ABELIAN IDEALS OF MAXIMAL DIMENSION FOR SOLVABLE LIE ALGEBRAS

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ABSTRACT. We compare the maximal dimension of abelian subalgebras and the maximal dimension of abelian ideals for finite-dimensional Lie algebras. We show that these dimensions coincide for solvable Lie algebras over an algebraically closed field of characteristic zero. We compute this invariant for all complex nilpotent Lie algebras of dimension $n \leq 7$. Furthermore we study the case where there exists an abelian subalgebra of codimension 2. Here we explicitly construct an abelian ideal of codimension 2 in case of nilpotent Lie algebras.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Denote by $\alpha(\mathfrak{g})$ the maximal dimension of an abelian subalgebra of $\mathfrak{g}$, and by $\beta(\mathfrak{g})$ the maximal dimension of an abelian ideal of $\mathfrak{g}$. Both invariants are important for many subjects. First of all they are very useful invariants in the study of Lie algebra contractions and degenerations. There is a large literature, in particular for low-dimensional Lie algebras, see [11, 3, 17, 21, 10], and the references given therein. Secondly, there are several results concerning the question of how big or small these maximal dimensions can be, compared to the dimension of the Lie algebra. For references see [20, 19, 15]. The results show, roughly speaking, that a Lie algebra of large dimension contains abelian subalgebras of large dimension. For example, the dimension of a nilpotent Lie algebra $\mathfrak{g}$ satisfying $\alpha(\mathfrak{g}) = \ell$ is bounded by $\dim(\mathfrak{g}) \leq \frac{\ell(\ell+1)}{2}$ [20, 19]. There is a better bound for 2-step nilpotent Lie algebras, see [16]. If $\mathfrak{g}$ is a complex solvable Lie algebra with $\alpha(\mathfrak{g}) = \ell$, then we have $\dim(\mathfrak{g}) \leq \frac{\ell(\ell+3)}{2}$, see [15]. In general, $\dim(\mathfrak{g}) \leq \frac{\ell(\ell+17)}{2}$ for any complex Lie algebra $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = \ell$, see [15].
For semisimple Lie algebras $\mathfrak{s}$ the invariant $\alpha(\mathfrak{s})$ has been completely determined by Malcev [9]. Since there are no nontrivial abelian ideals in $\mathfrak{s}$, we have $\beta(\mathfrak{s}) = 0$. Recently the study of abelian ideals in a Borel subalgebra $\mathfrak{b}$ of a simple complex Lie algebra $\mathfrak{s}$ has drawn considerable attention. We have indeed $\alpha(\mathfrak{s}) = \beta(\mathfrak{b})$, and this number can be computed purely in terms of certain root system invariants, see [22]. The result is reproduced for the interested reader in table 1. Furthermore Kostant found a relation of these invariants to discrete series representations of the corresponding Lie group, and to powers of the Euler product [12, 13]. In fact, there are many more results concerning the invariants α and β for simple Lie algebras and their Borel subalgebras.
In this paper, we want to point out the following interesting result: if $\mathfrak{g}$ is a solvable Lie algebra over an algebraically closed field of characteristic zero, then we have $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$. This means, given an abelian subalgebra of maximal dimension m there exists also an abelian ideal of dimension m.

TABLE 1. The invariant α for simple Lie algebras

$\mathfrak{s}$	$\dim(\mathfrak{s})$	$\alpha(\mathfrak{s})$
A_n, $n \geq 1$	$n(n+2)$	$\lfloor (\frac{n+1}{2})^2 \rfloor$
B_3	21	5
B_n, $n \geq 4$	$n(2n+1)$	$\frac{n(n-1)}{2} + 1$
C_n, $n \geq 2$	$n(2n+1)$	$\frac{n(n+1)}{2}$
D_n, $n \geq 4$	$n(2n-1)$	$\frac{n(n-1)}{2}$
G_2	14	3
F_4	52	9
E_6	78	16
E_7	133	27
E_8	248	36

For a given value of $\alpha(\mathfrak{g})$ the dimension of $\mathfrak{g}$ is bounded in terms of this value, as we have mentioned before. Hence it is natural to ask what we can say on n-dimensional Lie algebras $\mathfrak{g}$ where the value of $\alpha(\mathfrak{g})$ is close to n. Indeed, if $\alpha(\mathfrak{g}) = n$, then $\mathfrak{g}$ is abelian, $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$, and we are done. If $\alpha(\mathfrak{g}) = n - 1$, then also $\beta(\mathfrak{g}) = n - 1$. This means that $\mathfrak{g}$ has an abelian ideal of codimension 1 and is almost abelian. In particular, $\mathfrak{g}$ is 2-step solvable. In this case the structure of $\mathfrak{g}$, and even all its degenerations are quite well understood, see [10].

After these two easy cases it is reasonable to consider Lie algebras $\mathfrak{g}$ satisfying $\alpha(\mathfrak{g}) = n - 2$. Here we can classify all such non-solvable complex Lie algebras. However, for solvable Lie algebras we cannot expect to obtain a classification, not even in the nilpotent case. In fact, there exist even characteristically nilpotent Lie algebras $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = n - 2$. On the other hand we know that $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$, so that there is an abelian ideal of codimension 2. For many problems concerning the cohomology of nilpotent Lie algebras the subclass of those having an abelian ideal of codimension 1 or 2 is very important, see [1, 18] and the references given therein.

The structure of the paper is as follows. In section 2 we prove basic results concerning the invariants $\alpha(\mathfrak{g})$ and $\beta(\mathfrak{g})$. For a Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ of $\mathfrak{g}$ we show that $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) \leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r})$. The main result of this section is, as mentioned above, that $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$ for solvable Lie algebras over an algebraically closed field of characteristic zero. An example is given that the statement is not true in general for the field of real numbers.

In section 3 we construct an abelian ideal of codimension 1 for a Lie algebra $\mathfrak{g}$ satisfying $\alpha(\mathfrak{g}) = n - 1$. In section 4 we show that Lie algebras $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = n - 2$ are solvable or isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^\ell$ for some $\ell \geq 0$. In section 5 we study nilpotent Lie algebras $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = n - 2$ and explicitly construct an abelian ideal of codimension 2. Finally the invariants are computed for all complex nilpotent Lie algebras of dimension $n \leq 7$. This is accompanied by a remark on Lie algebra degenerations, where these invariants are really useful.

2. The invariants $\alpha(\mathfrak{g})$ and $\beta(\mathfrak{g})$

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra of dimension n over a field K. If not stated otherwise we assume that K is the field of complex numbers. Some results will also hold for other fields, but we are mainly interested in the case of complex numbers. Consider the following invariants of $\mathfrak{g}$:

$$\alpha(\mathfrak{g}) = \max\{\dim(\mathfrak{a}) \mid \mathfrak{a} \text{ is an abelian subalgebra of } \mathfrak{g}\},$$
$$\beta(\mathfrak{g}) = \max\{\dim(\mathfrak{b}) \mid \mathfrak{b} \text{ is an abelian ideal of } \mathfrak{g}\}.$$

An abelian subalgebra of maximal dimension is maximal abelian with respect to inclusion. However, a maximal abelian subalgebra need not be of maximal dimension:

Example 2.2. *Let $\mathfrak{f}_n$ be the standard graded filiform nilpotent Lie algebra of dimension n. Let $(e_1, \ldots, e_n)$ be a standard basis, such that $[e_1, e_i] = e_{i+1}$ for $2 \le i \le n-1$. Then $\mathfrak{a} = \langle e_1, e_n \rangle$ is a maximal abelian subalgebra of dimension 2, but $\alpha(\mathfrak{f}_n) = \beta(\mathfrak{f}_n) = n-1$.*

Clearly we have $\beta(\mathfrak{g}) \le \alpha(\mathfrak{g})$. In general, the two invariants are different. A complex semisimple Lie algebra $\mathfrak{s}$ has no abelian ideals, hence $\beta(\mathfrak{s}) = 0$. We already saw in table 1 that this is not true for the invariant $\alpha(\mathfrak{s})$. As mentioned before the following result holds, see [22]:

Proposition 2.3. *Let $\mathfrak{s}$ be a complex simple Lie algebra and $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{s}$. Then the maximal dimension of an abelian ideal in $\mathfrak{b}$ coincides with the maximal dimension of a commutative subalgebra of $\mathfrak{s}$, i.e., $\alpha(\mathfrak{s}) = \beta(\mathfrak{b})$. Furthermore the number of abelian ideals in $\mathfrak{b}$ is $2^{\mathrm{rank}(\mathfrak{s})}$.*

This implies $\alpha(\mathfrak{b}) = \beta(\mathfrak{b})$, because we have $\alpha(\mathfrak{b}) \le \alpha(\mathfrak{s}) = \beta(\mathfrak{b})$, since α is monotone:

Lemma 2.4. *The invariant α is monotone and additive: for a subalgebra $\mathfrak{h} \le \mathfrak{g}$ of $\mathfrak{g}$ we have $\alpha(\mathfrak{h}) \le \alpha(\mathfrak{g})$, and for two Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ we have $\alpha(\mathfrak{a} \oplus \mathfrak{b}) = \alpha(\mathfrak{a}) + \alpha(\mathfrak{b})$.*

The invariant β need not be monotone. For example, consider a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then $\beta(\mathfrak{h}) = 1 > 0 = \beta(\mathfrak{g})$. We also have the following result:

Lemma 2.5. *Let $\mathfrak{g}$ be a complex Lie algebra with Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$. Then $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) \le \alpha(\mathfrak{s}) + \alpha(\mathfrak{r})$.*

Proof. Let $\mathfrak{a}$ be an abelian subalgebra in $\mathfrak{g}$ of maximal dimension and $\pi \colon \mathfrak{s} \ltimes \mathfrak{r} \to \mathfrak{s}$, $(x, a) \to (x, 0)$ be the projection. Restricting this homomorphism of Lie algebras to $\mathfrak{a}$ yields

$$\dim(\mathfrak{a}) = \dim(\ker(\pi_{\mathfrak{a}})) + \dim(\mathrm{im}(\pi_{\mathfrak{a}})).$$

Since $\mathrm{im}(\pi_{\mathfrak{a}})$ is the homomorphic image of a subalgebra of $\mathfrak{s} \ltimes \mathfrak{r}$, we may assume that $\mathrm{im}(\pi_{\mathfrak{a}})$ is an abelian subalgebra of $\mathfrak{s}$. In particular we have $\dim(\mathrm{im}(\pi_{\mathfrak{a}})) \le \alpha(\mathfrak{s})$. Furthermore we have $\ker(\pi_{\mathfrak{a}}) = \mathfrak{a} \cap \mathfrak{r}$. Hence $\ker(\pi_{\mathfrak{a}})$ is an abelian subalgebra of $\mathfrak{r}$ and we have $\dim(\ker(\pi_{\mathfrak{a}})) \le \alpha(\mathfrak{r})$. Finally we obtain

$$\begin{aligned}
\alpha(\mathfrak{s} \ltimes \mathfrak{r}) &= \dim(\mathfrak{a}) \\
&= \dim(\ker(\pi_{\mathfrak{a}})) + \dim(\mathrm{im}(\pi_{\mathfrak{a}})) \\
&\le \alpha(\mathfrak{s}) + \alpha(\mathfrak{r}).
\end{aligned}$$

$\square$

The fact that $\alpha(\mathfrak{b}) = \beta(\mathfrak{b})$ for a Borel subalgebra $\mathfrak{b}$ of a complex simple Lie algebra can be generalized to all complex solvable Lie algebras.

Proposition 2.6. *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field K of characteristic zero. Then $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$.*

Proof. The result follows easily from the proof of Theorem 4.1 of [8]. For the convenience of the reader we give the details. Let G be the adjoint algebraic group of $\mathfrak{g}$. This is the smallest algebraic subgroup of $\operatorname{Aut}(\mathfrak{g})$ such that its Lie algebra $\operatorname{Lie}(G)$ contains $\operatorname{ad}(\mathfrak{g})$. Then $\operatorname{Lie}(G)$ is the algebraic hull of $\operatorname{ad}(\mathfrak{g})$. Since $\operatorname{ad}(\mathfrak{g})$ is solvable, so is $\operatorname{Lie}(G)$. Therefore G is a connected solvable algebraic group. Let $m = \alpha(\mathfrak{g})$. Consider the set $\mathcal{C}$ of all commutative subalgebras of $\mathfrak{g}$ of dimension m. This is, by assumtion, a non-empty set, which can be considered as a subset of the Grassmannian $Gr(\mathfrak{g}, m)$, which is an irreducible complete algebraic variety. Hence $\mathcal{C}$ is a non-empty complete variety, and G operates morphically on it, mapping each commutative subalgebra $\mathfrak{h}$ on $g(\mathfrak{h})$, for $g \in G$. By Borel's fixed point theorem, G has a fixed point I in $\mathcal{C}$, i.e., a subalgebra I of $\mathfrak{g}$ with $g(I) = I$ for all $g \in G$. In particular we have $\operatorname{ad}(x)(I) \subseteq I$ for all $x \in \mathfrak{g}$. Hence I is an abelian ideal of dimension m of $\mathfrak{g}$. $\qquad\square$

Borel's fixed point theorem relies on the closed orbit lemma. As a corollary one can also obtain the theorem of Lie-Kolchin. We note that the assumptions on K are really necessary. The next example shows that we need the field to be algebraically closed.

Example 2.7. *Let $\mathfrak{g}$ be the solvable Lie algebra of dimension 4 over $\mathbb{R}$ defined by*

$$[x_1, x_2] = x_2 - x_3, \quad [x_1, x_4] = 2x_4,$$
$$[x_1, x_3] = x_2 + x_3, \quad [x_2, x_3] = x_4$$

Then, over $\mathbb{R}$, we have $\alpha(\mathfrak{g}) = 2$, but $\beta(\mathfrak{g}) = 1$.

Let $\mathbb{K}$ be equal to $\mathbb{R}$ or $\mathbb{C}$. Obviously, $\langle x_3, x_4 \rangle$ is an abelian subalgebra of dimension 2 over $\mathbb{K}$. Assume that $\alpha(\mathfrak{g}) = 3$. Then $\mathfrak{g}$ is almost abelian, hence 2-step solvable. This is impossible, as $\mathfrak{g}$ is 3-step solvable. Hence $\alpha(\mathfrak{g}) = 2$ over $\mathbb{K}$.
Assume that I is a 2-dimensional abelian ideal over $\mathbb{K}$. It is easy to see that we can represent I as $\langle \alpha x_2 + \beta x_3, x_4 \rangle$ with $\alpha, \beta \in \mathbb{K}$. Obviously neither x_2 nor x_3 can belong to I. Hence $\alpha \neq 0$ and $\beta \neq 0$. We have $\alpha x_2 + \beta x_3 \in I$ and $[x_1, \alpha x_2 + \beta x_3] = (\alpha + \beta)x_2 + (\beta - \alpha)x_3 \in I$. This implies $(\alpha^2 + \beta^2)x_3 \in I$, hence $\alpha^2 + \beta^2 = 0$. This is a contradiction over $\mathbb{R}$, so that $\beta(\mathfrak{g}) = 1$ in this case. Over $\mathbb{C}$ we may take $\alpha = 1$ and $\beta = i$, and $I = \langle x_2 + ix_3, x_4 \rangle$ is a 2-dimensional abelian ideal.

The next two lemma's are well known. We state them just for further reference.

Lemma 2.8. *Let $\mathfrak{g}$ be a complex, non-abelian, nilpotent Lie algebra of dimension n. Then*

$$\frac{\sqrt{8n + 1} - 1}{2} \leq \alpha(\mathfrak{g}) \leq n - 1$$

Proof. The estimate is given in [8] for $\beta(\mathfrak{g})$. It also holds for $\alpha(\mathfrak{g})$ since $\alpha(\mathfrak{g}) \geq \beta(\mathfrak{g})$. $\qquad\square$

Lemma 2.9. *The center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is contained in any abelian subalgebra of maximal dimension.*

Proof. An abelian subalgebra $\mathfrak{a}$ of maximal dimension is self-centralizing, i.e., $\mathfrak{a} = Z_{\mathfrak{g}}(\mathfrak{a}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{a}] = 0\}$. Since $Z(\mathfrak{g}) \subset Z_{\mathfrak{g}}(\mathfrak{a})$, the claim follows. $\qquad\square$

3. Abelian subalgebras of codimension 1

Let $\mathfrak{g}$ be a Lie algebra satisfying $\alpha(\mathfrak{g}) = n - 1$. Such a Lie algebra is 2-step solvable, and their structure is well known (see [10], section 3). We will show that $\beta(\mathfrak{g}) = n - 1$ without using proposition 2.6. Our proof will be constructive. We do not only show the existence of an abelian ideal of dimension $n - 1$, but really construct such an ideal from a given abelian subalgebra of dimension $n - 1$. Note that Lie algebras $\mathfrak{g}$ with $\beta(\mathfrak{g}) = n - 1$ are called *almost abelian*.

Proposition 3.1. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra satisfying $\alpha(\mathfrak{g}) = n - 1$. Then we have $\beta(\mathfrak{g}) = n - 1$ and an ideal of codimension 1 can be constructed explicitly.*

Proof. Let $\mathfrak{a}$ be an abelian subalgebra of dimension $n - 1$. If $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$, then $\mathfrak{a}$ is also an abelian ideal, and we are done. Otherwise we choose a basis $(e_1, \ldots, e_n)$ for $\mathfrak{g}$ such that $\mathfrak{a} = \langle e_2, \ldots, e_n \rangle$. We have $[e_j, e_\ell] = 0$ for all $j, \ell \geq 2$. There exists a $k \geq 2$ such that $[e_1, e_k]$ is not contained in $\mathfrak{a}$. We may assume that $k = 2$ by relabelling e_2 and e_k. For $j \geq 2$ let

$$[e_1, e_j] = \alpha_{j1} e_1 + \alpha_{j2} e_2 + \cdots + \alpha_{jn} e_n.$$

We have $\alpha_{21} \neq 0$. Rescaling e_2 we may assume that $\alpha_{21} = 1$. Using the Jacobi identity we have for all $j \geq 2$

$$\begin{aligned}
0 &= [e_1, [e_2, e_j]] \\
&= -[e_2, [e_j, e_1]] - [e_j, [e_1, e_2]] \\
&= -\alpha_{j1}[e_1, e_2] + [e_1, e_j]
\end{aligned}$$

This implies $[e_1, e_j] = \alpha_{j1}[e_1, e_2]$ and $[e_1, \alpha_{j1} e_2 - e_j] = 0$ for all $j \geq 2$. Let $v_j = \alpha_{j1} e_2 - e_j$. Note that all v_j lie in the center of $\mathfrak{g}$, and that the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is 1-dimensional, generated by $[e_1, e_2]$. Now define

$$I := \langle [e_1, e_2], v_3, \ldots, v_n \rangle.$$

This is an abelian subalgebra of dimension $n - 1$ which contains the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$. Hence I is an abelian ideal of maximal dimension $n - 1$, and we have $\beta(\mathfrak{g}) = n - 1$. $\qquad\square$

4. Abelian subalgebras of codimension 2

Let $\mathfrak{g}$ be a complex Lie algebra of dimension n satisfying $\alpha(\mathfrak{g}) = n - 2$. We will show that $\mathfrak{g}$ must be solvable except for the cases $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^\ell$, for $\ell = n - 3 \geq 0$. We use the convention that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is included in this family, for $\ell = 0$.

Proposition 4.1. *Let $\mathfrak{g}$ be a complex Lie algebra with $\dim(\mathfrak{g}) = n$ and $\alpha(\mathfrak{g}) = n - 2$. Then either $\mathfrak{g}$ is isomorphic to one of the Lie algebras $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^\ell$, or $\mathfrak{g}$ is a solvable Lie algebra.*

Proof. Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition, where $\mathfrak{r}$ denotes the solvable radical of $\mathfrak{g}$. For a semisimple Levi subalgebra $\mathfrak{s}$ we have

$$\alpha(\mathfrak{s}) \leq \dim(\mathfrak{s}) - 2,$$

where equality holds if and only if $\mathfrak{s}$ is $\mathfrak{sl}_2(\mathbb{C})$. This follows from table 1 and lemma 2.4. By lemma 2.5 we have $\alpha(\mathfrak{s} \ltimes \mathfrak{r}) \leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r})$. Assume that $\mathfrak{s} \neq 0$. Then it follows that

$$\begin{aligned}
\alpha(\mathfrak{g}) &\leq \alpha(\mathfrak{s}) + \alpha(\mathfrak{r}) \\
&\leq \dim(\mathfrak{s}) - 2 + \dim(\mathfrak{r}) \\
&= n - 2.
\end{aligned}$$

Since we must have equality, it follows that $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and $\alpha(\mathfrak{r}) = \dim(\mathfrak{r})$. Therefore $\mathfrak{r}$ is abelian and $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C}) \ltimes_\varphi \mathbb{C}^\ell$ with a homomorphism $\varphi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathbb{C}^\ell)$. This Lie algebra contains an abelian subalgebra of codimension 2 if and only if φ is trivial. Indeed, the Lie bracket is given by $[(x,a),(y,b)] = ([x,y], \varphi(x)b - \varphi(y)a)$, for $x,y \in \mathfrak{sl}_2(\mathbb{C})$ and $a,b \in \mathbb{C}^\ell$. Since there is an abelian subalgebra of codimension 2, there must be a nonzero element $(x,0)$ commuting with all elements $(0,b)$, i.e., $(0,0) = [(x,0),(0,b)] = (0, \varphi(x)b)$ for all $b \in \mathbb{C}^\ell$. It follows that $\ker(\varphi)$ is non-trivial. Since $\mathfrak{sl}_2(\mathbb{C})$ is simple, $\varphi = 0$.
In the other remaining case we have $\mathfrak{s} = 0$. In that case, $\mathfrak{g}$ is solvable. $\square$

It is easy to classify such Lie algebras in low dimensions.

Proposition 4.2. *Let $\mathfrak{g}$ be a complex Lie algebra of dimension n and $\alpha(\mathfrak{g}) = n - 2$.*
  (1) *For $n = 3$ it follows $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$.*
  (2) *For $n = 4$, $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

$\mathfrak{g}$	*Lie brackets*
$\mathfrak{g}_1 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4$
$\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = -e_3,\ [e_2, e_3] = e_1$
$\mathfrak{g}_3$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = 2e_4,\ [e_2, e_3] = e_4$
$\mathfrak{g}_4(\alpha),\ \alpha \in \mathbb{C}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + \alpha e_3,\ [e_1, e_4] = (\alpha + 1)e_4,\ [e_2, e_3] = e_4$

Proof. The proof is straightforward, using a classification of low-dimensional Lie algebras, e.g., the one given in [5]. Note that $\mathfrak{g}_4(\alpha) \simeq \mathfrak{g}_4(\beta)$ if and only if $\alpha\beta = 1$ or $\alpha = \beta$. $\square$

5. NILPOTENT LIE ALGEBRAS

In a nilpotent Lie algebra $\mathfrak{g}$ any subalgebra of codimension 1 is automatically an ideal. Hence given an abelian subalgebra of maximal dimension $n-1$ we obtain an abelian ideal of dimension $n - 1$. In particular, $\alpha(\mathfrak{g}) = n - 1$ for a nilpotent Lie algebra implies $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$, and we can explicitly provide such ideals. We are able to extend this result to the case $\alpha(\mathfrak{g}) = n - 2$. Given an abelian subalgebra of dimension $n - 2$ we can construct an abelian ideal of dimension $n - 2$. This is non-trivial, since the abelian subalgebra of maximal dimension $n - 2$ need not be an ideal in general. Of course, the existence of such an ideal follows already from proposition 2.6, as does the equality $\alpha(\mathfrak{g}) = \beta(\mathfrak{g})$. However, the existence proof is not constructive. Our proof will be constructive and elementary, which might be more appropriate to our special situation.

Proposition 5.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n over a field of characteristic zero satisfying $\alpha(\mathfrak{g}) = n - 2$. Then there exists an algorithm to construct an abelian ideal of dimension $n - 2$ from an abelian subalgebra of dimension $n - 2$. In particular we have $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$.*

Proof. Let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{g}$ of maximal dimension $n - 2$. Choose a basis $(e_3, \ldots, e_n)$ for $\mathfrak{a}$. The normalizer of $\mathfrak{a}$,

$$N_\mathfrak{g}(\mathfrak{a}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{a}] \subseteq \mathfrak{a}\}$$

is a subalgebra strictly containing $\mathfrak{a}$. We may assume that $N_\mathfrak{g}(\mathfrak{a})$ has dimension $n - 1$, because otherwise $N_\mathfrak{g}(\mathfrak{a}) = \mathfrak{g}$, implying that $\mathfrak{a}$ is already an abelian ideal of maximal dimension $n - 2$.

We may extend the basis of $\mathfrak{a}$ to a basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$, such that $N_{\mathfrak{g}}(\mathfrak{a}) = \langle e_2, \ldots, e_n \rangle$. Because $N_{\mathfrak{g}}(\mathfrak{a})$ has codimension 1, it is an ideal in $\mathfrak{g}$. In particular we have

$$[e_1, N_{\mathfrak{g}}(\mathfrak{a})] \subseteq N_{\mathfrak{g}}(\mathfrak{a}).$$

On the other hand, $[e_1, \mathfrak{a}]$ is not contained in $\mathfrak{a}$, since e_1 is not in $N_{\mathfrak{g}}(\mathfrak{a})$. Hence there exists a vector e_k such that $[e_1, e_k]$ is not in $\mathfrak{a}$. By relabelling e_3 and e_k we may assume that $k = 3$. Hence writing

$$[e_1, e_j] = \alpha_{j2} e_2 + \cdots + \alpha_{jn} e_n$$

for $j \geq 2$, we may assume that $\alpha_{32} = 1$, i.e., $[e_1, e_3] = e_2 + \alpha_{33} e_3 + \cdots + \alpha_{3n} e_n$.

Lemma 5.2. *The following holds:*

(1) *We have $[e_2, e_j] = \alpha_{j2}[e_2, e_3]$ for all $j \geq 3$.*
(2) *The element $[e_2, e_3]$ is nonzero and contained in the center of $\mathfrak{g}$.*
(3) *The normalizer $N_{\mathfrak{g}}(\mathfrak{a})$ is two-step nilpotent.*
(4) *We have $[N_{\mathfrak{g}}(\mathfrak{a}), v_j] = 0$ for all $j \geq 3$, where $v_j = \alpha_{j2} e_3 - e_j$.*

Proof. The first statement follows from the Jacobi identity. We have, for all $j \geq 3$,

$$\begin{aligned}
0 &= [e_1, [e_3, e_j]] \\
&= -[e_3, [e_j, e_1]] - [e_j, [e_1, e_3]] \\
&= -\alpha_{j2}[e_2, e_3] + [e_2, e_j].
\end{aligned}$$

Concerning (2), assume first that $[e_2, e_3] = 0$. Then the subalgebra given by $\langle e_2, e_3, v_4, \ldots, v_n \rangle$ would be an abelian subalgebra of dimension $n - 1$, with the v_j defined as in (4). This is a contradiction to $\alpha(\mathfrak{g}) = n - 2$. Hence $[e_2, e_3]$ is non-zero. Since $e_2 \in N_{\mathfrak{g}}(\mathfrak{a})$, we have that $[e_2, e_3] \in \mathfrak{a}$. We write

$$[e_2, e_3] = \beta_{33} e_3 + \cdots + \beta_{3n} e_n.$$

We have $[e_3, [e_2, e_3]] = 0$. Now we have

$$[e_2, [e_2, e_3]] = (\beta_{33}\alpha_{32} + \cdots + \beta_{3n}\alpha_{n2})[e_2, e_3].$$

Since $\mathrm{ad}(e_2)$ is nilpotent, it follows $[e_2, [e_2, e_3]] = 0$. In the same way, $[e_1, [e_2, e_3]] = [e_2, [e_1, e_3]] - [e_3, [e_1, e_2]] = \lambda[e_2, e_3]$, so that $[e_1, [e_2, e_3]] = 0$, because $\mathrm{ad}(e_1)$ is nilpotent. Finally, $[e_j, [e_2, e_3]] = 0$ for all $j \geq 3$, since $[e_2, e_3] \in \mathfrak{a}$. It follows that $[e_2, e_3]$ lies in the center of $\mathfrak{g}$.
To show (3), note that $[N_{\mathfrak{g}}(\mathfrak{a}), N_{\mathfrak{g}}(\mathfrak{a})]$ is generated by $[e_2, e_3]$, so that

$$[N_{\mathfrak{g}}(\mathfrak{a}), N_{\mathfrak{g}}(\mathfrak{a})] \subseteq Z(\mathfrak{g}).$$

This proves (3).
The statement (4) follows from (1). $\qquad\square$

Now let $\mathfrak{a}_1 = \langle v_4, \ldots v_n \rangle$. This is an abelian subalgebra $\mathfrak{a}_1 \subseteq \mathfrak{a} \subseteq \mathfrak{g}$ of dimension $n - 3$. There exists an integer $\ell \geq 1$ satisfying

$$\begin{aligned}
\mathrm{ad}(e_1)^{\ell-1}(e_2) &\notin \mathfrak{a}_1, \\
\mathrm{ad}(e_1)^{\ell}(e_2) &\in \mathfrak{a}_1,
\end{aligned}$$

because $\mathrm{ad}(e_1)$ is nilpotent. We define

$$I := \langle \mathrm{ad}(e_1)^{\ell-1}(e_2), v_4, \ldots, v_n \rangle$$

We will show that I is an abelian ideal of maximal dimension $n-2$. First of all, I is a subalgebra of dimension $n - 2$. It is also abelian: because $N_{\mathfrak{g}}(\mathfrak{a})$ is an ideal, $\mathrm{ad}(e_1)^k(e_2) \in N_{\mathfrak{g}}(\mathfrak{a})$ for all $k \geq 0$. Then

$$[\mathrm{ad}(e_1)^k(e_2), v_j] = [\lambda_2 e_2 + \cdots \lambda_n e_n, \alpha_{j2} e_3 - e_j]$$
$$= \lambda_2 \alpha_{j2}[e_2, e_3] - \lambda_2[e_2, e_j]$$
$$= 0.$$

It remains to show that I is an ideal, i.e., that $\mathrm{ad}(e_i)(I) \subseteq I$ for all $i \geq 1$. We have

$$[e_1, \mathrm{ad}(e_1)^{\ell-1}(e_2)] = \mathrm{ad}(e_1)^{\ell}(e_2) \in \mathfrak{a}_1 \subseteq I,$$
$$[e_k, \mathrm{ad}(e_1)^{\ell-1}(e_2)] \in [N_{\mathfrak{g}}(\mathfrak{a}), N_{\mathfrak{g}}(\mathfrak{a})] \subseteq Z(\mathfrak{g}) \subseteq I,$$

for all $k \geq 2$. Here we have used lemma 2.9 to conclude that $Z(\mathfrak{g}) \subseteq I$. Also, $[e_k, v_j] = 0 \in I$ for all $k \geq 2$ and $j \geq 4$. It remains to show that

$$[e_1, v_j] \in I \text{ for all } j \geq 4.$$

We have

$$[e_2, [e_1, v_j]] = [e_1, [e_2, v_j]] + [v_j, [e_1, e_2]]$$
$$= 0.$$

This implies that $[e_1, v_j]$ commutes with all elements from I. If it were not in I, then $\langle [e_1, v_j], I \rangle$ would be an abelian subalgebra of dimension $n-1$, which is impossible. It follows that $[e_1, v_j] \in I$. $\qquad\square$

Remark 5.3. There is also an algorithm to compute $\alpha(\mathfrak{g})$ for an arbitrary complex Lie algebra of finite dimension, see [6].

In connection with the toral rank conjecture (TRC), which asserts that

$$\dim H^*(\mathfrak{g}, \mathbb{C}) \geq 2^{\dim Z(\mathfrak{g})}$$

for any finite-dimensional, complex nilpotent Lie algebra, there are interesting examples of nilpotent Lie algebras $\mathfrak{g}$ given, with $\beta(\mathfrak{g}) = n - 2$, of dimension $n \geq 10$, see [18]. These algebras also have the property that all its derivations are singular. An obvious question here is whether there exist characteristically nilpotent Lie algebras (CNLAs) $\mathfrak{g}$ of dimension n with $\alpha(\mathfrak{g}) = n - 2$. This is indeed the case.

Example 5.4. *The Lie algebra of dimension $n = 7$ defined by $[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$ and $[x_2, x_3] = x_6 + x_7$, $[x_2, x_4] = x_7$ is characteristically nilpotent, i.e., all of its derivations are nilpotent, see [14]. Furthermore it satisfies $\alpha(\mathfrak{g}) = n - 2 = 5$.*

We can find such examples in all dimensions $n \geq 7$. This suggests that nilpotent Lie algebras $\mathfrak{g}$ with $\alpha(\mathfrak{g}) = n - 2$ are not so easy to understand. The algebra in this example is *filiform nilpotent*, i.e., has maximal nilpotency class with respect to its dimension. In this case we can say something more on $\alpha(\mathfrak{g})$.

Definition 5.5. Let $\mathfrak{g}$ be a nilpotent Lie algebra, and $C^1(\mathfrak{g}) = \mathfrak{g}$, $C^i(\mathfrak{g}) = [\mathfrak{g}, C^{i-1}(\mathfrak{g})]$. Then $\mathfrak{g}$ is called *k-abelian*, if k is the smallest positive integer such that the ideal $C^k(\mathfrak{g})$ is abelian.

For a nilpotent k-abelian Lie algebra $\mathfrak{g}$ we have $\dim(C^k(\mathfrak{g})) \leq \beta(\mathfrak{g})$. In general equality does not hold. However, if $\mathfrak{g}$ is filiform nilpotent of dimension $n \geq 6$ with $k \geq 3$, then we do have equality:

Proposition 5.6. *Let $\mathfrak{g}$ be a k-abelian filiform Lie algebra of dimension $n \geq k + 3 \geq 6$. Then $\beta(\mathfrak{g}) = \alpha(\mathfrak{g}) = \dim(C^k(\mathfrak{g})) = n - k$, and $C^k(\mathfrak{g})$ is the unique abelian ideal of maximal dimension. We have*

$$\left\lceil \frac{n}{2} \right\rceil \leq \beta(\mathfrak{g}) \leq n - 3.$$

Proof. We may choose an adapted basis $(e_1, \ldots, e_n)$ for $\mathfrak{g}$, see [4]. Then $[e_1, e_i] = e_{i+1}$ for all $2 \leq i \leq n - 1$, and $C^j(\mathfrak{g}) = \langle e_{j+1}, \ldots, e_n \rangle$ with $\dim(C^j(\mathfrak{g})) = n - j$ for all $j \geq 2$. By assumption $C^k(\mathfrak{g})$ is abelian, but $C^{k-1}(\mathfrak{g})$ is not. We claim that every abelian ideal I in $\mathfrak{g}$ is contained in $C^k(\mathfrak{g})$. This will finish the proof. Suppose that there is an abelian ideal I which is not contained in $C^k(\mathfrak{g})$. We will show that this implies $C^{k-1}(\mathfrak{g}) \subseteq I$, so that I cannot be abelian, a contradiction. Let $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$ be a nontrivial element of I not lying in $C^k(\mathfrak{g})$, i.e., with $\alpha_i \neq 0$ for some $i < k + 1$. If $\alpha_1 \neq 0$, then for all $2 \leq j \leq n - 1$ we have $[e_j, x] = -\alpha_1 e_{j+1} + \alpha_2 [e_j, e_2] + \cdots + \alpha_n [e_j, e_n] \in I$. It follows that $\langle e_3, \cdots, e_n \rangle = C^2(\mathfrak{g}) \subseteq I$. Since $k > 2$ this implies that I is not abelian, a contradiction. Let $1 < i < k + 1$ be minimal such that $\alpha_i \neq 0$. Then for all $0 \leq j \leq n - i$ we have $\mathrm{ad}(e_1)^j(x) = \alpha_i e_{i+j} + \cdots + \alpha_{n-j} e_n \in I$. It follows that $\langle e_i, \ldots, e_n \rangle \subseteq I$. Indeed, for $j = n - i$ we have $\alpha_i e_n \in I$, then $\alpha_i e_{n-1} \in I$, and so on until $\alpha_i e_i \in I$. Since $i \leq k$ we have $C^{k-1}(\mathfrak{g}) \subseteq I$ and we are finished. Finally, we have $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$, so that we obtain the estimate on $\beta(\mathfrak{g})$. $\square$

Remark 5.7. If $\mathfrak{g}$ is filiform with $k = 2$, then $\mathfrak{g}$ is 2-step solvable and we have $\beta(\mathfrak{g}) = n - 1$ or $\beta(\mathfrak{g}) = n - 2$. Indeed, if $\mathfrak{g}$ is the standard graded filiform $\mathfrak{f}_n$ of dimension $n \geq 3$, then $\beta(\mathfrak{g}) = n - 1$ and $I = \langle e_2, \ldots, e_n \rangle$ is an abelian ideal of dimension $n - 1$. Otherwise $C^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is an abelian ideal of maximal dimension, so that $\beta(\mathfrak{g}) = n - 2$.

The invariant $\alpha(\mathfrak{g})$ for complex nilpotent Lie algebras has been determined up to dimension 6 in connection with degenerations [3],[21], and for real Lie algebras of dimension 6 in [14], appendix 4.4.

We want to give a list here, thereby correcting a few typos in [21]. In dimension 7 there is no list for $\alpha(\mathfrak{g})$, as far as we know. We use the classification of nilpotent Lie algebras up to dimension 7 by Magnin [14], and for dimension 6 also by de Graaf [7] and Seeley [21], to give tables for $\alpha(\mathfrak{g})$. The result for the indecomposable algebras in dimension $n \leq 5$ is as follows:

$\mathfrak{g}$	$\dim(\mathfrak{g})$	Lie brackets	$\alpha(\mathfrak{g})$
$\mathfrak{n}_3$	3	$[e_1, e_2] = e_3$	2
$\mathfrak{n}_4$	4	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$	3
$\mathfrak{g}_{5,6}$	5	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$	3
$\mathfrak{g}_{5,5}$	5	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$	4
$\mathfrak{g}_{5,3}$	5	$[e_1, e_2] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$	3
$\mathfrak{g}_{5,4}$	5	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$	3
$\mathfrak{g}_{5,2}$	5	$[e_1, e_2] = e_4, [e_1, e_3] = e_5$	4
$\mathfrak{g}_{5,1}$	5	$[e_1, e_3] = e_5, [e_2, e_4] = e_5$	3

For $n = 6$ we have:

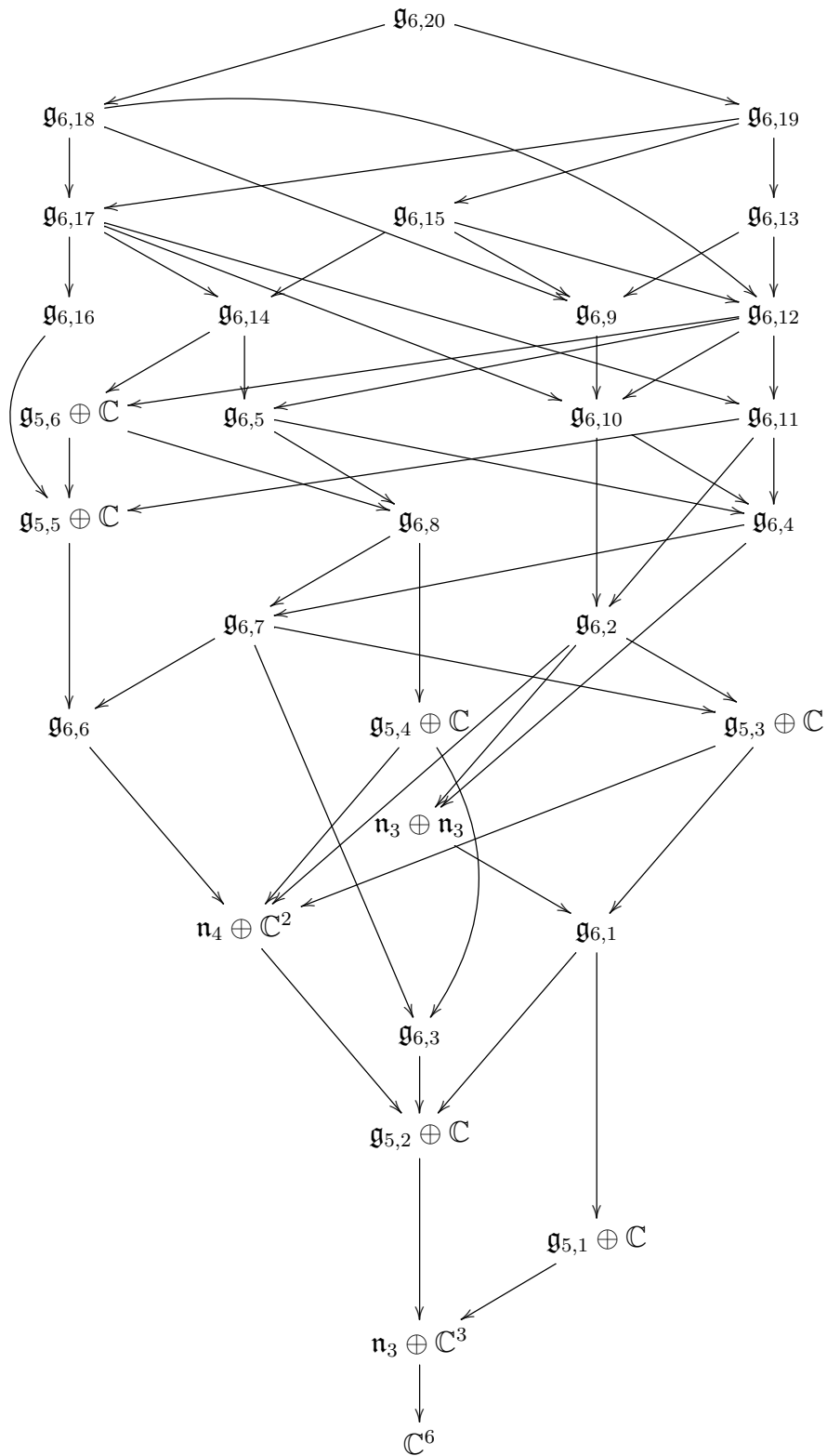| Magnin	de Graaf	Seeley	$\alpha(\mathfrak{g})$
$\mathfrak{g}_{6,20}$	$L_{6,14}$	12346_E	3
$\mathfrak{g}_{6,18}$	$L_{6,16}$	12346_C	3
$\mathfrak{g}_{6,19}$	$L_{6,15}$	12346_D	4
$\mathfrak{g}_{6,17}$	$L_{6,17}$	12346_B	4
$\mathfrak{g}_{6,15}$	$L_{6,21}(1)$	1346_C	4
$\mathfrak{g}_{6,13}$	$L_{6,13}$	1246	4
$\mathfrak{g}_{6,16}$	$L_{6,18}$	12346_A	5
$\mathfrak{g}_{6,14}$	$L_{6,21}(0)$	2346	4
$\mathfrak{g}_{6,9}$	$L_{6,19}(1)$	136_A	4
$\mathfrak{g}_{6,12}$	$L_{6,11}$	1346_B	4
$\mathfrak{g}_{5,6} \oplus \mathbb{C}$	$L_{6,6}$	$1 + 1235_B$	4
$\mathfrak{g}_{6,5}$	$L_{6,24}(1)$	246_E	4
$\mathfrak{g}_{6,10}$	$L_{6,20}$	136_B	4
$\mathfrak{g}_{6,11}$	$L_{6,12}$	1346_A	4
$\mathfrak{g}_{5,5} \oplus \mathbb{C}$	$L_{6,7}$	$1 + 1235_A$	5
$\mathfrak{g}_{6,8}$	$L_{6,24}(0)$	246_D	4
$\mathfrak{g}_{6,4}$	$L_{6,19}(0)$	246_B	4
$\mathfrak{g}_{6,7}$	$L_{6,23}$	246_C	4
$\mathfrak{g}_{6,2}$	$L_{6,10}$	146	4
$\mathfrak{g}_{6,6}$	$L_{6,25}$	246_A	5
$\mathfrak{g}_{5,4} \oplus \mathbb{C}$	$L_{6,9}$	$1 + 235$	4
$\mathfrak{g}_{5,3} \oplus \mathbb{C}$	$L_{6,5}$	$1 + 135$	4
$\mathfrak{n}_3 \oplus \mathfrak{n}_3$	$L_{6,22}(1)$	$13 + 13$	4
$\mathfrak{n}_4 \oplus \mathbb{C}^2$	$L_{6,3}$	$2 + 124$	5
$\mathfrak{g}_{6,1}$	$L_{6,22}(0)$	26	4
$\mathfrak{g}_{6,3}$	$L_{6,26}$	36	4
$\mathfrak{g}_{5,2} \oplus \mathbb{C}$	$L_{6,8}$	$1 + 25$	5
$\mathfrak{g}_{5,1} \oplus \mathbb{C}$	$L_{6,4}$	$1 + 15$	4
$\mathfrak{n}_3 \oplus \mathbb{C}^3$	$L_{6,2}$	$3 + 13$	5
$\mathbb{C}^6$	$L_{6,1}$	0	6

The Hasse diagram for the degenerations of nilpotent Lie algebras in dimension 6 is given in the end of the paper. For more details on degenerations see [5]. There are some typos in [21] which we found by computing all degenerations again. This is not part of this paper, but will be published elsewhere. The Hasse diagram is only listed for the interested reader as an appendix, to demonstrate that the computation of $\alpha(\mathfrak{g})$ has interesting applications. The diagram also gives a good control for the computation of $\alpha(\mathfrak{g})$, since it is well known that $\alpha(\mathfrak{g}) \leq \alpha(\mathfrak{h})$ if $\mathfrak{g} \rightarrow_{\deg} \mathfrak{h}$, see [5].

In dimension 7 we use the classification of Magnin to compute $\alpha(\mathfrak{g})$ for all indecomposable, complex nilpotent Lie algebras of dimension 7, using an algorithm of [6]. Note that $4 \leq \alpha(\mathfrak{g}) \leq 6$ in this case, see lemma 2.8. The result is as follows:

$\alpha(\mathfrak{g}) = 4:$ $\quad \mathfrak{g} = \mathfrak{g}_{7,0.1},\ \mathfrak{g}_{7,0.4(\lambda)},\ \mathfrak{g}_{7,0.5},\ \mathfrak{g}_{7,0.6},\ \mathfrak{g}_{7,0.7},\ \mathfrak{g}_{7,0.8},\ \mathfrak{g}_{7,1.02},\ \mathfrak{g}_{7,1.03},\ \mathfrak{g}_{7,1.1(i_\lambda),\lambda\neq1},\ \mathfrak{g}_{7,1.1(ii)},$
$\mathfrak{g}_{7,1.1(iii)},\ \mathfrak{g}_{7,1.1(iv)},\ \mathfrak{g}_{7,1.1(v)},\ \mathfrak{g}_{7,1.1(vi)},\ \mathfrak{g}_{7,1.2(i_\lambda),\lambda\neq1},\ \mathfrak{g}_{7,1.2(ii)},\ \mathfrak{g}_{7,1.2(iii)},\ \mathfrak{g}_{7,1.2(iv)},$
$\mathfrak{g}_{7,1.3(i_\lambda),\lambda\neq0},\ \mathfrak{g}_{7,1.3(ii)},\ \mathfrak{g}_{7,1.3(iii)},\ \mathfrak{g}_{7,1.3(iv)},\ \mathfrak{g}_{7,1.3(v)},\ \mathfrak{g}_{7,1.5},\ \mathfrak{g}_{7,1.8},\ \mathfrak{g}_{7,1.11},\ \mathfrak{g}_{7,1.14},$
$\mathfrak{g}_{7,1.17},\ \mathfrak{g}_{7,1.19},\ \mathfrak{g}_{7,1.20},\ \mathfrak{g}_{7,1.21},\ \mathfrak{g}_{7,2.1(i_\lambda),\lambda\neq0,1},\ \mathfrak{g}_{7,2.1(ii)},\ \mathfrak{g}_{7,2.1(iii)},\ \mathfrak{g}_{7,2.1(iv)},\ \mathfrak{g}_{7,2.1(v)},$
$\mathfrak{g}_{7,2.2},\ \mathfrak{g}_{7,2.4},\ \mathfrak{g}_{7,2.5},\ \mathfrak{g}_{7,2.6},\ \mathfrak{g}_{7,2.10},\ \mathfrak{g}_{7,2.12},\ \mathfrak{g}_{7,2.13},\ \mathfrak{g}_{7,2.17},\ \mathfrak{g}_{7,2.23},\ \mathfrak{g}_{7,2.26},\ \mathfrak{g}_{7,2.28},$
$\mathfrak{g}_{7,2.29},\ \mathfrak{g}_{7,2.30},\ \mathfrak{g}_{7,2.34},\ \mathfrak{g}_{7,2.35},\ \mathfrak{g}_{7,2.37},\ \mathfrak{g}_{7,3.1(i_\lambda),\lambda\neq0,1},\ \mathfrak{g}_{7,3.1(iii)},\ \mathfrak{g}_{7,3.13},\ \mathfrak{g}_{7,3.18},$
$\mathfrak{g}_{7,3.22},\ \mathfrak{g}_{7,4.4}.$

$\alpha(\mathfrak{g}) = 5:$ $\quad \mathfrak{g} = \mathfrak{g}_{7,0.2},\ \mathfrak{g}_{7,0.3},\ \mathfrak{g}_{7,1.01(i)},\ \mathfrak{g}_{7,1.01(ii)},\ \mathfrak{g}_{7,1.1(i_\lambda),\lambda=1},\ \mathfrak{g}_{7,1.2(i_\lambda),\lambda=1},\ \mathfrak{g}_{7,1.3(i_\lambda),\lambda=0},\ \mathfrak{g}_{7,1.4},$
$\mathfrak{g}_{7,1.6},\ \mathfrak{g}_{7,1.7},\ \mathfrak{g}_{7,1.9},\ \mathfrak{g}_{7,1.10},\ \mathfrak{g}_{7,1.12},\ \mathfrak{g}_{7,1.13},\ \mathfrak{g}_{7,1.15},\ \mathfrak{g}_{7,1.16},\ \mathfrak{g}_{7,1.18},\ \mathfrak{g}_{7,2.1(i_\lambda),\lambda=0,1},$
$\mathfrak{g}_{7,2.7},\ \mathfrak{g}_{7,2.8},\ \mathfrak{g}_{7,2.9},\ \mathfrak{g}_{7,2.11},\ \mathfrak{g}_{7,2.14},\ \mathfrak{g}_{7,2.15},\ \mathfrak{g}_{7,2.16},\ \mathfrak{g}_{7,2.18},\ \mathfrak{g}_{7,2.19},\ \mathfrak{g}_{7,2.20},\ \mathfrak{g}_{7,2.21},$
$\mathfrak{g}_{7,2.22},\ \mathfrak{g}_{7,2.24},\ \mathfrak{g}_{7,2.25},\ \mathfrak{g}_{7,2.27},\ \mathfrak{g}_{7,2.31},\ \mathfrak{g}_{7,2.32},\ \mathfrak{g}_{7,2.33},\ \mathfrak{g}_{7,2.36},\ \mathfrak{g}_{7,2.38},\ \mathfrak{g}_{7,2.39},$
$\mathfrak{g}_{7,2.40},\ \mathfrak{g}_{7,2.41},\ \mathfrak{g}_{7,2.42},\ \mathfrak{g}_{7,2.43},\ \mathfrak{g}_{7,2.44},\ \mathfrak{g}_{7,2.45},\ \mathfrak{g}_{7,3.1(i_\lambda),\lambda=0,1},\ \mathfrak{g}_{7,3.3},\ \mathfrak{g}_{7,3.4},\ \mathfrak{g}_{7,3.5},$
$\mathfrak{g}_{7,3.6},\ \mathfrak{g}_{7,3.7},\ \mathfrak{g}_{7,3.8},\ \mathfrak{g}_{7,3.9},\ \mathfrak{g}_{7,3.10},\ \mathfrak{g}_{7,3.11},\ \mathfrak{g}_{7,3.12},\ \mathfrak{g}_{7,3.14},\ \mathfrak{g}_{7,3.15},\ \mathfrak{g}_{7,3.16},\ \mathfrak{g}_{7,3.17},$
$\mathfrak{g}_{7,3.19},\ \mathfrak{g}_{7,3.21},\ \mathfrak{g}_{7,3.23}\ \mathfrak{g}_{7,3.24},\ \mathfrak{g}_{7,4.1},\ \mathfrak{g}_{7,4.3}.$

$\alpha(\mathfrak{g}) = 6:$ $\quad \mathfrak{g} = \mathfrak{g}_{7,2.3},\ \mathfrak{g}_{7,3.2},\ \mathfrak{g}_{7,3.20},\ \mathfrak{g}_{7,4.2}.$

$\mathfrak{g}_{6,20}$

$\mathfrak{g}_{6,18}$

$\mathfrak{g}_{6,19}$

$\mathfrak{g}_{6,17}$

$\mathfrak{g}_{6,15}$

$\mathfrak{g}_{6,13}$

$\mathfrak{g}_{6,16}$

$\mathfrak{g}_{6,14}$

$\mathfrak{g}_{6,9}$

$\mathfrak{g}_{6,12}$

$\mathfrak{g}_{5,6} \oplus \mathbb{C}$

$\mathfrak{g}_{6,5}$

$\mathfrak{g}_{6,10}$

$\mathfrak{g}_{6,11}$

$\mathfrak{g}_{5,5} \oplus \mathbb{C}$

$\mathfrak{g}_{6,8}$

$\mathfrak{g}_{6,4}$

$\mathfrak{g}_{6,7}$

$\mathfrak{g}_{6,2}$

$\mathfrak{g}_{6,6}$

$\mathfrak{g}_{5,4} \oplus \mathbb{C}$

$\mathfrak{g}_{5,3} \oplus \mathbb{C}$

$\mathfrak{n}_3 \oplus \mathfrak{n}_3$

$\mathfrak{n}_4 \oplus \mathbb{C}^2$

$\mathfrak{g}_{6,1}$

$\mathfrak{g}_{6,3}$

$\mathfrak{g}_{5,2} \oplus \mathbb{C}$

$\mathfrak{g}_{5,1} \oplus \mathbb{C}$

$\mathfrak{n}_3 \oplus \mathbb{C}^3$

$\mathbb{C}^6$

6. Acknowledgment

We would like to thank Wolfgang Moens for helpful discussions.

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