

FAITHFUL LIE ALGEBRA MODULES AND QUOTIENTS OF THE UNIVERSAL ENVELOPING ALGEBRA

DIETRICH BURDE AND WOLFGANG ALEXANDER MOENS

ABSTRACT. We describe a new method to determine faithful representations of small dimension for a finite-dimensional nilpotent Lie algebra. We give various applications of this method. In particular we find a new upper bound on the minimal dimension of a faithful module for the Lie algebras being counterexamples to a well-known conjecture of J. Milnor.

1. INTRODUCTION

Let \mathfrak{g} be a finite-dimensional complex Lie algebra. Denote by $\mu(\mathfrak{g})$ the minimal dimension of a faithful \mathfrak{g} -module. This is an invariant of \mathfrak{g} , which is finite by Ado's theorem. Indeed, Ado's theorem asserts that there exists a faithful linear representation of finite dimension for \mathfrak{g} . There are many reasons why it is interesting to study $\mu(\mathfrak{g})$, and to find good upper bounds for it. One important motivation comes from questions on fundamental groups of complete affine manifolds and left-invariant affine structures on Lie groups. A famous problem of Milnor in this area is related to the question whether or not $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$ holds for all solvable Lie algebras. For the history of this problem, and the counterexamples to it see [9], [2] and the references given therein.

It is also interesting to find new proofs and refinements for Ado's theorem. We want to mention the work of Neretin [10], who gave a proof of Ado's theorem, which appears to be more natural than the classical ones. This gives also a new insight into upper bounds for arbitrary Lie algebras.

From a computational view, it is also very interesting to construct faithful representations of small degree for a given nilpotent Lie algebra \mathfrak{g} . In [6] we have given various methods for such constructions. In this paper we present another method using quotients of the universal enveloping algebra, which has many applications and gives even better results than the previous constructions. We obtain new upper bounds on the invariant $\mu(\mathfrak{g})$ for complex filiform nilpotent Lie algebras \mathfrak{g} . In particular, we find new upper bounds on $\mu(\mathfrak{g})$ for the counterexamples to Milnor's conjecture in dimension 10.

Date: October 6, 2010.

2000 Mathematics Subject Classification. Primary 17B10, 17B25.

The authors were supported by the FWF, Projekt P21683. The second author was also supported by a Junior Research Fellowship of the ESI, Vienna.

The paper is organized as follows. After some basic properties we give estimates on $\mu(\mathfrak{g})$ in terms of $\dim(\mathfrak{g})$ according to the structure of the solvable radical of \mathfrak{g} . In the third section we describe the new construction of faithful modules by quotients of the universal enveloping algebra. We decompose the Lie algebra \mathfrak{g} as a semidirect product $\mathfrak{g} = \mathfrak{d} \ltimes \mathfrak{n}$, for some ideal \mathfrak{n} and a subalgebra $\mathfrak{d} \subseteq \text{Der}(\mathfrak{n})$, and then constructing faithful $\mathfrak{d} \ltimes \mathfrak{n}$ -submodules of $U(\mathfrak{n})$. This is illustrated with two easy examples.

In the fourth section we give some applications of this construction. First we prove a bound on $\mu(\mathfrak{g})$ for an arbitrary Lie algebra \mathfrak{g} in terms of the $\dim(\mathfrak{g}/\mathfrak{n})$ and $\dim(\mathfrak{r})$, where \mathfrak{n} denotes the nilradical of \mathfrak{g} , and \mathfrak{r} the solvable radical. Then we apply the construction to show that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})$ for all 2-step nilpotent Lie algebras. Finally we apply the method to obtain new estimates on $\mu(\mathfrak{g})$ for filiform Lie algebras \mathfrak{g} , in particular for $\dim(\mathfrak{g}) = 10$. As for the counterexamples to Milnor's conjecture in dimension 10, we give an example in Example 4.13. It is quite difficult to see that this Lie algebra satisfies $\mu(\mathfrak{g}) \geq 12$, so that it does not admit an affine structure, see [2]. On the other hand, it was known that $\mu(\mathfrak{g}) \leq 22$. Our new method gives $\mu(\mathfrak{g}) \leq 18$, which is up to now the best known upper bound.

2. DEFINITIONS AND BASIC PROPERTIES

All Lie algebras are assumed to be complex and finite-dimensional, if not stated otherwise. Denote by c the nilpotency class of a nilpotent Lie algebra.

Definition 2.1. Let \mathfrak{g} be a Lie algebra. We denote by $\mu(\mathfrak{g})$ the minimal dimension of a faithful \mathfrak{g} -module, and by $\tilde{\mu}(\mathfrak{g})$ the minimal dimension of a faithful nilpotent \mathfrak{g} -module.

Note that $\tilde{\mu}(\mathfrak{g})$ is only well defined, if \mathfrak{g} is nilpotent. On the other hand, every nilpotent Lie algebra admits a faithful nilpotent \mathfrak{g} -module of finite dimension [1]. Recall the following lemma from [5].

Lemma 2.2. *Let \mathfrak{h} be a subalgebra of \mathfrak{g} . Then $\mu(\mathfrak{h}) \leq \mu(\mathfrak{g})$. Furthermore, if \mathfrak{a} and \mathfrak{b} are two Lie algebras, then $\mu(\mathfrak{a} \oplus \mathfrak{b}) \leq \mu(\mathfrak{a}) + \mu(\mathfrak{b})$.*

Definition 2.3. Denote by \mathfrak{b}_m the subalgebra of $\mathfrak{gl}_m(\mathbb{C})$ consisting of all upper-triangular matrices, by $\mathfrak{n}_m = [\mathfrak{b}_m, \mathfrak{b}_m]$ the subalgebra of all strictly upper-triangular matrices, and by \mathfrak{t}_m the subalgebra of diagonal matrices.

The following result is in principle well known. However, it appears in different formulations, e.g., compare with Theorem 2.2 in [7].

Proposition 2.4. *Let \mathfrak{n} be a nilpotent Lie algebra and $\rho: \mathfrak{n} \rightarrow \mathfrak{gl}(V)$ be a linear representation of \mathfrak{n} of degree m . Then ρ can be written as the sum of two representations $\rho = \delta + \nu$, such that*

- (1) $\delta(\mathfrak{n}) \subseteq \mathfrak{t}_m$ and $\nu(\mathfrak{n}) \subseteq \mathfrak{n}_m$ relative to a suitable choice of basis of V .
- (2) $\delta([\mathfrak{n}, \mathfrak{n}]) = 0$, and δ and ν commute.

(3) $[\rho(x), \rho(y)] = [\nu(x), \nu(y)]$ for all $x, y \in \mathfrak{n}$.

Proof. By the weight space decomposition for modules of nilpotent Lie algebras we can write

$$V = \bigoplus_{i=1}^s V^{\lambda_i}(\mathfrak{n}),$$

where $\lambda \in \text{Hom}(\mathfrak{n}, \mathbb{C})$ are the different weights of ρ , and $V^{\lambda_i}(\mathfrak{n})$ are the weight spaces. In an appropriate basis of V the operators $\rho(x)$ are given by block matrices with blocks

$$\begin{pmatrix} \lambda_i(x) & & * \\ & \ddots & \\ 0 & & \lambda_i(x) \end{pmatrix}.$$

Then let $\delta(x)$ be the diagonal part given by $\bigoplus_i \lambda_i(x) \text{id}_{V^{\lambda_i}}$, and put $\nu = \rho - \delta$. Now it is easy to see that δ and ν are representations. In fact, the λ_i are characters, so that $\delta([\mathfrak{n}, \mathfrak{n}]) = 0$. Also, δ commutes with ν , since it is a multiple of the identity on each block. This shows (1) and (2), which in turn imply (3). \square

The next proposition gives a lower bound on $\mu(\mathfrak{n})$ in terms of the nilpotency class of \mathfrak{n} . As a special case we recover the well-known estimate $n \leq \mu(\mathfrak{f})$ for a filiform Lie algebra \mathfrak{f} of dimension n .

Proposition 2.5. *Let \mathfrak{n} be a nilpotent Lie algebra of class c and dimension $n \geq 2$. Then we have $c + 1 \leq \mu(\mathfrak{n})$.*

Proof. If \mathfrak{n} is abelian, then $\mu(\mathfrak{n}) \geq \lceil 2\sqrt{n-1} \rceil \geq 2 = c + 1$ by proposition 2.4 of [5]. Assume now that \mathfrak{n} is not abelian. Consider a faithful representation $\rho: \mathfrak{n} \hookrightarrow \mathfrak{gl}(V)$ of degree m . Let $\rho = \delta + \nu$ be a decomposition according to proposition 2.4. Then $[\rho(x), \rho(y)] = [\nu(x), \nu(y)]$ for all $x, y \in \mathfrak{n}$. Hence the non-trivial nilpotent Lie algebras $\rho(\mathfrak{n})$ and $\nu(\mathfrak{n})$ have the same nilpotency class c . Since the nilpotency class of \mathfrak{n}_m is $m - 1$, and $\nu(\mathfrak{n}) \subseteq \mathfrak{n}_m$, it follows $c \leq m - 1$. If we take ρ to be of minimal degree, we obtain $c + 1 \leq \mu(\mathfrak{n})$. \square

Corollary 2.6. *Let \mathfrak{f} be a filiform nilpotent Lie algebra of dimension n . Then $n \leq \mu(\mathfrak{f})$.*

There has been some interest lately in determining $\tilde{\mu}(\mathfrak{n})$ for nilpotent Lie algebras \mathfrak{n} . We find that $\tilde{\mu}(\mathfrak{n})$ coincides with $\mu(\mathfrak{n})$ for a broad class of nilpotent Lie algebras.

Lemma 2.7. *Let \mathfrak{n} be a nilpotent Lie algebra satisfying $Z(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{n}]$. Consider a linear representation ρ of \mathfrak{n} with above decomposition $\rho = \delta + \nu$. Then ρ is faithful if and only if ν is.*

Proof. A representation of a nilpotent Lie algebra \mathfrak{n} is faithful if and only if the center $Z(\mathfrak{n})$ acts faithfully. Since $\rho(x) = \nu(x)$ for all $x, y \in [\mathfrak{n}, \mathfrak{n}]$, and $Z(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{n}]$, ρ and ν coincide on $Z(\mathfrak{n})$. Hence the center acts faithfully by ρ if and only if it acts faithfully by ν . \square

Corollary 2.8. *Let \mathfrak{n} be a nilpotent Lie algebra satisfying $Z(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{n}]$. Then $\mu(\mathfrak{n}) = \tilde{\mu}(\mathfrak{n})$.*

Remark 2.9. The condition $Z(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{n}]$ on nilpotent Lie algebras \mathfrak{n} is not too restrictive. In fact, \mathfrak{n} always splits as $\mathbb{C}^\ell \oplus \mathfrak{m}$ with $Z(\mathfrak{m}) \subseteq [\mathfrak{m}, \mathfrak{m}]$. In particular, if the center is 1-dimensional, or if \mathfrak{n} is indecomposable, the condition is satisfied. This includes \mathfrak{n} being filiform nilpotent.

We are also interested in estimating $\mu(\mathfrak{g})$ in terms of $\dim(\mathfrak{g})$. We present results which depend on the structure of the solvable radical of \mathfrak{g} . A first result is the following.

Lemma 2.10. *For any Lie algebra \mathfrak{g} we have $\sqrt{\dim(\mathfrak{g})} \leq \mu(\mathfrak{g})$.*

Proof. Suppose that \mathfrak{g} can be embedded into some $\mathfrak{gl}_m(\mathbb{C})$, then

$$\dim(\mathfrak{g}) \leq \dim(\mathfrak{gl}_m(\mathbb{C})) = m^2.$$

In particular this holds for $m = \mu(\mathfrak{g})$. □

Lemma 2.11. *Let \mathfrak{g} be represented as $\mathfrak{b} \rtimes_{\delta} \mathfrak{a}$ for a Lie algebra \mathfrak{b} and an abelian Lie algebra \mathfrak{a} , such that the homomorphism $\delta: \mathfrak{b} \rightarrow \mathfrak{gl}(\mathfrak{a})$ is faithful. Then we have*

$$\mu(\mathfrak{g}) \leq \dim(\mathfrak{a}) + 1.$$

Proof. Let $\dim(\mathfrak{a}) = r$ and $\mathfrak{aff}(\mathfrak{a}) = \mathfrak{gl}_r(\mathbb{C}) \rtimes_{\text{id}} \mathbb{C}^r \subseteq \mathfrak{gl}_{r+1}(\mathbb{C})$ be the Lie algebra of affine transformations of $\mathfrak{a} = \mathbb{C}^r$. Define

$$\varphi: \mathfrak{b} \rtimes_{\delta} \mathfrak{a} \rightarrow \mathfrak{aff}(\mathfrak{a}), \quad (b, a) \mapsto (\delta(b), a).$$

Then it is obvious that φ is faithful if and only if δ is faithful. Moreover the degree of the representation is $r + 1$. □

Denote by $\text{rad}(\mathfrak{g})$ the solvable radical of \mathfrak{g} .

Proposition 2.12. *Let \mathfrak{g} be a Lie algebra such that $\text{rad}(\mathfrak{g})$ is abelian. Then we have*

$$\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}),$$

and the only Lie algebras which satisfy equality are the abelian Lie algebras of dimension $n \leq 4$ and the Lie algebras $\mathfrak{e}_8 \oplus \cdots \oplus \mathfrak{e}_8$.

Proof. The claim is clear for simple and abelian Lie algebras, see [5]. Since the μ -invariant is subadditive, it also follows for reductive Lie algebras. Now suppose that \mathfrak{g} is not reductive. Then we can even show that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) - 2$. Let $\mathfrak{a} = \text{rad}(\mathfrak{g})$, and $\mathfrak{s} \rtimes_{\delta} \mathfrak{a}$ be a Levi decomposition, where the homomorphism $\delta: \mathfrak{s} \rightarrow \mathfrak{gl}(\mathfrak{a})$ is given by $\delta(x) = \text{ad}(x)|_{\mathfrak{a}}$. Since \mathfrak{s} is semisimple we can choose an ideal \mathfrak{s}' in \mathfrak{s} such that $\mathfrak{s} = \ker(\delta) \oplus \mathfrak{s}'$ and $\mathfrak{g} = \ker(\delta) \oplus (\mathfrak{s}' \rtimes_{\delta'} \mathfrak{a})$, where $\delta' = \delta|_{\mathfrak{s}'}$. Note that $\delta': \mathfrak{s}' \rightarrow \mathfrak{gl}(\mathfrak{a})$ is faithful. Now \mathfrak{s}' is non-trivial, since otherwise $\mathfrak{g} = \ker(\delta) \oplus \mathfrak{a}$ would be reductive.

This implies $\dim(\mathfrak{s}') \geq 3$ and $\dim(\ker(\delta)) = \dim(\mathfrak{s}) - \dim(\mathfrak{s}') \leq \dim(\mathfrak{s}) - 3$. Since $\ker(\delta)$ is semisimple, and by lemma 2.11 we obtain

$$\begin{aligned} \mu(\mathfrak{g}) &\leq \mu(\ker(\delta)) + \mu(\mathfrak{s}' \rtimes_{\delta'} \mathfrak{a}) \\ &\leq \dim(\ker(\delta)) + \dim(\mathfrak{a}) + 1 \\ &\leq \dim(\mathfrak{s}) - 3 + \dim(\mathfrak{a}) + 1 \\ &= \dim(\mathfrak{g}) - 2. \end{aligned}$$

Finally we assume that $\mu(\mathfrak{g}) = \dim(\mathfrak{g})$. By the above inequality, \mathfrak{g} needs to be reductive. If \mathfrak{g} is simple, then only $\mathfrak{g} = \mathfrak{e}_8$ satisfies the condition, see [5]. For a semisimple Lie algebra $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell$ we have $\mu(\mathfrak{s}) = \sum_i \mu(\mathfrak{s}_i)$ and $\mu(\mathfrak{s}_i) \leq \dim(\mathfrak{s}_i)$. This implies that the only semisimple Lie algebras \mathfrak{s} satisfying $\mu(\mathfrak{s}) = \dim(\mathfrak{s})$ are direct sums of \mathfrak{e}_8 . Also, the only abelian Lie algebras satisfying the condition are the ones of dimension $n \leq 4$. On the other hand, any reductive Lie algebra \mathfrak{g} satisfying $\mu(\mathfrak{g}) = \dim(\mathfrak{g})$ must be either semisimple or abelian: if $\mathfrak{g} = \mathfrak{s} \oplus \mathbb{C}^{\ell+1}$ with $\ell \geq 0$ and a non-trivial semisimple Lie algebra \mathfrak{s} , then $\mu(\mathfrak{s} \oplus \mathbb{C}) = \mu(\mathfrak{s})$, see [5], and

$$\begin{aligned} \mu(\mathfrak{g}) &\leq \mu(\mathfrak{s} \oplus \mathbb{C}) + \mu(\mathbb{C}^\ell) \\ &\leq \mu(\mathfrak{s}) + \ell \\ &\leq \dim(\mathfrak{s}) + \ell \\ &\leq \dim(\mathfrak{g}) - 1. \end{aligned}$$

This is a contradiction, and we are done. \square

Our next result is that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$ for any Lie algebra with $\text{rad}(\mathfrak{g})$ abelian or 2-step nilpotent. We need the following two lemmas.

Lemma 2.13. *Let \mathfrak{g} be a nilpotent Lie algebra and D a derivation of \mathfrak{g} that induces an isomorphism on the center. Then $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$.*

Proof. The center $Z(\mathfrak{g})$ is a nonzero characteristic ideal of \mathfrak{g} , such that $D(Z(\mathfrak{g})) \subseteq Z(\mathfrak{g})$. Denote by \mathfrak{d} the 1-dimensional Lie algebra generated by D , and form the split extension $\mathfrak{d} \rtimes \mathfrak{g}$. By assumption this is a Lie algebra of dimension $\dim(\mathfrak{g}) + 1$ with trivial center. Hence its adjoint representation $\text{ad}: \mathfrak{d} \rtimes \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{d} \rtimes \mathfrak{g})$ is faithful. Together with the embedding $\mathfrak{g} \hookrightarrow \mathfrak{d} \rtimes \mathfrak{g}$ we obtain a faithful representation of \mathfrak{g} of degree $\dim(\mathfrak{g}) + 1$. \square

Lemma 2.14. *Let \mathfrak{g} be a Lie algebra with Levi decomposition $\mathfrak{g} = \mathfrak{s} \rtimes \mathfrak{r}$, such that $\mathfrak{s} \leq \text{Der}(\mathfrak{r})$. Suppose D is a derivation of the radical \mathfrak{r} . Then the map $\pi: \mathfrak{s} \rtimes \mathfrak{r} \rightarrow \mathfrak{s} \rtimes \mathfrak{r}$ given by $(X, t) \mapsto (0, D(t))$ is a derivation of \mathfrak{g} if and only if $[D, \mathfrak{s}] = 0$.*

Proof. Consider any pair $a = (X, t)$ and $b = (Y, s)$ of elements in \mathfrak{g} . We need to show that $\pi([a, b]) = [\pi(a), b] + [a, \pi(b)]$. The commutator of a and b is given by $[(X, t), (Y, s)] = ([X, Y], X(s) - Y(t) + [t, s])$ so that

$$\begin{aligned} \pi([(X, t), (Y, s)]) &= (0, D([t, s]) + (D \circ X)(s) - (D \circ Y)(t)) \\ &= (0, [D(t), s] + [t, D(s)] + (D \circ X)(s) - (D \circ Y)(t)). \end{aligned}$$

We have $\pi((X, t)) = (0, D(t))$ and $\pi((Y, s)) = (0, D(s))$, hence

$$\begin{aligned} [\pi((X, t)), (Y, s)] + [(X, t), \pi((Y, s))] &= [(0, D(t)), (Y, s)] + [(X, t), (0, D(s))] \\ &= (0, [D(t), s] + [t, D(s)] + (X \circ D)(s) \\ &\quad - (Y \circ D)(t)). \end{aligned}$$

We see that π is a derivation of \mathfrak{g} if and only if these two expressions coincide for all $X, Y \in \mathfrak{s}$ and all $s, t \in \mathfrak{n}$. This is the case iff $[D, X](s) = 0$ for all $X \in \mathfrak{s}$ and all $s \in \mathfrak{n}$. This finishes the proof. \square

Proposition 2.15. *Let \mathfrak{g} be a Lie algebra such that $\text{rad}(\mathfrak{g})$ is nilpotent of class at most two. Then we have $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$.*

Proof. Let $\mathfrak{s} \ltimes \mathfrak{n}$ be a Levi decomposition for \mathfrak{g} . If $\text{rad}(\mathfrak{g})$ is abelian, the claim follows from proposition 2.12. Now assume that \mathfrak{n} is nilpotent of class two. As in the proof of proposition 2.12 we may assume that \mathfrak{s} acts faithfully on \mathfrak{n} and that $\mathfrak{s} \subseteq \text{Der}(\mathfrak{n})$. Now $\mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}]$ is an \mathfrak{s} -submodule of \mathfrak{n} , since \mathfrak{s} acts on \mathfrak{n} by derivations, and \mathfrak{n}_2 is invariant under these derivations, because it is a characteristic ideal. Since \mathfrak{s} is semisimple, there exists an \mathfrak{s} -invariant complement \mathfrak{n}_1 to $[\mathfrak{n}, \mathfrak{n}]$. The \mathfrak{s} -module decomposition $\mathfrak{n}_1 + \mathfrak{n}_2$ of \mathfrak{n} defines a linear transformation D of \mathfrak{n} as follows: $D|_{\mathfrak{n}_1} = \text{id}_{\mathfrak{n}_1}$ and $D|_{\mathfrak{n}_2} = 2\text{id}_{\mathfrak{n}_2}$. This is in fact a derivation of \mathfrak{n} . Note that D commutes with \mathfrak{s} in $\text{Der}(\mathfrak{n})$. The derivation D then extends to a derivation π of $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ by lemma 2.14. Since D is an isomorphism, $\pi|_{Z(\mathfrak{g})}$ is also an isomorphism. By lemma 2.13, we may then conclude that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$. \square

3. QUOTIENTS OF THE UNIVERSAL ENVELOPING ALGEBRA

3.1. Order and length functions. Let \mathfrak{n} be a nilpotent Lie algebra of dimension n and class c . Consider a strictly descending filtration of \mathfrak{n} of the following form

$$\mathfrak{n} = \mathfrak{n}^{[1]} \supset \mathfrak{n}^{[2]} \supset \dots \supset \mathfrak{n}^{[C+1]} = 0,$$

where the $\mathfrak{n}^{[i]}$ are subalgebras satisfying $[\mathfrak{n}^{[i]}, \mathfrak{n}^{[j]}] \subseteq \mathfrak{n}^{[i+j]}$ for all $1 \leq i, j \leq C + 1$. We say that the filtration is of *length* C , and we call it an *adapted filtration*. For example, such a filtration is given by the descending central series \mathfrak{n}^i for \mathfrak{n} of length c . To any such filtration associate an *order function*

$$o : \mathfrak{n} \rightarrow \mathbb{N} \cup \{\infty\}, \quad x \mapsto \max_{t \in \mathbb{N}} \{x \in \mathfrak{n}^{[t]}\}.$$

If we let $\mathfrak{n}^{[t]} = 0$ for all $t \geq C + 1$, then it makes sense to define $o(0) = \infty$. It is easy to see that the order function o satisfies the following two properties

$$\begin{aligned} o(x + y) &\geq \min\{o(x), o(y)\}, \\ o([x, y]) &\geq o(x) + o(y) \end{aligned}$$

for all $x, y \in \mathfrak{n}$.

For a given subalgebra \mathfrak{m} of \mathfrak{n} satisfying $\mathfrak{m} \supset \mathfrak{n}^{[2]}$ we obtain an induced filtration

$$\mathfrak{m} \supset \mathfrak{n}^{[2]} \supset \dots \supset \mathfrak{n}^{[C+1]} = 0,$$

and an associated order function. We extend the order function to the universal enveloping algebra $U(\mathfrak{n})$ of \mathfrak{n} as follows. Choose a basis x_1, \dots, x_n of \mathfrak{n} such that the first n_1 elements span a complement of $\mathfrak{n}^{[2]}$ in \mathfrak{n} , the next n_2 elements span a complement of $\mathfrak{n}^{[3]}$ in $\mathfrak{n}^{[2]}$, and so on. We identify the basis elements x_i of \mathfrak{n} with the images X_i in $U(\mathfrak{n})$ by the natural embedding. The Poincaré-Birkhoff-Witt theorem states that the monomials $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ form a basis for $U(\mathfrak{n})$. Now we set $o(X^\alpha) = \sum_{j=1}^n \alpha_j o(X_j)$. For a linear combination $W = \sum_{\alpha} c_{\alpha} X^{\alpha}$ we define $o(W) = \min_{\alpha} \{o(X^{\alpha}) \mid c_{\alpha} \neq 0\}$.

Furthermore we define a *length function*

$$\lambda: U(\mathfrak{n}) \rightarrow \mathbb{N} \cup \{\infty\}$$

by $\lambda(0) = \infty$, $\lambda(1) = 0$ and $\lambda(X^\alpha) = \lambda(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = \sum_{i=1}^n \alpha_i$. Here 1 denotes the unit element of $U(\mathfrak{n})$. For a linear combination $W = \sum_{\alpha} c_{\alpha} X^{\alpha}$ we set $\lambda(W) = \min_{\alpha} \{\lambda(X^{\alpha}) \mid c_{\alpha} \neq 0\}$.

The following result is well known for functions o and λ with respect to the standard filtration of \mathfrak{n} . It easily generalizes to all adapted filtrations we have defined.

Lemma 3.1. *For all $X, Y \in U(\mathfrak{n})$ we have the following inequalities:*

- (1) $o(X + Y) \geq \min\{o(X), o(Y)\}$.
- (2) $o(XY) \geq o(X) + o(Y)$.
- (3) $\lambda(X + Y) \geq \min\{\lambda(X), \lambda(Y)\}$.
- (4) $\lambda(X) \leq o(X)$.

Note that all nonzero elements of \mathfrak{n} have length 1. Let

$$V_t = \{X \in U(\mathfrak{n}) \mid o(X) \geq t\}.$$

This is an \mathfrak{n} -submodule of $U(\mathfrak{n})$, where the action is given by left-multiplication. Furthermore we have $\mathfrak{n} \cap V_t = \{0\}$ for all $t \geq C + 1$.

3.2. Actions on $U(\mathfrak{n})$. The Lie algebra \mathfrak{n} acts naturally on $U(\mathfrak{n})$ by left multiplications. We denote this action by xY , for $x \in \mathfrak{n}$ and $Y \in U(\mathfrak{n})$. We will show that semidirect products $\mathfrak{d} \ltimes \mathfrak{n}$ for subalgebras $\mathfrak{d} \leq \text{Der}(\mathfrak{n})$ also act naturally on $U(\mathfrak{n})$. First of all, \mathfrak{d} acts on \mathfrak{n} by derivations. Thus we already have an action of \mathfrak{d} on \mathfrak{n} . For $D \in \text{Der}(\mathfrak{n})$ let $D(1) = 0$ and define recursively $D(XY) = D(X)Y + XD(Y)$ for all $X, Y \in U(\mathfrak{n})$. Then the action of $\mathfrak{d} \ltimes \mathfrak{n}$ on $U(\mathfrak{n})$ is given by

$$(D, x).Y = D(Y) + xY$$

for all $(D, x) \in \mathfrak{d} \ltimes \mathfrak{n}$, and all $Y \in U(\mathfrak{n})$. This is well defined, and we have the following useful lemma concerning faithful quotients.

Lemma 3.2. *Suppose that W is a $\mathfrak{d} \ltimes \mathfrak{n}$ -submodule of $U(\mathfrak{n})$ such that $W \cap \mathfrak{n} = 0$. Then the quotient module $U(\mathfrak{n})/W$ is faithful.*

Consider a nilpotent Lie algebra \mathfrak{n} together with the standard filtration given by the lower central series. We have the following result.

Proposition 3.3. *Let \mathfrak{n} be a nilpotent Lie algebra of dimension n and nilpotency class c . Let \mathfrak{d} be a subalgebra of $\text{Der}(\mathfrak{n})$ acting completely reducibly on \mathfrak{n} . Then V_{c+1} is a $\mathfrak{d} \times \mathfrak{n}$ -submodule of $U(\mathfrak{n})$ such that the quotient module $U(\mathfrak{n})/V_{c+1}$ is faithful of dimension at most $\frac{3}{\sqrt{n}}2^n$.*

Proof. Choose a basis for \mathfrak{n} associated to the standard filtration of \mathfrak{n} as in section 3.1, but with the additional requirement that each complement $C^{[i]}$ to $\mathfrak{n}^{[i]}$ is also invariant under the action of \mathfrak{d} , i.e., $D(C^{[i]}) \subseteq C^{[i]}$ for all $D \in \mathfrak{d}$. This is possible since the $\mathfrak{n}^{[i]}$ are characteristic ideals, hence invariant under \mathfrak{d} , so that they are submodules, which have a complementary submodule by the complete reducibility. Associate a PBW-basis for $U(\mathfrak{n})$ as before. Consider a basis element $x_j \in C^{[i]}$. Then $o(x_j) = i$ and $o(D(x_j)) \geq i$, since $D(x_j)$ is again in $C^{[i]}$, so has order i or ∞ . Hence it follows that $o(D(W)) \geq o(W)$ for all $W \in U(\mathfrak{n})$. This means that V_{c+1} is a \mathfrak{d} -submodule of $U(\mathfrak{n})$. Since we already know that V_{c+1} is a \mathfrak{n} -submodule, it is a $\mathfrak{d} \times \mathfrak{n}$ -submodule of $U(\mathfrak{n})$. The quotient is faithful by lemma 3.2. Its dimension is bounded by $\frac{3}{\sqrt{n}}2^n$, which was shown in [3], where it was considered just as an \mathfrak{n} -module. \square

3.3. The construction of faithful quotients. Let \mathfrak{n} be a nilpotent Lie algebra, together with some adapted filtration $\mathfrak{n}^{[t]}$ of length C , and a subalgebra $\mathfrak{d} \leq \text{Der}(\mathfrak{n})$.

Definition 3.4. An ideal J of \mathfrak{n} is called *compatible*, with respect to $\mathfrak{n}^{[t]}$ and \mathfrak{d} , if it satisfies

- (1) $D(J) \subseteq J$ for all $D \in \mathfrak{d}$,
- (2) J is abelian.
- (3) $\mathfrak{n}^{[t]} \subseteq J \subseteq \mathfrak{n}^{[t+1]}$ for some $t \geq 0$.

Denote by $\langle\langle J \rangle\rangle$ the linear subspace of $U(\mathfrak{n})$ generated by all Xy for $X \in U(\mathfrak{n})$ and $y \in J$. By assumption J satisfies

$$\mathfrak{n} = \mathfrak{n}^{[1]} \supset \dots \supset \mathfrak{n}^{[t]} \supseteq J \supseteq \mathfrak{n}^{[t+1]} \supset \dots \supset \mathfrak{n}^{[C+1]} = 0.$$

For the rest of this section choose a basis x_1, \dots, x_n of \mathfrak{n} such that the first n_1 elements span a complement of $\mathfrak{n}^{[2]}$ in \mathfrak{n} , the next n_2 elements span a complement of $\mathfrak{n}^{[3]}$ in $\mathfrak{n}^{[2]}$, and so on, including a basis of a complement of J in $\mathfrak{n}^{[t]}$, and a complement of $\mathfrak{n}^{[t+1]}$ in J . A basis for J is then of the form x_m, \dots, x_n for some $m \geq 1$. By the PBW-theorem we obtain standard monomials X^α in $U(\mathfrak{n})$ according to this basis.

Lemma 3.5. *Let J be a compatible ideal in \mathfrak{n} . Then $\langle\langle J \rangle\rangle$ is the linear span of the standard monomials $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with $(\alpha_m, \dots, \alpha_n) \neq (0, \dots, 0)$. For any $W \in U(\mathfrak{n})$ and any $y \in J$ we have $\lambda(Wy) \geq \lambda(W) + 1$.*

Proof. First note that the monomials $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with $(\alpha_m, \dots, \alpha_n) \neq (0, \dots, 0)$ belong to $\langle\langle J \rangle\rangle$. They even span $\langle\langle J \rangle\rangle$: assume that $T = X_{i_1} \dots X_{i_\ell}$ is a standard monomial of length ℓ , and x_k be a basis vector of J , i.e., $m \leq k$. If $i_\ell \leq k$ then Tx_k is one of our fixed standard monomials of length $\ell+1$, and obviously contained in $\langle\langle J \rangle\rangle$. Otherwise there exists a minimal i_r such that $i_{r-1} \leq k < i_r$. Then, by definition

of our basis for \mathfrak{n} , all $X_{i_r}, \dots, X_{i_\ell}$ are in J . Since J is abelian, $X_{i_r} \cdots X_{i_\ell} x_k = x_k X_{i_r} \cdots X_{i_\ell}$. Then we obtain $Tx_k = X_{i_1} \cdots X_{i_{r-1}} x_k X_{i_r} \cdots X_{i_\ell}$. This is a standard monomial as above, contained in $\langle\langle J \rangle\rangle$, and of length $\ell + 1$. For an arbitrary element $W = \sum c_\alpha X^\alpha$ in $U(\mathfrak{n})$ we have, using (3) of lemma 3.1,

$$\begin{aligned} \lambda(Wx_k) &= \lambda\left(\sum_{\alpha} c_{\alpha} X^{\alpha} x_k\right) \\ &\geq \min_{\alpha} \{\lambda(X^{\alpha} x_k)\} \\ &\geq \min_{\alpha} \{\lambda(X^{\alpha}) + 1\} \\ &= \lambda(W) + 1. \end{aligned}$$

Since the standard monomials $T = X^\alpha$ span $U(\mathfrak{n})$ as a vector space, the claim follows by a similar computation. \square

We define a subset

$$\mathcal{L}_2 = \langle W \in U(\mathfrak{n}) \mid \lambda(W) \geq 2 \rangle$$

of $U(\mathfrak{n})$. Note that it is a vector space since a linear combination of elements of it is an element again of length at least two. We have $\mathfrak{n} \cap \mathcal{L}_2 = 0$, since the nonzero elements of \mathfrak{n} have length 1.

Lemma 3.6. *Let J be a compatible ideal in \mathfrak{n} , and \mathfrak{d} be a subalgebra of $\text{Der}(\mathfrak{n})$. Then*

$$W_J = \langle\langle J \rangle\rangle \cap \mathcal{L}_2$$

is a $\mathfrak{d} \ltimes \mathfrak{n}$ -submodule of $U(\mathfrak{n})$, such that the quotient $U(\mathfrak{n})/W_J$ is faithful.

Proof. By the above remark, W_J is a vector space. Let $x \in \mathfrak{n}$, $W \in U(\mathfrak{n})$ and $x_k \in J$ such that $Wx_k \in W_J$. We want to show that $x(Wx_k) = (xW)x_k$ again is in W_J . By definition it is in $\langle\langle J \rangle\rangle$. For the length we obtain, using lemma 3.5, $\lambda((xW)x_k) \geq 1 + \lambda(xW) \geq 2$. Hence W_J is invariant under the action of \mathfrak{n} . Now we will show that W_J is invariant under \mathfrak{d} , so that it is a $\mathfrak{d} \ltimes \mathfrak{n}$ -submodule of $U(\mathfrak{n})$. Let $D \in \mathfrak{d}$ be a derivation. Then $D(Wx_k) = D(W)x_k + WD(x_k)$. Both terms on the RHS are in $\langle\langle J \rangle\rangle$ by definition, and since $D(x_k) \in J$. It remains to show that their length is at least 2. Since by assumption $Wx_k \in W_J$, we have $\lambda(W) \geq 1$. This implies $\lambda(D(W)) \geq 1$, and $\lambda(D(W)x_k) \geq \lambda(D(W)) + 1 \geq 2$. For the second term we obtain $\lambda(WD(x_k)) \geq \lambda(W) + 1 \geq 2$. Since the sum of two elements of length at least 2 has length at least 2, we obtain $D(Wx_k) \in W_J$. Finally, we show that the quotient $U(\mathfrak{n})/W_J$ is faithful. By lemma 3.2 it suffices to show that $\mathfrak{n} \cap W_J = 0$. This follows from $\mathfrak{n} \cap W_J \subseteq \mathfrak{n} \cap \mathcal{L}_2 = 0$. \square

We remark that the above quotient module will not yet be finite-dimensional in general. We will achieve this by enlarging the submodule via V_C , where again $V_t = \{X \in U(\mathfrak{n}) \mid o(X) \geq t\}$, and C denotes the length of the filtration attached to a compatible ideal J .

Proposition 3.7. *Let J be a compatible ideal in \mathfrak{n} . Suppose that $o(D(x)) \geq o(x) + 1$ for all $x \in \mathfrak{n}$ and all $D \in \mathfrak{d}$. Then*

$$Z_J = \langle W_J, V_C \cap \mathcal{L}_2 \rangle$$

is a $\mathfrak{d} \times \mathfrak{n}$ -submodule of $U(\mathfrak{n})$, such that the quotient $U(\mathfrak{n})/Z_J$ is faithful and finite-dimensional.

Proof. We first show that $\langle V_C \cap \mathcal{L}_2 \rangle$ is a $\mathfrak{d} \times \mathfrak{n}$ -submodule of $U(\mathfrak{n})$. The assumption also implies that $o(D(W)) \geq o(W) + 1$ for all $W \in U(\mathfrak{n})$. Then for every (D, x) in $\mathfrak{d} \times \mathfrak{n}$ and every $W \in V_C \cap \mathcal{L}_2$ we have

$$\begin{aligned} o((D, x).W) &= o(D(W) + xW) \\ &\geq \min\{o(D(W)), o(xW)\} \\ &\geq o(W) + 1 \\ &\geq C + 1. \end{aligned}$$

Hence $\langle V_C \cap \mathcal{L}_2 \rangle$ is mapped into V_{C+1} under the action of $\mathfrak{d} \times \mathfrak{n}$. But we have $V_{C+1} \subseteq V_C \cap \mathcal{L}_2$, because $V_{C+1} \subseteq V_C$ and $V_{C+1} \subseteq \mathcal{L}_2$. For the latter inclusion we note that every nonzero element of length at most one has order at most C , so that all elements of order at least $C + 1$ have length at least two. Hence all $(D, x).W$ are contained in $\langle V_C \cap \mathcal{L}_2 \rangle$. This implies that Z_J is a $\mathfrak{d} \times \mathfrak{n}$ -submodule, using lemma 3.6. Since $V_{C+1} \subseteq Z_J$ we have $\dim(U(\mathfrak{n})/Z_J) \leq \dim(U(\mathfrak{n})/V_{C+1})$. Since the latter dimension is finite, we obtain that $U(\mathfrak{n})/Z_J$ is finite-dimensional. Finally we show that the quotient module is faithful. Since $\mathfrak{n} \cap Z_J \subseteq \mathfrak{n} \cap \mathcal{L}_2 = 0$ it follows from lemma 3.2. \square

3.4. Algorithmic construction. We want to apply proposition 3.7 to construct faithful modules of small dimension for a given nilpotent Lie algebra \mathfrak{g} . The *input* is the Lie algebra \mathfrak{g} with a given basis, together with a decomposition $\mathfrak{g} = \mathfrak{d} \times \mathfrak{n}$, for some ideal \mathfrak{n} , a subalgebra $\mathfrak{d} \subseteq \text{Der}(\mathfrak{n})$, and choices of an admissible filtration $\mathfrak{n}^{[t]}$, a compatible ideal J , and so on, such that the assumptions of the proposition are satisfied. The *output* will be a faithful \mathfrak{g} -module of finite dimension. How small this dimension is, will depend on clever choices of $\mathfrak{n}, \mathfrak{d}, J, \mathfrak{g}^{[t]}$, and so on. The algorithmic construction can be derived from proposition 3.7. Let us illustrate this explicitly for the standard filiform Lie algebra \mathfrak{g} of dimension 4, with two different choices. We choose a basis x_1, \dots, x_4 of \mathfrak{g} such that $[x_1, x_i] = x_{i+1}$ for $i = 2, 3$.

Example 3.8. *Write $\mathfrak{g} = \mathfrak{d} \times \mathfrak{n}$ with $\mathfrak{n} = \langle x_1, x_3, x_4 \rangle$ and $\mathfrak{d} = \langle \text{ad}(x_2)|_{\mathfrak{n}} \rangle$. Choose the filtration $\mathfrak{n} = \mathfrak{n}^{[1]} \supset \mathfrak{n}^{[2]} \supset \mathfrak{n}^{[3]} \supset \mathfrak{n}^{[4]} = 0$ of length $C = 3$ by $\mathfrak{n}^{[2]} = \langle x_3, x_4 \rangle$ and $\mathfrak{n}^{[3]} = \langle x_4 \rangle$. Choose $J = \mathfrak{n}^{[2]}$ as the compatible ideal. Then all conditions of the proposition are satisfied, and we obtain a faithful \mathfrak{g} -module of dimension 5.*

First note that we really have a filtration, J is indeed a compatible ideal, and the assumption for the derivations in \mathfrak{d} is satisfied. Now the basis elements of order at most 3 in $U(\mathfrak{n})$ are given as follows: 1 has order 0; X_1 has order 1; X_3, X_1^2 have

order 2, and X_1^3, X_1X_3, X_4 have order 3. Also, 1 has length 0, and X_1, X_3, X_4 have length 1. Then we obtain

$$\begin{aligned} U(\mathfrak{n}) &= \langle 1, X_1, X_3, X_1^2, X_1^3, X_1X_3, X_4 \rangle + V_4, \\ \langle\langle J \rangle\rangle &= \langle X_3, X_1X_3, X_4 \rangle + V_4', \\ W_J &= \langle X_1X_3 \rangle + V_4'' \\ Z_J &= \langle X_1X_3, X_1^3 \rangle + V_4. \end{aligned}$$

where V_4', V_4'' are subspaces of V_4 . Hence we obtain that

$$U(\mathfrak{n})/Z_J = \langle \bar{1}, \bar{X}_1, \bar{X}_3, \bar{X}_1^2, \bar{X}_4 \rangle$$

where the bar denotes the cosets. This is a faithful \mathfrak{g} -module of dimension 5. We can compute it explicitly, giving the action of the generators x_1, x_2 of \mathfrak{g} .

$$\begin{aligned} x_1 \cdot \bar{1} &= \bar{X}_1, \quad x_1 \cdot \bar{X}_1 = \bar{X}_1^2, \quad x_1 \cdot \bar{X}_3 = \bar{0}, \quad x_1 \cdot \bar{X}_1^2 = \bar{0}, \quad x_1 \cdot \bar{X}_4 = \bar{0}, \\ x_2 \cdot \bar{1} &= \bar{0}, \quad x_2 \cdot \bar{X}_1 = [X_2, X_1] = -\bar{X}_3, \quad x_2 \cdot \bar{X}_3 = \bar{0}, \quad x_2 \cdot \bar{X}_1^2 = \bar{X}_4, \quad x_2 \cdot \bar{X}_4 = \bar{0}. \end{aligned}$$

Here we have

$$\begin{aligned} x_2 \cdot X_1^2 &= [X_2, X_1^2] \\ &= [X_2, X_1]X_1 + X_1[X_2, X_1] \\ &= -X_3X_1 - X_1X_3 \\ &= -[X_3, X_1] - 2X_1X_3 \\ &= X_4 - 2X_1X_3, \end{aligned}$$

so that $x_2 \cdot \bar{X}_1^2 = \bar{X}_4$. Note that this \mathfrak{g} -module has a submodule, generated by \bar{X}_1^2 with a faithful quotient of dimension 4. Since $\mu(\mathfrak{g}) = 4$, the result is optimal.

In the second example we will directly obtain a faithful 4-dimensional \mathfrak{g} -module. It will not be isomorphic to the above quotient module.

Example 3.9. Write $\mathfrak{g} = \mathfrak{d} \ltimes \mathfrak{n}$ with $\mathfrak{n} = \langle x_2, x_3, x_4 \rangle$ and $\mathfrak{d} = \langle \text{ad}(x_1)|_{\mathfrak{n}} \rangle$. Choose the filtration $\mathfrak{n} = \mathfrak{n}^{[1]} \supset \mathfrak{n}^{[2]} \supset \mathfrak{n}^{[3]} \supset \mathfrak{n}^{[4]} = 0$ of length $C = 3$ by $\mathfrak{n}^{[2]} = \langle x_3, x_4 \rangle$ and $\mathfrak{n}^{[3]} = \langle x_4 \rangle$. Choose $J = \mathfrak{n}^{[1]}$ as the compatible ideal. Then all conditions of the proposition are satisfied, and we obtain a faithful \mathfrak{g} -module of dimension 4.

Note that J is an abelian ideal of codimension 1 in \mathfrak{g} . With $D = \text{ad}(x_1)|_{\mathfrak{n}}$ we have $D(x_2) = x_3$ and $D(x_3) = x_4$. The elements of order at most 3 in $U(\mathfrak{n})$ are given as follows: 1 has order 0; X_2 has order 1; X_3, X_2^2 have order 2, and X_2^3, X_2X_3, X_4 have order 3. Then we obtain

$$\begin{aligned} U(\mathfrak{n}) &= \langle 1, X_2, X_3, X_2^2, X_2^3, X_2X_3, X_4 \rangle + V_4, \\ \langle\langle J \rangle\rangle &= \langle X_2, X_3, X_2^2, X_2^3, X_2X_3, X_4 \rangle + V_4', \\ W_J &= \langle X_2^2, X_2^3, X_2X_3 \rangle + V_4'' \\ Z_J &= \langle X_2^2, X_2^3, X_2X_3 \rangle + V_4. \end{aligned}$$

where V_4', V_4'' are subspaces of V_4 . Hence we obtain that

$$U(\mathfrak{n})/Z_J = \langle \bar{1}, \bar{X}_2, \bar{X}_3, \bar{X}_4 \rangle.$$

This is a faithful \mathfrak{g} -module of dimension 4. It is given by

$$\begin{aligned} x_1 \cdot \bar{1} &= \bar{0}, & x_1 \cdot \bar{X}_2 &= \bar{X}_3, & x_1 \cdot \bar{X}_3 &= \bar{X}_4, & x_1 \cdot \bar{X}_4 &= \bar{0}, \\ x_2 \cdot \bar{1} &= \bar{X}_2, & x_2 \cdot \bar{X}_2 &= \bar{0}, & x_2 \cdot \bar{X}_3 &= \bar{0}, & x_2 \cdot \bar{X}_4 &= \bar{0}. \end{aligned}$$

4. APPLICATIONS

4.1. A general bound. It is interesting to ask for good estimates on $\mu(\mathfrak{g})$ for arbitrary Lie algebras. So far, general bounds have only been given for nilpotent Lie algebras. For example, if \mathfrak{g} is nilpotent of dimension r and of class c , then $\mu(\mathfrak{g}) \leq \binom{r+c}{c}$, see [8]. Independently of c we have $\mu(\mathfrak{g}) \leq \frac{3}{\sqrt{r}} 2^r$, see [3]. There have been some attempts to find similar estimates for solvable Lie algebras. We will present here such a bound for arbitrary Lie algebras \mathfrak{g} . Denote by \mathfrak{n} the nilradical of \mathfrak{g} , and by \mathfrak{r} its solvable radical. We may assume that \mathfrak{r} is non-trivial, because otherwise the adjoint representation is faithful. Hence let $\dim(\mathfrak{r}) = r \geq 1$. We will show that $\mu(\mathfrak{g}) \leq \mu(\mathfrak{g}/\mathfrak{n}) + \frac{3}{\sqrt{r}} \cdot 2^r$.

We start with the following result of Neretin [10], which we have slightly reformulated for our purposes.

Proposition 4.1. *Let \mathfrak{g} be a complex Lie algebra with solvable radical \mathfrak{r} and Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$. Let \mathfrak{p} be a reductive subalgebra of \mathfrak{g} and \mathfrak{m} a nilpotent ideal satisfying the following properties:*

- (a) $\mathfrak{p} \cap \mathfrak{m} = 0$,
- (b) $[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{m}$ and $\mathfrak{s} \subseteq \mathfrak{p}$,
- (c) \mathfrak{p} acts completely reducibly on \mathfrak{m} .

Then there exists a nilpotent Lie algebra \mathfrak{h} of dimension $\dim(\mathfrak{g}) - \dim(\mathfrak{p})$ such that \mathfrak{g} embeds into a Lie algebra $(\mathfrak{p} \oplus \mathbb{C}^\ell) \ltimes \mathfrak{h}$, with $\ell = \dim(\mathfrak{g}/(\mathfrak{p} \ltimes \mathfrak{m}))$, and the action of $\mathfrak{p} \oplus \mathbb{C}^\ell$ on \mathfrak{h} is completely reducible.

We note the following corollary.

Corollary 4.2. *Let \mathfrak{g} be a complex Lie algebra with solvable radical \mathfrak{r} and nilradical \mathfrak{n} . Then there exists a nilpotent Lie algebra \mathfrak{h} of dimension $\dim(\mathfrak{r})$ such that \mathfrak{g} embeds into a Lie algebra $(\mathfrak{g}/\mathfrak{n}) \ltimes \mathfrak{h}$, and the action of $\mathfrak{g}/\mathfrak{n}$ on \mathfrak{h} is completely reducible.*

Proof. In the notation of the above proposition write $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ and choose $\mathfrak{p} = \mathfrak{s}$, and $\mathfrak{m} = \mathfrak{n}$. Then the conditions (a) – (c) are satisfied. Indeed, $\mathfrak{s} \cap \mathfrak{n} \subseteq \mathfrak{s} \cap \mathfrak{r} = 0$. Furthermore $[\mathfrak{g}, \mathfrak{r}]$ is a nilpotent ideal, hence is contained in \mathfrak{n} . Finally, \mathfrak{s} acts completely reducibly on \mathfrak{n} , because \mathfrak{s} is semisimple. The result follows. \square

We obtain the following bound on $\mu(\mathfrak{g})$:

Proposition 4.3. *Let \mathfrak{g} be a complex Lie algebra with nilradical \mathfrak{n} and solvable radical \mathfrak{r} . Assume that $\dim(\mathfrak{r}) = r \geq 1$. Then we have*

$$\mu(\mathfrak{g}) \leq \mu(\mathfrak{g}/\mathfrak{n}) + \frac{3}{\sqrt{r}} \cdot 2^r$$

Proof. We can embed \mathfrak{g} into a Lie algebra $(\mathfrak{g}/\mathfrak{n}) \ltimes \mathfrak{h}$ as in the corollary, where \mathfrak{h} is a nilpotent Lie algebra of dimension $\dim(\mathfrak{r})$, and $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ is reductive. This means $\mathfrak{g} \subseteq \mathfrak{q} \ltimes \mathfrak{h}$, and hence $\mu(\mathfrak{g}) \leq \mu(\mathfrak{q} \ltimes \mathfrak{h})$ by lemma 2.2. Now we want to apply proposition 3.3 to $\mathfrak{q} \ltimes \mathfrak{h}$. For that we need that \mathfrak{q} is a subalgebra of $\text{Der}(\mathfrak{h})$, or equivalently, that \mathfrak{q} acts faithfully on \mathfrak{h} . However, we may always decompose the reductive Lie algebra \mathfrak{q} as $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2$, where \mathfrak{q}_1 commutes with \mathfrak{h} , and \mathfrak{q}_2 acts faithfully and completely reducibly on \mathfrak{h} . Again by lemma 2.2, we obtain $\mu(\mathfrak{q} \ltimes \mathfrak{h}) \leq \mu(\mathfrak{q}_1) + \mu(\mathfrak{q}_2 \ltimes \mathfrak{h})$. We have $\mu(\mathfrak{q}_1) \leq \mu(\mathfrak{q})$ because of $\mathfrak{q}_1 \subseteq \mathfrak{q}$. Furthermore we have $\mu(\mathfrak{q}) \leq \dim(\mathfrak{q})$ by proposition 2.12. Now proposition 3.3 can be applied to $\mathfrak{q}_2 \ltimes \mathfrak{h}$, and we obtain

$$\begin{aligned} \mu(\mathfrak{g}) &\leq \mu(\mathfrak{q} \ltimes \mathfrak{h}) \\ &\leq \mu(\mathfrak{q}_1) + \mu(\mathfrak{q}_2 \ltimes \mathfrak{h}) \\ &\leq \dim(\mathfrak{q}) + \frac{3}{\sqrt{r}} \cdot 2^r \end{aligned}$$

□

4.2. Two-step nilpotent Lie algebras. It is well known that we have $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$ for all two-step nilpotent Lie algebras \mathfrak{g} , see [3]. As an application of our algorithm we are able to refine the result slightly. Of course, for special classes of two-step nilpotent Lie algebras (such as for Heisenberg Lie algebras) much more can be shown. We prove the following result.

Proposition 4.4. *It holds $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})$ for all two-step nilpotent Lie algebras \mathfrak{g} .*

Proof. We can write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $Z(\mathfrak{g}_2) \subseteq [\mathfrak{g}_2, \mathfrak{g}_2]$ and \mathfrak{g}_1 abelian. Assume that we already know that $\mu(\mathfrak{g}_2) \leq \dim(\mathfrak{g}_2)$. Then, by lemma 2.2, it follows $\mu(\mathfrak{g}) \leq \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{g}_2) + \mu(\mathfrak{g}_2) \leq \dim(\mathfrak{g})$. Hence we may assume that \mathfrak{g} satisfies $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$. Let $\dim(\mathfrak{g}) = n$ and choose an ideal $\mathfrak{n} \subseteq \mathfrak{g}$ of codimension 1 containing the commutator of \mathfrak{g} . Let x_1, \dots, x_n be a basis of \mathfrak{g} , such that x_2, \dots, x_n span \mathfrak{n} . Then $\mathfrak{g} = \langle x_1 \rangle \oplus \mathfrak{n}$ as a vector space. Let $\mathfrak{d} = \langle \text{ad}(x_1)|_{\mathfrak{n}} \rangle$, and we may write $\mathfrak{g} = \mathfrak{d} \ltimes \mathfrak{n}$. Let $\mathfrak{n}^{[1]} \supset \mathfrak{n}^{[2]} \supset 0$ be the filtration of length $C = 2$ given by $\mathfrak{n}^{[1]} = \mathfrak{n}$ and $\mathfrak{n}^{[2]} = Z(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Recall here that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Choose $J = Z(\mathfrak{g})$ as a compatible ideal. It satisfies the conditions of definition 3.4, since it is invariant under all derivations of \mathfrak{d} , and it is abelian. Note that we have $D(\mathfrak{n}^{[1]}) \subseteq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}^{[2]}$ for all $D \in \mathfrak{d}$, so that $o(D(x)) \geq o(x) + 1$ for all $x \in \mathfrak{n}$. Now we can apply proposition 3.7 with these choices. We obtain a faithful module $U(\mathfrak{n})/Z_J = U(\mathfrak{n})/\mathcal{L}_2$, which has dimension n , since it is spanned by the classes of $1, x_2, \dots, x_n$. □

4.3. Filiform nilpotent Lie algebras. We wish to apply proposition 3.7 to filiform nilpotent Lie algebras \mathfrak{f} of dimension n in order to improve the known upper bounds for $\mu(\mathfrak{f})$. Let $\mathfrak{f}^1 = \mathfrak{f}$ and $\mathfrak{f}^i = [\mathfrak{f}, \mathfrak{f}^{i-1}]$. Let $\beta(\mathfrak{f})$ be the maximal dimension of an abelian ideal of \mathfrak{f} . It is well known that $n/2 \leq \beta(\mathfrak{f}) \leq n - 1$. Denote by $p_k(j)$ the number of partitions of j in which each term does not exceed k . Let $p_k(0) = 1$ for all $k \geq 0$ and $p_0(j) = 0$ for all $j \geq 1$.

Proposition 4.5. *Let \mathfrak{f} be a filiform nilpotent Lie algebra of dimension n having an abelian ideal J of dimension $1 \leq \beta \leq n - 1$. Then we have $\mu(\mathfrak{f}) \leq f(n, \beta)$, where*

$$f(n, \beta) = \beta + \sum_{j=0}^{n-2} p_{n-1-\beta}(j).$$

Proof. Let x_1, \dots, x_n be an adapted basis of \mathfrak{f} in the sense of [11]. Then choose $\mathfrak{n} = \langle x_2, \dots, x_n \rangle$ and $\mathfrak{d} = \langle \text{ad}(x_1)|_{\mathfrak{n}} \rangle$, so that $\mathfrak{f} = \mathfrak{d} \ltimes \mathfrak{n}$. Define a filtration $\mathfrak{n}^{[1]} \supset \mathfrak{n}^{[2]} \supset \dots \supset \mathfrak{n}^{[C]} \supset 0$ of length $C = n - 1$ by $\mathfrak{n}^{[1]} = \mathfrak{n}$ and $\mathfrak{n}^{[i]} = \mathfrak{f}^i$ for $i \geq 2$. We may write $J = \langle x_m, \dots, x_n \rangle$ with $m \geq 2$ and $n - m + 1 = \beta$. It is easy to see that J is a compatible ideal in the sense of definition 3.4. Furthermore we have $o(D(x)) \geq o(x) + 1$ for all $x \in \mathfrak{n}$ and all $D \in \mathfrak{d}$. Now we can apply proposition 3.7. We obtain a faithful module $U(\mathfrak{n})/Z_J$. We will show that its dimension is $\beta + \sum_{j=0}^{n-2} p_{n-1-\beta}(j)$. It is generated by the classes

$$\{\overline{X_m}, \dots, \overline{X_n}\} \cup \{\overline{X^\alpha} = \overline{X_2^{\alpha_2} \dots X_{m-1}^{\alpha_{m-1}}} \mid o(X^\alpha) \leq n - 2\}.$$

There are β monomials in the first set. The cardinality of the second set is given by

$$\begin{aligned} & \#\{(\alpha_2, \dots, \alpha_{m-1}) \in \mathbb{Z}_{\geq 0}^{m-2} \mid 1 \cdot \alpha_2 + 2 \cdot \alpha_3 + \dots + (m-2) \cdot \alpha_{m-1} \leq n - 2\} \\ &= \sum_{j=0}^{n-2} \#\{(\alpha_2, \dots, \alpha_{m-1}) \mid 1 \cdot \alpha_2 + 2 \cdot \alpha_3 + \dots + (m-2) \cdot \alpha_{m-1} = j\} \\ &= \sum_{j=0}^{n-2} p_{m-2}(j). \end{aligned}$$

Since $m - 2 = n - 1 - \beta$ we obtain the required dimension. \square

Note that for $\beta = 1$ we obtain the bound from [3]:

$$\mu(\mathfrak{f}) \leq f(n, 1) = 1 + \sum_{j=0}^{n-2} p(j) < 1 + e^{\sqrt{2\pi(n-1)/3}}.$$

Here $p(j)$ denotes the unrestricted partition function, and $p(0) = 1$. The following result shows that our bound from the above proposition yields an improvement.

Proposition 4.6. *Let $n \geq 3$. Then $f(n, \beta)$ is monotonic in β , i.e., it holds*

$$f(n, n - 1) \leq f(n, n - 2) \leq \dots \leq f(n, 2) = f(n, 1),$$

with equality for $\beta = 1$ and $\beta = 2$.

The proof is easy, and we leave it to the reader. We can also determine $f(n, \beta)$ explicitly for large β :

Proposition 4.7. *Let $n \geq 4$. Then it holds*

$$\begin{aligned} f(n, n-1) &= n, \\ f(n, n-2) &= 2n-3, \\ f(n, n-3) &= \frac{n^2 + 3n - 12 + 2\lfloor n/2 \rfloor}{4}. \end{aligned}$$

If $\beta = n-1$, then $\beta = \beta(\mathfrak{f})$, and \mathfrak{f} is the standard graded filiform Lie algebra. Then the bound $\mu(\mathfrak{f}) \leq f(n, n-1) = n$ is optimal, since we already know that $\mu(\mathfrak{f}) = n$ in this case. See also example 3.9 for the case $n = 4$.

Remark 4.8. It is also easy to show that

$$f(n, \beta) \leq \beta + \frac{(2n - \beta - 3)^{n-\beta-1}}{(n - \beta - 1)!}$$

for all $n \geq 3$ and all $1 \leq \beta \leq n-1$.

We can also derive a bound on $\mu(\mathfrak{f})$ which only depends on n . For this we take the smallest possible $\beta = \beta(\mathfrak{f})$ in terms of n , which is given by $\beta = \lceil n/2 \rceil$. Then $n-1-\beta = \lfloor n/2 \rfloor - 1$, and we obtain the following result:

Corollary 4.9. *Let \mathfrak{f} be a filiform nilpotent Lie algebra of dimension $n \geq 3$. Then*

$$\mu(\mathfrak{f}) \leq n-1 + \sum_{j=0}^{n-2} p_{\lfloor \frac{n}{2} \rfloor - 1}(j).$$

4.4. Filiform Lie algebras of dimension 10. We may represent all complex filiform Lie algebras of dimension 10 with respect to an adapted basis (x_1, \dots, x_{10}) as a family of Lie algebras $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$, with 13 parameters satisfying the following polynomial equations:

$$\begin{aligned}\alpha_{11}(2\alpha_1 + \alpha_7) - 3\alpha_7^2 &= 0, \\ \alpha_{13}(2\alpha_1 - \alpha_7 - \alpha_{11}) &= 0, \\ \alpha_{13}(2\alpha_3 + \alpha_9) - \alpha_{12}(2\alpha_1 + \alpha_7) &= 3\alpha_{11}(\alpha_2 + \alpha_8) - 7\alpha_7\alpha_8.\end{aligned}$$

We call the parameters *admissible*, if they define a Lie algebra, i.e., if they satisfy these equations. Note that we obtain other equations as consequences, such as

$$\alpha_{13}(\alpha_1^2 - \alpha_7^2) = 0.$$

The explicit Lie brackets are given as follows:

$$\begin{aligned}[x_1, x_i] &= x_{i+1}, \quad 2 \leq i \leq 9 \\ [x_2, x_3] &= \alpha_1 x_5 + \alpha_2 x_6 + \alpha_3 x_7 + \alpha_4 x_8 + \alpha_5 x_9 + \alpha_6 x_{10} \\ [x_2, x_4] &= \alpha_1 x_6 + \alpha_2 x_7 + \alpha_3 x_8 + \alpha_4 x_9 + \alpha_5 x_{10} \\ [x_2, x_5] &= (\alpha_1 - \alpha_7)x_7 + (\alpha_2 - \alpha_8)x_8 + (\alpha_3 - \alpha_9)x_9 + (\alpha_4 - \alpha_{10})x_{10} \\ [x_2, x_6] &= (\alpha_1 - 2\alpha_7)x_8 + (\alpha_2 - 2\alpha_8)x_9 + (\alpha_3 - 2\alpha_9)x_{10} \\ [x_2, x_7] &= (\alpha_1 - 3\alpha_7 + \alpha_{11})x_9 + (\alpha_2 - 3\alpha_8 + \alpha_{12})x_{10} \\ [x_2, x_8] &= (\alpha_1 - 4\alpha_7 + 3\alpha_{11})x_{10} \\ [x_2, x_9] &= -\alpha_{13}x_{10} \\ [x_3, x_4] &= \alpha_7 x_7 + \alpha_8 x_8 + \alpha_9 x_9 + \alpha_{10} x_{10} \\ [x_3, x_5] &= \alpha_7 x_8 + \alpha_8 x_9 + \alpha_9 x_{10} \\ [x_3, x_6] &= (\alpha_7 - \alpha_{11})x_9 + (\alpha_8 - \alpha_{12})x_{10} \\ [x_3, x_7] &= (\alpha_7 - 2\alpha_{11})x_{10} \\ [x_3, x_8] &= \alpha_{13}x_{10} \\ [x_4, x_5] &= \alpha_{11}x_9 + \alpha_{12}x_{10} \\ [x_4, x_6] &= \alpha_{11}x_{10} \\ [x_4, x_7] &= -\alpha_{13}x_{10} \\ [x_5, x_6] &= \alpha_{13}x_{10}\end{aligned}$$

We want to determine as good as possible upper bounds on $\mu(\mathfrak{f})$, for all Lie algebras $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$. The results will depend on the parameters, and we have to introduce a case distinction. For each case we choose a particular construction which yields a faithful \mathfrak{f} -module V of some dimension $10 \leq \dim(V) \leq 18$. This improves the known bound $10 \leq \mu(\mathfrak{f}) \leq 22$ from [2] for such Lie algebras. We can also construct a faithful \mathfrak{f} -module $V = V(\alpha_1, \dots, \alpha_{13})$, which does not depend on a case distinction for the parameters. In other words, such a module gives an upper

bound on $\mu(\mathfrak{f})$ for all admissible parameters at the same time. We call such a module a *general \mathfrak{f} -module*. We will give such a module explicitly.

Proposition 4.10. *There is a general faithful \mathfrak{f} -module $V_{58} = V_{58}(\alpha_1, \dots, \alpha_{13})$ of dimension 58.*

Proof. The faithful \mathfrak{f} -module V_{58} is obtained by proposition 4.5 as follows. Take $J = \langle x_6, \dots, x_{10} \rangle$ as compatible ideal. This means $\beta = 5$ and the construction yields a module with a basis consisting of $f(10, 5) = 58$ monomials. The computation of $f(10, 5)$ uses $(p_4(0), \dots, p_4(8)) = (1, 1, 2, 3, 5, 6, 9, 11, 15)$. The basis consists of the following standard monomials, writing x_i for \bar{X}_i .

order	monomials
0	1
1	x_2
2	x_3, x_2^2
3	x_4, x_2x_3, x_2^3
4	$x_5, x_2x_4, x_3^2, x_2^2x_3, x_2^4$
5	$x_6, x_3x_4, x_2x_5, x_2^2x_4, x_2x_3^2x_3, x_2^3x_3, x_2^5$
6	$x_7, x_4^2, x_3x_5, x_2x_3x_4, x_3^3, x_2^2x_5, x_2^3x_4, x_2^2x_3^2, x_2^4x_3, x_2^6$
7	$x_8, x_4x_5, x_2x_4^2, x_2^2x_4, x_2x_3x_5, x_2^2x_3x_4, x_2x_3^3, x_2^3x_5, x_2^4x_4, x_2^3x_5^2, x_2^5x_3, x_2^7$
8	$x_9, x_3x_4^2, x_5^2, x_2x_4x_5, x_3^2x_5, x_2^2x_4^2, x_2x_3^2x_4, x_3^4, x_2^2x_3x_5, x_2^3x_3x_4$
9	$x_2^2x_3^3, x_2^4x_5, x_2^5x_4, x_2^4x_3^2, x_2^6x_3, x_2^8$ x_{10}

Denote this basis by v_1, \dots, v_{58} , ordered lexicographically. Note that $v_{58} = x_{10}$ generates the center of \mathfrak{f} . The module is determined by the action of the generators x_1 and x_2 of the Lie algebra $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$. It is given by

$$\begin{aligned}
 x_1.v_1 &= 0, \\
 x_1.v_2 &= v_3, \\
 x_1.v_3 &= v_5, \\
 x_1.v_4 &= 2v_6 - \alpha_1v_8 - \alpha_2v_{13} - \alpha_3v_{20} - \alpha_4v_{30} - \alpha_5v_{42} - \alpha_6v_{58}, \\
 x_1.v_5 &= v_8, \\
 x_1.v_6 &= v_9 + v_{10}, \\
 x_1.v_7 &= 3v_{11} - 3\alpha_1v_{15} + \alpha_1(\alpha_1 - \alpha_7)v_{20} + (2\alpha_1\alpha_2 - 2\alpha_2\alpha_7 - \alpha_1\alpha_8)v_{30} \\
 &\quad + (2\alpha_1\alpha_3 - \alpha_1\alpha_9 + \alpha_{11}\alpha_3 + \alpha_2^2 - 2\alpha_2\alpha_8 - 3\alpha_3\alpha_7)v_{42} + (2\alpha_1\alpha_4 - \alpha_1\alpha_{10} \\
 &\quad + 3\alpha_{11}\alpha_4 + \alpha_{12}\alpha_3 - \alpha_{13}\alpha_5 + 2\alpha_2\alpha_3 - 2\alpha_2\alpha_9 - 3\alpha_3\alpha_8 - 4\alpha_4\alpha_7)v_{58},
 \end{aligned}$$

$$\begin{aligned}
x_1.v_8 &= v_{13}, \\
x_1.v_9 &= v_{14} + v_{15}, \\
x_1.v_{10} &= 2v_{14} - \alpha_7v_{20} - \alpha_8v_{30} - \alpha_9v_{42} - \alpha_{10}v_{58}, \\
x_1.v_{11} &= v_{16} + 2v_{17} - \alpha_1v_{22} + \alpha_1\alpha_7v_{30} + (\alpha_1\alpha_8 - \alpha_{11}\alpha_2 + \alpha_2\alpha_7)v_{42} \\
&\quad + (\alpha_1\alpha_9 - 2\alpha_{11}\alpha_3 - \alpha_{12}\alpha_2 + \alpha_{13}\alpha_4 + \alpha_2\alpha_8 + \alpha_3\alpha_7)v_{58}, \\
x_1.v_{12} &= 4v_{18} - 6\alpha_1v_{25} + \alpha_1(4\alpha_1\alpha_7 - \alpha_1^2 - 3\alpha_1\alpha_{11})v_{42} \\
&\quad + (4\alpha_1^2\alpha_8 - \alpha_1^2\alpha_{12} - 3\alpha_1^2\alpha_2 - 6\alpha_1\alpha_{11}\alpha_2 + 3\alpha_1\alpha_{11}\alpha_8 + \alpha_1\alpha_{12}\alpha_7 + 2\alpha_1\alpha_{13}\alpha_3 \\
&\quad - \alpha_1\alpha_{13}\alpha_9 + 11\alpha_1\alpha_2\alpha_7 - 7\alpha_1\alpha_7\alpha_8 + \alpha_{11}\alpha_{13}\alpha_3 + 6\alpha_{11}\alpha_2\alpha_7 + \alpha_{13}\alpha_2^2 \\
&\quad - 2\alpha_{13}\alpha_2\alpha_8 - 3\alpha_{13}\alpha_3\alpha_7 - 8\alpha_2\alpha_7^2)v_{58}, \\
x_1.v_{13} &= v_{20}, \\
x_1.v_{14} &= v_{21} + v_{22}, \\
x_1.v_{15} &= v_{22}, \\
x_1.v_{16} &= 2v_{23} + v_{25} - \alpha_1v_{31} + \alpha_1\alpha_{11}v_{42} + (\alpha_1\alpha_{12} + \alpha_{11}\alpha_2 - \alpha_{13}\alpha_3)v_{58}, \\
x_1.v_{17} &= 2v_{23} + v_{24}, \\
x_1.v_{18} &= v_{26} + 3v_{27} - 3\alpha_1v_{34} + (2\alpha_1^2\alpha_{11} - \alpha_1^2\alpha_7 - 2\alpha_1\alpha_{11}\alpha_7 - 2\alpha_1\alpha_{13}\alpha_2 \\
&\quad + \alpha_1\alpha_{13}\alpha_8 + \alpha_1\alpha_7^2 + 2\alpha_{13}\alpha_2\alpha_7)v_{58}, \\
x_1.v_{19} &= 5v_{28} - 10\alpha_1v_{37} + \alpha_1\alpha_{13}(4\alpha_1\alpha_7 - \alpha_1^2 - 3\alpha_1\alpha_{11})v_{58}, \\
x_1.v_{20} &= v_{30}, \\
x_1.v_{21} &= 2v_{31} - \alpha_{11}v_{42} - \alpha_{12}v_{58}, \\
x_1.v_{22} &= v_{31}, \\
x_1.v_{23} &= v_{32} + v_{33} + v_{34}, \\
x_1.v_{24} &= 3v_{33} + (\alpha_7^2 - 2\alpha_{11}\alpha_7 + \alpha_{13}\alpha_8)v_{58}, \\
x_1.v_{25} &= 2v_{34} - \alpha_1v_{44} + \alpha_{13}\alpha_2v_{58}, \\
x_1.v_{26} &= 3v_{35} + v_{37} - 3\alpha_1v_{45} + \alpha_1\alpha_{13}(\alpha_1 - \alpha_7)v_{58}, \\
x_1.v_{27} &= 2v_{35} + 2v_{36} - \alpha_1v_{46} - \alpha_1\alpha_{13}\alpha_7v_{58}, \\
x_1.v_{28} &= v_{38} + 4v_{39} - 6\alpha_1v_{50}, \\
x_1.v_{29} &= 6v_{40} - 15\alpha_1v_{53}, \\
x_1.v_{30} &= v_{42}, \\
x_1.v_{31} &= v_{44}, \\
x_1.v_{32} &= v_{43} + 2v_{45}, \\
x_1.v_{33} &= 2v_{43} + v_{46} - \alpha_{13}\alpha_7v_{58}, \\
x_1.v_{34} &= v_{45} + v_{46}, \\
x_1.v_{35} &= v_{47} + 2v_{48} + v_{50},
\end{aligned}$$

$$\begin{aligned}
x_1.v_{36} &= 3v_{48} + v_{49}, \\
x_1.v_{37} &= 3v_{50}, \\
x_1.v_{38} &= 4v_{51} + v_{53}, \\
x_1.v_{39} &= 2v_{51} + 3v_{52}, \\
x_1.v_{40} &= v_{54} + 5v_{55}, \\
x_1.v_{41} &= 7v_{56}, \\
x_1.v_{42} &= v_{58}, \\
x_1.v_{43} &= 0, \\
x_1.v_{44} &= -\alpha_{13}v_{58}, \\
x_1.v_{45} &= \cdots = x_1.v_{58} = 0.
\end{aligned}$$

$$\begin{aligned}
x_2.v_1 &= v_2, & x_2.v_2 &= v_4, & x_2.v_3 &= v_6, & x_2.v_4 &= v_7, & x_2.v_5 &= v_9, \\
x_2.v_6 &= v_{11}, & x_2.v_7 &= v_{12}, & x_2.v_8 &= v_{15}, & x_2.v_9 &= v_{16}, & x_2.v_{10} &= v_{17}, \\
x_2.v_{11} &= v_{18}, & x_2.v_{12} &= v_{19}, & x_2.v_{13} &= 0, & x_2.v_{14} &= v_{23}, & x_2.v_{15} &= v_{25}, \\
x_2.v_{16} &= v_{26}, & x_2.v_{17} &= v_{27}, & x_2.v_{18} &= v_{28}, & x_2.v_{19} &= v_{29}, & x_2.v_{20} &= 0, \\
x_2.v_{21} &= v_{32}, & x_2.v_{22} &= v_{34}, & x_2.v_{23} &= v_{35}, & x_2.v_{24} &= v_{36}, & x_2.v_{25} &= v_{37}, \\
x_2.v_{26} &= v_{38}, & x_2.v_{27} &= v_{39}, & x_2.v_{28} &= v_{40}, & x_2.v_{29} &= v_{41}, & x_2.v_{30} &= 0, \\
x_2.v_{31} &= v_{45}, & x_2.v_{32} &= v_{47}, & x_2.v_{33} &= v_{48}, & x_2.v_{34} &= v_{50}, & x_2.v_{35} &= v_{51}, \\
x_2.v_{36} &= v_{52}, & x_2.v_{37} &= v_{53}, & x_2.v_{38} &= v_{54}, & x_2.v_{39} &= v_{55}, & x_2.v_{40} &= v_{56}, \\
x_2.v_{41} &= v_{57}, & x_2.v_{42} &= \cdots = x_2.v_{58} &= 0.
\end{aligned}$$

□

Corollary 4.11. *There is a general faithful \mathfrak{f} -module $V_{20} = V_{20}(\alpha_1, \dots, \alpha_{13})$ of dimension 20.*

Proof. We apply the algorithm **Quotient** from [6] to the module V_{58} . This works as follows. The space of invariants is given by

$$V_{58}^{\mathfrak{f}} = \langle \alpha_{13}v_{42} + v_{44}, v_{43}, v_{45}, \dots, v_{58} \rangle,$$

with $\dim(V_{58}^{\mathfrak{f}}) = 16$ for all parameters $\alpha_1, \dots, \alpha_{13}$. We choose a complement U of $Z(\mathfrak{f}) = \langle v_{58} \rangle$ in $V_{58}^{\mathfrak{f}}$ by taking the above basis for $V_{58}^{\mathfrak{f}}$ except for v_{58} . Then U is a submodule such that the quotient $V_{43} = V_{58}/U$ is a faithful module of dimension 43. For the quotient, we may write the following relations

$$\begin{aligned}
v_{43} &= 0, \\
v_{44} &= -\alpha_{13}v_{42}, \\
v_{45} &= \cdots = v_{57} = 0.
\end{aligned}$$

In other words, we may view $v_1, \dots, v_{42}, v_{58}$ as a basis of V_{43} . Now we repeat this procedure. We have

$$V_{43}^{\mathfrak{f}} = \langle \alpha_{13}v_{30} + v_{31}, v_{32}, v_{33} + \alpha_7\alpha_{13}v_{42}, v_{34}, \dots, v_{41}, v_{58} \rangle,$$

with $\dim(V_{43}^{\mathfrak{f}}) = 12$ for all parameters $\alpha_1, \dots, \alpha_{13}$. We choose U from $V_{43}^{\mathfrak{f}}$ by omitting v_{58} , and obtain a faithful quotient $V_{32} = V_{43}^{\mathfrak{f}}/U$ of dimension 32. We can take the following quotient relations

$$\begin{aligned} v_{31} &= -\alpha_{13}v_{30}, \\ v_{32} &= 0, \\ v_{33} &= -\alpha_7\alpha_{13}v_{42}, \\ v_{34} &= \dots = v_{41} = 0. \end{aligned}$$

In the next step we obtain $\dim(V_{32}^{\mathfrak{f}}) = 10$ for all parameters $\alpha_1, \dots, \alpha_{13}$. Choosing a complement U as above we obtain a faithful module $V_{23} = V_{32}^{\mathfrak{f}}/U$ of dimension 23, where the relations are given by

$$\begin{aligned} v_{21} &= -2\alpha_{13}v_{20} - \alpha_{11}v_{30} - \alpha_{12}v_{42}, \\ v_{22} &= -\alpha_{13}v_{20}, \\ &\vdots = \vdots \\ v_{29} &= 0. \end{aligned}$$

The dimension of the space of invariants $V_{23}^{\mathfrak{f}}$ however does depend on the parameters. It can be of dimension 5, 6 or 7, depending on certain case distinctions. Without case distinction we can still choose some subspace U of invariants not containing v_{58} , which need not be a maximal with this property. This way we arrive at a faithful quotient V_{20} of dimension 20. If we continue with case distinctions we obtain many different faithful quotients V of dimensions $10 \leq \dim(V) \leq 18$. The quotient algorithm stops if the space of invariants is 1-dimensional, spanned by v_{58} . Then there is no faithful quotient of lower dimension. \square

Remark 4.12. Note that the choice of the complements U in the quotient algorithm is not unique. For our choice we obtained faithful modules of dimensions 58, 43, 32 and 23. In general, the dimensions might depend on U . However, taking quotients by invariants is no restriction. In fact, the following result is easy to show: let \mathfrak{n} be a nilpotent Lie algebra, and V be a nilpotent \mathfrak{n} -module. Then every faithful quotient of V can be obtained by taking successive quotients by invariants.

Example 4.13. Consider the Lie algebra $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ with

$$(\alpha_1, \dots, \alpha_{13}) = (1, 0, 0, 0, 0, 0, -1, 1, 0, 0, 3, -16, 1).$$

We have $\mu(\mathfrak{f}) \geq 12$, and \mathfrak{f} admits no affine structure, see [2]. The above algorithm yields a faithful quotient of V_{58} of dimension 18. Hence we have $\mu(\mathfrak{f}) \leq 18$, and this is up to now the best known estimate.

Note that the above Lie algebra has minimal β -invariant, namely $\beta(\mathfrak{f}) = 5$. The Betti numbers are given by $(b_0, \dots, b_{10}) = (1, 2, 3, 5, 6, 6, 6, 5, 3, 2, 1)$.

We come back to finding as good as possible estimates on $\mu(\mathfrak{f})$ for all filiform Lie algebras $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ of dimension 10. Therefore we need to consider different choices of admissible parameters, which give well defined classes of filiform Lie algebras. The cases are as follows:

Case 1: $2\alpha_1 + \alpha_7 = 0$.

Case 2: $2\alpha_1 + \alpha_7 \neq 0$.

Case 2a: $\alpha_{13} \neq 0, \alpha_7^2 = \alpha_1^2 \neq 0$.

Case 2a1: $\alpha_7 = \alpha_1$.

Case 2a2: $\alpha_7 = -\alpha_1$.

Case 2a2a: $3\alpha_2 + \alpha_8 = 0$.

Case 2a2b: $3\alpha_2 + \alpha_8 \neq 0$.

Case 2b: $\alpha_{13} = 0$.

Case 2b1: $\alpha_7^2 \neq \alpha_1^2$.

Case 2b2: $\alpha_7^2 = \alpha_1^2$.

Case 2b2a: $\alpha_7 = \alpha_1$.

Case 2b2b: $\alpha_7 = -\alpha_1$.

Case 2b2b1: $3\alpha_2 + \alpha_8 = 0$.

Case 2b2b2: $3\alpha_2 + \alpha_8 \neq 0$.

Lemma 4.14. *All above conditions are isomorphism invariants. In particular, algebras of different cases are non-isomorphic.*

Proof. Using the β -invariant we have

$$\alpha_1 = 0 \Leftrightarrow \beta(\mathfrak{f}/\mathfrak{f}^5) = 4,$$

$$\alpha_7 = 0 \Leftrightarrow \beta(\mathfrak{f}^2/\mathfrak{f}^7) = 5,$$

$$\alpha_{11} = 0 \Leftrightarrow \beta(\mathfrak{f}^3/\mathfrak{f}^9) = 6,$$

$$\alpha_{13} = 0 \Leftrightarrow \beta(\mathfrak{f}^4/\mathfrak{f}^{11}) = 6,$$

$$\alpha_7 = \alpha_1 \Leftrightarrow \beta(\mathfrak{f}/\mathfrak{f}^2 \ltimes \mathfrak{f}^4/\mathfrak{f}^7) = 4.$$

The Lie algebras of case 1 satisfy $2\alpha_1 + \alpha_7 = 0$, which is equivalent to $\alpha_1 = \alpha_7 = 0$. The above table shows that these conditions are isomorphism invariants. Hence the Lie algebras of case 1 and case 2 are well defined. The same applies to case 2a and case 2b, because $\alpha_{13} \neq 0$ and $\alpha_{13} = 0$ are isomorphism invariants. Recall that $\alpha_{13} \neq 0$ implies $\alpha_7^2 = \alpha_1^2$. The claim is also clear for the cases 2a1, 2a2. Note that $\alpha_7 = \alpha_1 \neq 0$ is also equivalent to the conditions $\beta(\mathfrak{f}^2/\mathfrak{f}^7) \neq 5$ and $[\mathfrak{f}^2, \mathfrak{f}^5] = \mathfrak{f}^9$. As we will see in proposition 4.18, the Lie algebras of case 2b1 are well defined. Finally, for the cases with $\alpha_7 = -\alpha_1 \neq 0$ the condition $3\alpha_2 + \alpha_8 = 0$ is equivalent to the

fact, that the Lie algebra $\mathfrak{f}/\mathfrak{f}^8$ admits an invertible derivation. Hence this condition is also an isomorphism invariant. \square

For each case we have a result on $\mu(\mathfrak{f})$. Let us start with the first case.

Proposition 4.15. *Let $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 satisfying $2\alpha_1 + \alpha_7 = 0$. Then $\mu(\mathfrak{f}) = 10$.*

Proof. The parameters are admissible iff $\alpha_1 = \alpha_7 = 0$ and $\alpha_{11}(\alpha_2 + \alpha_8) = 0$. To construct a module for \mathfrak{f} we need to find two operators $L(x_1)$ and $L(x_2)$, which define $L(x_i) := [L(x_1), L(x_{i-1})]$ for $i \geq 3$, so that the conditions $L([x_i, x_j]) = [L(x_i), L(x_j)]$ are satisfied for all $i, j \geq 1$. This module is faithful if and only if $L(x_{10})$ is nonzero. It is easy to see that we can always find such operators, by taking $L(x_1) = \text{ad}(x_1)$ and $L(x_2)$ some 10×10 lower-triangular matrix. However, the construction depends on different cases, such as $\alpha_{13} \neq 0$, or $\alpha_{13} = 0$ with $\alpha_{11} \neq 0, \alpha_2 \neq 0$, with $\alpha_{11} \neq 0, \alpha_2 = 0$, or with $\alpha_{11} = 0$. For more details see [2]. \square

For case 2a we have the following results, see [2]:

Proposition 4.16. *Let $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 satisfying $2\alpha_1 + \alpha_7 \neq 0, \alpha_{13} \neq 0$ and $\alpha_7 = \alpha_1$. Then $\mu(\mathfrak{f}) \leq 11$.*

Proposition 4.17. *Let $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 satisfying $2\alpha_1 + \alpha_7 \neq 0, \alpha_{13} \neq 0$ and $\alpha_7 = -\alpha_1$. Then $\mu(\mathfrak{f}) \leq 11$ if and only if $3\alpha_2 + \alpha_8 = 0$. Otherwise we have $\mu(\mathfrak{f}) \leq 18$.*

In this case the module V_{58} from proposition 4.10 always has a faithful quotient of dimension 18. This can be seen by applying the quotient algorithm as in corollary 4.11. For $3\alpha_2 + \alpha_8 \neq 0$ this is the best bound known so far. The example given in 4.13 belongs to this class.

For case 2b we have the following results, see [2] and [4]:

Proposition 4.18. *Let $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 satisfying $2\alpha_1 + \alpha_7 \neq 0$. Then \mathfrak{f} admits a central extension $0 \rightarrow Z(\mathfrak{h}) \rightarrow \mathfrak{h} \rightarrow \mathfrak{f} \rightarrow 0$ by some filiform nilpotent Lie algebra \mathfrak{h} if and only if $\alpha_{13} = 0$ and $\alpha_1^2 \neq \alpha_7^2$, in which case we have $\mu(\mathfrak{f}) = 10$.*

Proposition 4.19. *Let $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 satisfying $2\alpha_1 + \alpha_7 \neq 0, \alpha_{13} = 0$ and $\alpha_7 = \alpha_1$. Then $\mu(\mathfrak{f}) \leq 11$.*

Proposition 4.20. *Let $\mathfrak{f} = \mathfrak{f}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 satisfying $2\alpha_1 + \alpha_7 \neq 0, \alpha_{13} = 0$ and $\alpha_7 = -\alpha_1$. Then $\mu(\mathfrak{f}) \leq 11$ if and only if $3\alpha_2 + \alpha_8 = 0$. Otherwise we have $\mu(\mathfrak{f}) \leq 15$.*

Here we use proposition 4.10 for the subcase $3\alpha_2 + \alpha_8 \neq 0$. Then the module V_{58} has a faithful quotient of dimension 15. In fact, for some cases, it even has a faithful quotient of dimension 12, 13 or 14.

REFERENCES

- [1] G. Birkhoff: *Representability of Lie algebras and Lie groups by matrices*. Annals of Mathematics **38** (1937), 526–532.
- [2] D. Burde: *Affine structures on nilmanifolds*. Int. J. of Math. **7** (1996), no. 5, 599–616.
- [3] D. Burde: *A refinement of Ado's Theorem*. Archiv Math. **70** (1998), 118–127.
- [4] D. Burde: *Affine cohomology classes for filiform Lie algebras*. Contemporary Mathematics **262** (2000), 159–170.
- [5] D. Burde, W. Moens: *Minimal faithful representations of reductive Lie algebras*. Archiv der Mathematik **89** (2007), no. 6, 513–523.
- [6] D. Burde, B. Eick, W. de Graaf: *Computing faithful representations for nilpotent Lie algebras*. J. Algebra **22** (2009), no. 3, 602–612.
- [7] L. Cagliero, N. Rojas: *Faithful representations of minimal dimension of current Heisenberg Lie algebras*. Internat. J. Math. **20** (2009), no. 11, 1347–1362.
- [8] W. A. de Graaf: *Constructing faithful matrix representations of Lie algebras*. Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, ISSAC'97, (1997), 54–59. ACM Press New York.
- [9] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. Math. **25** (1977), 178–187.
- [10] Y. Neretin: *A construction of finite-dimensional faithful representation of Lie algebra*. Rend. Circ. Mat. Palermo (2) Suppl. No. **71** (2003), 159–161.
- [11] M. Vergne: *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes*. Bull. Soc. Math. France **98** (1970), 81–116.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, 1090 WIEN,
AUSTRIA

E-mail address: `dietrich.burde@univie.ac.at`

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, 1090 WIEN,
AUSTRIA

E-mail address: `wolfgang.moens@univie.ac.at`