ we have that $N_1^{\mathfrak{g}_5}(a)$ is isomorphic to $N_1^{\mathfrak{g}_4}(ia)$. Over $\mathbb{R}$ such an isomorphism does not exist as the underlying Lie algebras are not isomorphic.
- Table 8 gives a list of all Novikov algebras over $\mathbb{R}$, with Lie algebra $\mathfrak{g}_5$. Over $\mathbb{C}$ there would be an extra algebra isomorphic to $N_2^{\mathfrak{g}_4}$; however over $\mathbb{R}$ this algebra does not exist.
- Also here $N_1^{\mathfrak{g}_5}(a)$ is isomorphic to $N_1^{\mathfrak{g}_5}(b)$ if and only if $a = b$ or $a = -b$.

5. Novikov algebras over $\mathbb{C}$ of dimension four with a nilpotent Lie algebra

In this section we give the classification of the 4-dimensional Novikov algebras over $\mathbb{C}$ such that the associated Lie algebra is nilpotent.

First of all there are the Novikov algebras that have an abelian associated Lie algebra. Again those are commutative associative algebras. They can be classified using the same procedure as in Section 3, using the classification of nilpotent CAA's up to dimension 4. This can be obtained from [14], or [12]. For the nilpotent CAA's of dimensions up to 3 we use the multiplication tables of Table 1. The multiplication tables of the nilpotent CAA's of dimension 4 are taken from [12]. We get the following list of CAA's of dimension 4.

Table 9: CAA's of dimension 4.

name	multiplication table
$A_{4,1}$	
$A_{4,2}$	$e_1^2 = e_2$
$A_{4,3}$	$e_1^2 = e_3$, $e_2^2 = e_3$
$A_{4,4}$	$e_1^2 = e_2$, $e_1e_2 = e_3$
$A_{4,5}$	$e_1^2 = -e_3$, $e_1e_2 = e_4$, $e_2^2 = e_3$
$A_{4,6}$	$e_1e_2 = e_4$, $e_2^2 = e_3$
$A_{4,7}$	$e_1^2 = e_4$, $e_2^2 = e_4$, $e_3^2 = e_4$
$A_{4,8}$	$e_1^2 = e_2$, $e_1e_2 = e_4$, $e_3^2 = e_4$
$A_{4,9}$	$e_1^2 = e_2$, $e_1e_2 = e_3$, $e_1e_3 = e_4$, $e_2^2 = e_4$
$4\widetilde{A}_0$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_3^2 = e_3$, $e_4^2 = e_4$
$2\widetilde{A}_0 + \widetilde{A}_1$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_3^2 = e_3$, $e_3e_4 = e_4$
$2\widetilde{A}_1$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_3^2 = e_3$, $e_3e_4 = e_4$
$\widetilde{A}_0 + \widetilde{A}_{2,1}$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_2e_3 = e_3$, $e_2e_4 = e_4$
$\widetilde{A}_0 + \widetilde{A}_{2,2}$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_2e_3 = e_3$, $e_2e_4 = e_4$, $e_3^2 = e_4$
$\widetilde{A}_{3,1}$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_1e_4 = e_4$
$\widetilde{A}_{3,2}$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_1e_4 = e_4$, $e_2^2 = e_3$
$\widetilde{A}_{3,3}$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_1e_4 = e_4$, $e_2^2 = e_3$, $e_2e_3 = e_4$
$\widetilde{A}_{3,4}$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_1e_4 = e_4$ $e_2^2 = e_4$, $e_3^2 = e_4$
$3\widetilde{A}_0 + A_1$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_3^2 = e_3$
$\widetilde{A}_0 + \widetilde{A}_1 + A_1$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_2e_3 = e_3$
$\widetilde{A}_{2,1} + A_1$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_1e_3 = e_3$
$\widetilde{A}_{2,2} + A_1$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_2^2 = e_3$
$2\widetilde{A}_0 + A_{2,1}$	$e_1^2 = e_1$, $e_2^2 = e_2$
$2\widetilde{A}_0 + A_{2,2}$	$e_1^2 = e_1$, $e_2^2 = e_2$, $e_3^2 = e_4$
$\widetilde{A}_1 + A_{2,1}$	$e_1^2 = e_1$, $e_1e_2 = e_2$
$\widetilde{A}_1 + A_{2,2}$	$e_1^2 = e_1$, $e_1e_2 = e_2$, $e_3^2 = e_4$
$\widetilde{A}_0 + A_{3,1}$	$e_1^2 = e_1$
$\widetilde{A}_0 + A_{3,2}$	$e_1^2 = e_1$, $e_2e_3 = e_4$
$\widetilde{A}_0 + A_{3,3}$	$e_1^2 = e_1$, $e_2^2 = e_3$
$\widetilde{A}_0 + A_{3,4}$	$e_1^2 = e_1$, $e_2^2 = e_3$, $e_2e_3 = e_4$

There are the following non-abelian nilpotent Lie algebras of dimension 4:

name	nonzero brackets
$\mathfrak{h}_1$	$[e_1, e_2] = e_3$
$\mathfrak{h}_2$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$

Tabel 10 contains the Novikov algebras with associated Lie algebra $\mathfrak{h}_1$. Table 11 has those with Lie algebra $\mathfrak{h}_2$.

Table 10: Novikov algebras with Lie algebra $\mathfrak{h}_1$.

name	multiplication table
$N_1^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3$
$N_2^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3, e_2e_2=e_4$
$N_3^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3, e_2e_2=e_1$
$N_4^{\mathfrak{h}_1}$	$e_1e_1=e_3,\ e_1e_2=e_3, e_2e_2=e_4$
$N_5^{\mathfrak{h}_1}$	$e_1e_2=e_3,\ e_2e_2=e_4,\ e_2e_4=e_3,\ e_4e_2=e_3$
$N_6^{\mathfrak{h}_1}$	$e_1e_2=e_3,\ e_2e_4=e_3,\ e_4e_2=e_3$
$N_7^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=e_3,\ e_2e_2=e_1+\alpha e_4,\ e_2e_4=e_3,\ e_4e_2=e_3$
$N_8^{\mathfrak{h}_1}$	$e_1e_1=e_3,\ e_1e_2=e_3,\ e_2e_2=e_4,\ e_2e_4=e_3,\ e_4e_2=e_3$
$N_9^{\mathfrak{h}_1}$	$e_1e_1=e_3,\ e_1e_2=e_3,\ e_2e_4=e_3,\ e_4e_2=e_3$
$N_{10}^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3,\ e_4e_4=e_3$
$N_{11}^{\mathfrak{h}_1}$	$e_1e_2=\frac{1}{2}e_3,\ e_2e_1=-\frac{1}{2}e_3,\ e_2e_2=e_3,\ e_4e_4=e_3$
$N_{12}^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3, e_2e_2=e_1,\ e_4e_4=e_3$
$N_{13}^{\mathfrak{h}_1}$	$e_1e_2=e_3+e_4,\ e_2e_1=e_4$
$N_{14}^{\mathfrak{h}_1}$	$e_1e_2=e_3+e_4,\ e_2e_1=e_4,\ e_2e_2=e_1$
$N_{15}^{\mathfrak{h}_1}(\alpha)$	$e_1e_1=e_3,\ e_1e_2=e_3+e_4,\ e_2e_1=e_4,\ e_2e_2=\alpha e_3$
$N_{16}^{\mathfrak{h}_1}$	$e_1e_1=e_3,\ e_1e_2=e_3+e_4,\ e_2e_1=e_4,\ e_2e_2=e_1,\ e_2e_4=e_3,\ e_4e_2=e_3$
$N_{17}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_2e_1=e_2,\ e_2e_2=e_3,\ e_3e_1=e_3$
$N_{18}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_2e_1=e_2,\ e_3e_1=e_3$
$N_{19}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_3e_1=e_3,\ e_4e_1=e_4$
$N_{20}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_2e_2=e_2,\ e_3e_1=e_3,\ e_4e_1=e_4$
$N_{21}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_2e_2=e_4,\ e_3e_1=e_3,\ e_4e_1=e_4$
$N_{22}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_2e_4=e_3,\ e_3e_1=e_3,\ e_4e_1=e_4,\ e_4e_2=e_3$
$N_{23}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_2e_2=e_4,\ e_2e_4=e_3,\ e_3e_1=e_3,\ e_4e_1=e_4,\ e_4e_2=e_3$
$N_{24}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_2e_2=e_3,\ e_3e_1=e_3,\ e_4e_1=e_4,\ e_4e_4=e_3$
$N_{25}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_1e_4=e_4,\ e_2e_1=e_2,\ e_3e_1=e_3,\ e_4e_1=e_4,\ e_4e_4=e_3$
$N_{26}^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3,\ e_2e_2=e_1,\ e_4e_4=e_4$
$N_{27}^{\mathfrak{h}_1}(\alpha)$	$e_1e_2=(\alpha+1)e_3,\ e_2e_1=\alpha e_3,\ e_4e_4=e_4$
$N_{28}^{\mathfrak{h}_1}$	$e_1e_2=\frac{1}{2}e_3,\ e_2e_1=-\frac{1}{2}e_3,\ e_2e_2=e_3,\ e_4e_4=e_4$
$N_{29}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_2e_1=e_2,\ e_2e_2=e_3,\ e_3e_1=e_3,\ e_4e_4=e_4$
$N_{30}^{\mathfrak{h}_1}$	$e_1e_1=e_1,\ e_1e_2=e_2+e_3,\ e_1e_3=e_3,\ e_2e_1=e_2,\ e_3e_1=e_3,\ e_4e_4=e_4$

Among these algebras there are precisely the following isomorphisms:

- $N_1^{\mathfrak{h}_1}(\alpha)\cong N_1^{\mathfrak{h}_1}(\beta)$ if and only if $\alpha=\beta$ or $\beta=-\alpha-1$. In the latter case, $\phi(e_1)=-y_2$, $\phi(e_2)=y_1$, $\phi(e_i)=y_i$, $i=3,4$ defines an isomorphism $\phi: N_1^{\mathfrak{h}_1}(\alpha)\to N_1^{\mathfrak{h}_1}(-\alpha-1)$.

- $N_{10}^{\mathfrak{h}_1}(\alpha) \cong N_{10}^{\mathfrak{h}_1}(\beta)$ if and only if $\alpha = \beta$ or $\beta = -\alpha - 1$. In the latter case, $\phi(e_1) = -y_2$, $\phi(e_2) = y_1$, $\phi(e_i) = y_i$, $i = 3, 4$ defines an isomorphism $\phi : N_{10}^{\mathfrak{h}_1}(\alpha) \to N_{10}^{\mathfrak{h}_1}(-\alpha - 1)$.

Table 11: Novikov algebras with Lie algebra $\mathfrak{h}_2$.

name	multiplication table
$N_1^{\mathfrak{h}_2}$	$e_1e_2 = e_3,\ e_1e_3 = e_4$
$N_2^{\mathfrak{h}_2}$	$e_1e_1 = e_2,\ e_1e_2 = e_3,\ e_1e_3 = e_4$
$N_3^{\mathfrak{h}_2}$	$e_1e_2 = e_3 + e_4,\ e_1e_3 = e_4,\ e_2e_1 = e_4$
$N_4^{\mathfrak{h}_2}$	$e_1e_1 = e_2,\ e_1e_2 = e_3 + e_4,\ e_1e_3 = e_4,\ e_2e_1 = e_4$
$N_5^{\mathfrak{h}_2}$	$e_1e_3 = \frac{1}{2}e_4,\ e_2e_1 = -e_3,\ e_3e_1 = -\frac{1}{2}e_4$
$N_6^{\mathfrak{h}_2}$	$e_1e_1 = e_4,\ e_1e_3 = \frac{1}{2}e_4,\ e_2e_1 = -e_3,\ e_3e_1 = -\frac{1}{2}e_4$
$N_7^{\mathfrak{h}_2}$	$e_1e_1 = e_2,\ e_1e_3 = \frac{1}{2}e_4,\ e_2e_1 = -e_3,\ e_3e_1 = -\frac{1}{2}e_4$
$N_8^{\mathfrak{h}_2}(\alpha)$	$e_1e_1 = (2\alpha^2 + \alpha)e_2,\ e_1e_2 = (2\alpha + 1)e_3,\ e_1e_3 = (\alpha + 1)e_4,\ e_2e_1 = 2\alpha e_3,\ e_2e_2 = e_4,$ $e_3e_1 = \alpha e_4$
$N_9^{\mathfrak{h}_2}$	$e_1e_1 = e_3,\ e_1e_2 = e_3,\ e_1e_3 = e_4,\ e_2e_2 = e_4$
$N_{11}^{\mathfrak{h}_2}$	$e_1e_1 = e_3,\ e_1e_3 = \frac{1}{2}e_4,\ e_2e_1 = -e_3,\ e_2e_2 = e_4,\ e_3e_1 = -\frac{1}{2}e_4$
$N_{12}^{\mathfrak{h}_2}$	$e_1e_1 = e_4,\ e_1e_2 = e_3,\ e_1e_3 = e_4,\ e_2e_2 = 2e_3,\ e_2e_3 = e_4,\ e_3e_2 = e_4$
$N_{13}^{\mathfrak{h}_2}$	$e_1e_2 = e_3,\ e_1e_3 = e_4,\ e_2e_2 = 2e_3,\ e_2e_3 = e_4,\ e_3e_2 = e_4$
$N_{14}^{\mathfrak{h}_2}(\alpha)$	$e_1e_1 = \alpha e_4,\ e_1e_2 = e_3,\ e_1e_3 = e_4,\ e_2e_2 = 2e_3 + e_4,\ e_2e_3 = e_4,\ e_3e_2 = e_4$
$N_{15}^{\mathfrak{h}_2}$	$e_1e_1 = e_1,\ e_1e_2 = e_2 + e_3,\ e_1e_3 = e_3 + e_4,\ e_1e_4 = e_4,\ e_2e_1 = e_2,\ e_3e_1 = e_3,$ $e_4e_1 = e_4$
$N_{16}^{\mathfrak{h}_2}$	$e_1e_1 = e_1,\ e_1e_2 = e_2 + e_3,\ e_1e_3 = e_3 + e_4,\ e_1e_4 = e_4,\ e_2e_1 = e_2,\ e_2e_2 = e_4,$ $e_3e_1 = e_3,\ e_4e_1 = e_4$
$N_{17}^{\mathfrak{h}_2}$	$e_1e_1 = e_1,\ e_1e_2 = e_2 + e_3,\ e_1e_3 = e_3 + e_4,\ e_1e_4 = e_4,\ e_2e_1 = e_2,\ e_2e_2 = 2e_3 + \alpha e_4,$ $e_2e_3 = e_4,\ e_3e_1 = e_3,\ e_3e_2 = e_4,\ e_4e_1 = e_4$

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Fakultät für Mathematik, Universität Wien, Nordbergstr. 15, 1090 Wien, Austria
E-mail address: dietrich.burde@univie.ac.at

Dipartimento di Matematica, Via Sommarive 14, I-38050 Povo (Trento), Italy
E-mail address: degraaf@science.unitn.it