DEGENERATIONS OF PRE-LIE ALGEBRAS

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ABSTRACT. We consider the variety of pre-Lie algebra structures on a given n-dimensional vector space. The group $GL_n(K)$ acts on it, and we study the closure of the orbits with respect to the Zariski topology. This leads to the definition of pre-Lie algebra degenerations. We give fundamental results on such degenerations, including invariants and necessary degeneration criteria. We demonstrate the close relationship to Lie algebra degenerations. Finally we classify all orbit closures in the variety of complex 2-dimensional pre-Lie algebras.

1. Introduction

Contractions of Lie algebras are limiting processes between Lie algebras, which have been studied first in physics [13],[9]. For example, classical mechanics is a limiting case of quantum mechanics as $\hbar \to 0$, described by a contraction of the Heisenberg-Weyl Lie algebra to the abelian Lie algebra of the same dimension.

In mathematics, often a more general definition of contractions is used, so called degenerations. Here one considers the variety of n-dimensional Lie algebra structures and the orbit closures with respect to the Zariski topology of $GL_n(K)$-orbits. There is a large literature on degenerations, see for example [11] and the references cited therein. Degenerations are related to deformations. They have been studied also for commutative algebras, associative algebras and Leibniz algebras, see for example [8], [12]. Of course, orbit closures and hence degenerations can be considered for all algebras. However, we are particularly interested in so called *pre-Lie algebras*, which have very interesting applications in geometry and physics, see [4] for a survey. This class of algebras also includes Novikov algebras. For applications of Novikov algebras in physics, see [2].

The aim of this article is to provide a degeneration theory for pre-Lie algebras, and to find interesting invariants, which are preserved under the process of degeneration. It turns out, that among other things such invariants are given by polynomial operator identities $T(x,y)=0$ in the operators $L(x), L(y), R(x), R(y)$, the left and right multiplications of the pre-Lie algebra. For example, the identity $T(x,y)=L(x)R(y)-R(y)L(x)=0$ for all $x,y \in A$ says that the algebra A is associative. This is preserved under degeneration.

On the other hand we find semi-continuous functions on the variety of n-dimensional pre-Lie algebra structures. An example is given by the dimension of the center $\mathcal{Z}(A)$ of a pre-Lie algebra. If a pre-Lie algebra A degenerates to a pre-Lie algebra B, then $\dim \mathcal{Z}(A) \leq \dim \mathcal{Z}(B)$. The function $f(\lambda)=\dim \mathcal{Z}(\lambda)$ is an upper semi-continuous function on the variety of pre-Lie algebra structures. We may also consider the dimensions of left and right annihilators, or in fact of various other spaces, like certain subalgebras and cohomology spaces. Since a pre-Lie algebra in general is not anti-commutative, we often have two possibilities (like right and left annihilators), where we had only one in the Lie algebra case.

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Furthermore we introduce generalized derivation algebras. The dimension of these spaces are again upper semi-continuous functions. More generally, certain generalized cohomology spaces can be studied in this context.
Finally we apply our results to classify all orbit closures in the variety of 2-dimensional pre-Lie algebras. The resulting Hasse diagram shows how complicated the situation is already in dimension 2.

2. The variety of pre-Lie algebras

Pre-Lie algebras, or left-symmetric algebras arise in many areas of mathematics and physics. It seems that A. Cayley [5] in 1896 was the first one to introduce pre-Lie algebras, in the context of rooted tree algebras. From 1960 onwards they became widely known by the work of Vinberg, Koszul and Milnor in connection with convex homogeneous cones and affinely flat manifolds. Around 1990 they appeared in renormalization theory and quantum mechanics, starting with the work of Connes and Kreimer [6]. For the details and the references see [4]. The definition is as follows:

Definition 2.1. A K-algebra A together with a bilinear product $(x,y)\mapsto x\cdot y$ is called a *pre-Lie algebra*, if the identity

$$(x\cdot y)\cdot z - x\cdot(y\cdot z) = (y\cdot x)\cdot z - y\cdot(x\cdot z) \tag{1}$$

holds for all $x,y,z\in A$. A pre-Lie algebra is called a *Novikov algebra*, if the identity

$$(x\cdot y)\cdot z = (x\cdot z)\cdot y \tag{2}$$

holds for all $x,y,z\in A$.

The commutator $[x,y]=x\cdot y-y\cdot x$ in a pre-Lie algebra defines a Lie bracket. We denote the associated Lie algebra by $\mathfrak{g}_A$.

Remark 2.2. If $(x,y,z)=(x\cdot y)\cdot z-x\cdot(y\cdot z)$ denotes the associator in A, then A is a pre-Lie algebra, if and only if $(x,y,z)=(y,x,z)$ for all x,y,z in A. For this reason, A is also called a *left-symmetric algebra.*

In analogy to the variety $\mathcal{L}_n(K)$ of n-dimensional Lie algebra structures we can define the variety of arbitrary non-associative algebras. We want to focus here on pre-Lie algebras. Let V be a vector space of dimension n over a field K. Fix a basis $(e_1,\dots,e_n)$ of V. If $(x,y)\mapsto x\cdot y$ is a pre-Lie algebra product on V with $e_i\cdot e_j=\sum_{k=1}^n c_{ij}^k e_k$, then $(c_{ij}^k)\in K^{n^3}$ is called a *pre-Lie algebra structure* on V.

Definition 2.3. Let V be an n-dimensional vector space over a field K. Denote by $\mathcal{P}_n(K)$ the set of all pre-Lie algebra structures on V. This is called the *variety of pre-Lie algebra structures.*

The set $\mathcal{P}_n(K)$ is an affine algebraic set, since the identity (1) is given by polynomials in the structure constants c_{ij}^k. It need not be irreducible, however. Denote by μ a pre-Lie algebra product on V. The general linear group $GL_n(K)$ acts on $\mathcal{P}_n(K)$ by

$$(g\circ\mu)(x,y)=g(\mu(g^{-1}x,g^{-1}y))$$

for $g\in GL_n(K)$ and $x,y\in V$.
Denote by $O(\mu)$ the orbit of μ under this action, and by $\overline{O(\mu)}$ the closure of the orbit with respect to the Zariski topology. The orbits in $\mathcal{P}_n(K)$ under this action correspond to isomorphism classes of n-dimensional pre-Lie algebras.

Definition 2.4. Let $\lambda, \mu \in \mathcal{P}_n(K)$ be two pre-Lie algebra laws. We say that λ *degenerates* to μ, if $\mu \in \overline{O(\lambda)}$. This is denoted by $\lambda \to_{\deg} \mu$. We say that the degeneration $\lambda \to_{\deg} \mu$ is *proper*, if $\mu \in \overline{O(\lambda)} \setminus O(\lambda)$, i.e., if λ and μ are not isomorphic.

The existence of a pre-Lie algebra degeneration $A \to_{\deg} B$ means the following: the algebra B is represented by a structure μ which lies in the Zariski closure of the $GL_n(K)$-orbit of some structure λ which represents A.

The following important result is due to Borel [1]:

Proposition 2.5. *If G is a complex algebraic group and X is a complex algebraic variety with regular action, then each orbit $G(x)$, $x \in X$ is a smooth algebraic variety, open in its closure $\overline{G(x)}$. Its boundary $\overline{G(x)} \setminus G(x)$ is a union of orbits of strictly lower dimension. Each orbit $G(x)$ is a constructible set, hence $\overline{G(x)}$ coincides with the closure $\overline{G(x)}^d$ in the standard Euclidean topology.*

Recall that a subset $Y \subseteq X$ is called constructible if it is a finite union of locally closed sets. The result has some interesting consequences:

Corollary 2.6. *Denote by $\mathbb{C}(t)$ the field of fractions of the polynomial ring $\mathbb{C}[t]$. If there exists an operator $g_t \in GL_n(\mathbb{C}(t))$ such that $\lim_{t\to 0} g_t \circ \lambda = \mu$ for $\lambda, \mu \in \mathcal{P}_n(\mathbb{C})$, then $\lambda \to_{\deg} \mu$.*

Proof. We have $\mu \in \overline{O(\lambda)}^d$ by assumption, which implies $\mu \in \overline{O(\lambda)}$. □

Example 2.7. *Any n-dimensional complex pre-Lie algebra λ degenerates to the zero pre-Lie algebra $\mathbb{C}^n$.*

Let $g_t = t^{-1}E_n$, where E_n is the identity matrix. Then we have

$$(g_t \circ \lambda)(x, y) = t^{-1}\lambda(tx, ty) = t\lambda(x, y),$$

hence λ degenerates to the zero product, i.e., $\lim_{t\to 0} g_t \circ \lambda = \mathbb{C}^n$.

Remark 2.8. Borel's result implies also the following (the argument is the same as the one given in [11] for Lie algebras). Every degeneration of complex pre-Lie algebras can be realized by a so called sequential contraction, i.e., $A \to_{\deg} B$ is equivalent to the fact that we have

$$\lim_{\varepsilon\to 0} g_\varepsilon \circ \lambda = \mu$$

where $g_\varepsilon \in GL_n(\mathbb{C})$, $\varepsilon > 0$ and $\lambda, \mu \in \mathcal{P}_n(\mathbb{C})$, such that λ and μ represent the pre-Lie algebras A and B respectively.

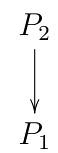
Corollary 2.9. *The process of degeneration in $\mathcal{P}_n(\mathbb{C})$ defines a partial order on the orbit space of n-dimensional pre-Lie algebra structures, given by $O(\mu) \leq O(\lambda) \iff \mu \in \overline{O(\lambda)}$.*

Proof. The relation is clearly reflexive. The transitivity follows from the fact that $O(\lambda) \subseteq \overline{O(\mu)} \iff \overline{O(\lambda)} \subseteq \overline{O(\mu)}$. Finally, antisymmetry follows from the fact, that any orbit is open in its closure. □

The transitivity is very useful. If $\lambda \to_{\deg} \mu$ and $\mu \to_{\deg} \nu$, then $\lambda \to_{\deg} \nu$. For $\lambda \in \mathcal{P}_n(K)$ we have $\dim O(\lambda) = \dim \operatorname{End}(V) - \dim \operatorname{Der}(\lambda) = n^2 - \dim \operatorname{Der}(\lambda)$. Here $\operatorname{Der}(\lambda)$ denotes the derivation algebra of the algebra A represented by λ.

Corollary 2.10. *Let $\lambda \to_{\deg} \mu$ be a proper degeneration in $\mathcal{P}_n(\mathbb{C})$. Then $\dim O(\lambda) > \dim O(\mu)$ and $\dim \operatorname{Der}(\lambda) < \dim \operatorname{Der}(\mu)$.*

We can represent the degenerations in $\mathcal{P}_n(K)$ with respect to the above partial order in a diagram: order the pre-Lie algebras by orbit dimension in $\mathcal{P}_n(K)$, in each row the algebras with the same orbit dimension, on top the ones with the biggest orbit dimension. Draw a directed arrow between two algebras A and B, if A degenerates to B. This diagram is called the *Hasse diagram* of degenerations in $\mathcal{P}_n(K)$. It shows the classification of orbit closures.
A rather trivial example is the case of 1-dimensional complex pre-Lie algebras. Let (e_1) be a basis of $\mathbb{C}$. Then there are two pre-Lie algebras. Denote by P_1 the algebra with zero product, and by P_2 the algebra with $e_1 \cdot e_1 = e_1$. Then $\dim \operatorname{Der}(P_1) = 1$ and $\dim \operatorname{Der}(P_2) = 0$. The Hasse diagram for $\mathcal{P}_1(\mathbb{C})$ is given as follows:



3. Criteria for degenerations

Given two pre-Lie algebras A and B we want to decide whether A degenerates to B or not. Suppose that the answer is yes. Then we would like to find a $g_t \in GL_n(\mathbb{C}(t))$ realizing such a degeneration. If the answer is no, we need an argument to show that such a degeneration is impossible. One way is to find an invariant for A, i.e., a polynomial in terms of the structure constants which is zero on the whole orbit of A, so that it must be also zero on the orbit closure of A. If B does not satisfy this polynomial equation, then B cannot lie in the orbit closure of A.
For example, commutativity of A is such an invariant. If $L(x)$ resp. $R(x)$ denotes the left resp. right multiplication operators in $\operatorname{End}(A)$, then commutativity of A means that the operator $T(x) = L(x) - R(x)$ satisfies $T(x) = 0$ for all $x \in A$. This is clearly such a polynomial invariant on the orbit of A. Hence a degeneration $A \to_{\text{deg}} B$ is impossible, if A is commutative, but B is not. Another operator identity is $T(x, y) = [L(x), R(y)] = L(x)R(y) - R(y)L(x) = 0$ for all $x, y \in A$, which says that the algebra A is associative.

Lemma 3.1. *Let A and B be two pre-Lie algebras of dimension n, and $T(x_1, \dots, x_n)$ be a polynomial in the operators $L(x_1), \dots, L(x_n)$ and $R(x_1), \dots, R(x_n)$. Suppose that $T(x_1, \dots, x_n) = 0$ for A but not for B. Then A cannot degenerate to B.*

Proof. Let $\varphi\colon A \to A'$ be an isomorphism of pre-Lie algebras, and denote by $L(x), R(x)$ the left resp. right multiplications in A, and by $\ell(x), r(x)$ the ones in A'. Then $\varphi(x \cdot y) = \varphi(x).\varphi(y)$ implies

$$L(x) = \varphi^{-1} \circ \ell(\varphi(x)) \circ \varphi, \quad R(x) = \varphi^{-1} \circ r(\varphi(x)) \circ \varphi.$$

If a polynomial T in the left- and right multiplications of A vanishes, then the same is true for the left-and right multiplications of all pre-Lie algebra structures in the GL_n-orbit representing A, since a base change just induces a conjugation of the operator polynomial. It follows that the operator identity holds also for all pre-Lie algebra structures in the orbit closure. This implies the claim. □

We can also consider invariants of $\lambda \in \mathcal{P}_n(K)$ defining upper (or lower) semi-continuous functions $f\colon \mathcal{L}_n(k) \to \mathbb{Z}_{\geq 0}$. Then $\lambda \to_{\text{deg}} \mu$ implies $f(\lambda) \leq f(\mu)$ or $f(\lambda) \geq f(\mu)$. Recall the following definition:

Definition 3.2. Let X be a topological space. A function $f\colon X \to \mathbb{Z}_{\geq 0}$ is called *upper semi-continuous*, if $f^{-1}(]-\infty, n[)$ is open in X for all $n \in \mathbb{Z}_{\geq 0}$. It is called *lower semi-continuous*, if $f^{-1}(]n, \infty[)$ is open in X for all $n \in \mathbb{Z}_{\geq 0}$.

In the case of Lie algebras, for example, $f(\lambda) = \dim Z(\lambda)$ is a upper semi-continuous function on the variety of n-dimensional Lie algebra structures, and satisfies $f(\lambda) \leq f(\mu)$ for $\lambda \to_{\deg} \mu$. There are more such invariants yielding semi-continuous functions, for example the dimensions of cocycle spaces and Lie algebra cohomology groups.
It is very natural to consider similar invariants for pre-Lie algebras A. Define the left and right annihilator, and the center of A by

$$\begin{aligned}
\mathcal{L}(A) &= \{x \in A \mid x \cdot A = 0\},\\
\mathcal{R}(A) &= \{x \in A \mid A \cdot x = 0\},\\
\mathcal{Z}(A) &= \{x \in A \mid x \cdot A = A \cdot x = 0\}.
\end{aligned}$$

There are also cohomology groups $H^n_{pre}(A, M)$ for pre-Lie algebras A, with an A-bimodule M, see [7]. The case, where $M = A$ is the regular module goes already back to Nijenhuis. In this case we have $Z^1(A, A) = \mathrm{Der}(A)$, and $Z^2(A, A)$ describes infinitesimal pre-Lie algebra deformations of A. The cohomology of pre-Lie algebras is related to Lie algebra cohomology as follows, see [7]:

$$H^n_{pre}(A, M) \cong H^{n-1}(\mathfrak{g}_A, \mathrm{Hom}_K(A, M)).$$

The various dimensions of these spaces define semi-upper continuous functions:

Lemma 3.3. *If $A \to_{\deg} B$ then*

$$\begin{aligned}
\dim Z^n_{pre}(A, A) &\leq \dim Z^n_{pre}(B, B)\\
\dim H^n_{pre}(A, A) &\leq \dim H^n_{pre}(B, B)\\
\dim \mathcal{L}(A) &\leq \dim \mathcal{L}(B)\\
\dim \mathcal{R}(A) &\leq \dim \mathcal{R}(B)\\
\dim \mathcal{Z}(A) &\leq \dim \mathcal{Z}(B)
\end{aligned}$$

The proof is similar to the proof in the Lie algebra case. A crucial lemma here is the following, see [10]:

Lemma 3.4. *Let G be a complex reductive algebraic group with Borel subgroup B. If G acts regularly on an affine variety X, then for all $x \in X$,*

$$\overline{G \cdot x} = G \cdot (\overline{B \cdot x}).$$

We have also the following easy result, which shows that pre-Lie algebra degenerations in a sense refine the ones for Lie algebras.

Lemma 3.5. *If $A \to_{\deg} B$ then $\mathfrak{g}_A \to_{\deg} \mathfrak{g}_B$ for the associated Lie algebras.*

Proof. Let $(e_1, \ldots, e_n)$ be a basis of the underlying vector space V. Denote the product in A by $e_i \cdot e_j$, in B by $e_i . e_j$. The Lie bracktes are given by $[e_i, e_j]_A = e_i \cdot e_j - e_j \cdot e_i$ and

$[e_i, e_j]_B = e_i.e_j - e_j.e_i$. We have

$$\begin{aligned}\lim_{\varepsilon\to 0} g_\varepsilon([g_\varepsilon^{-1}(e_i), g_\varepsilon^{-1}(e_j)]_A) &= \lim_{\varepsilon\to 0} g_\varepsilon\left(g_\varepsilon^{-1}(e_i)\cdot g_\varepsilon^{-1}(e_j) - g_\varepsilon^{-1}(e_j)\cdot g_\varepsilon^{-1}(e_i)\right)\\ &= \lim_{\varepsilon\to 0} g_\varepsilon\left(g_\varepsilon^{-1}(e_i)\cdot g_\varepsilon^{-1}(e_j)\right) - \lim_{\varepsilon\to 0} g_\varepsilon\left(g_\varepsilon^{-1}(e_j)\cdot g_\varepsilon^{-1}(e_i)\right)\\ &= e_i.e_j - e_j.e_i\\ &= [e_i, e_j]_B.\end{aligned}$$

This implies that $\mathfrak{g}_A \to_{\deg} \mathfrak{g}_B$, because of remark 2.8. □

Remark 3.6. With more theory at hand there is a shorter, but less elementary proof of the above lemma: since the map $A \mapsto \mathfrak{g}_A$ is a $GL(V)$-equivariant morphism, it maps orbit closures into orbit closures.

We can also generalize the trace invariants of Lie algebras to the case of pre-Lie algebras. For $x, y \in A$ and $i, j \in \mathbb{N}$ consider the expression

$$c_{i,j}(A) = \frac{\operatorname{tr}(L(x)^i)\cdot \operatorname{tr}(L(y)^j)}{\operatorname{tr}(L(x)^i L(y)^j)}.$$

If this is well-defined for all $x, y \neq 0$ and finite, then it is an interesting invariant of A. Just like in the Lie algebra case, we obtain the following result.

Lemma 3.7. *Suppose A degenerates to B and both values $c_{i,j}(A)$ and $c_{i,j}(B)$ are well-defined for all $x, y \neq 0$, then $c_{i,j}(A) = c_{i,j}(B)$.*

We can generalize the definition of pre-Lie algebra derivations as follows.

Definition 3.8. Let $\alpha, \beta, \gamma \in \mathbb{C}$ and define $\operatorname{Der}_{(\alpha,\beta,\gamma)}(A)$ to be the space of all $D \in \operatorname{End}(A)$ satisfying

$$\alpha D(x\cdot y) = \beta D(x)\cdot y + \gamma x\cdot D(y)$$

for all $x, y \in A$. We call the elements $D \in \operatorname{Der}_{(\alpha,\beta,\gamma)}(A)$ also (α, β, γ)-derivations.

Lemma 3.9. *If $A \to_{\deg} B$, then $\dim \operatorname{Der}_{(\alpha,\beta,\gamma)}(A) \le \dim \operatorname{Der}_{(\alpha,\beta,\gamma)}(B)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.*

Proof. Let $\lambda, \mu \in \mathcal{P}_n(\mathbb{C})$ represent A and B. Fix a basis $(e_1, \dots, e_n)$ of the underlying vector space. Then $\lim_{\varepsilon\to 0}(g_\varepsilon \circ \lambda)(e_i, e_j) = \mu(e_i, e_j)$ for operators $g_\varepsilon \in GL_n(\mathbb{C})$. For $D \in \operatorname{Der}_{\alpha,\beta,\gamma}(A)$ we write $D = (d_{ij})_{1\le i,j\le n}$, and $D(e_i) = \sum_{l=1}^n d_{li}e_l$. We have $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ in A, with the structure constants c_{ij}^k. Since D is an (α, β, γ)-derivation we have

$$\sum_{l=1}^n (\alpha c_{ij}^l d_{kl} - \beta c_{lj}^k d_{li} - \gamma c_{il}^k d_{lj}) = 0$$

for all i, j, k. We can rewrite these n^3 equations as a matrix equation $Md = 0$ where d is the vector formed by the columns of the matrix $D = (d_{ij})$, and M is a $n^3 \times n^2$ matrix depending on c_{ij}^k and α, β, γ. Thus we have $\ker(M) = \operatorname{Der}_{(\alpha,\beta,\gamma)}(A)$. If A degenerates to B via g_ε we obtain a sequence of matrices M_ε with $\lim_{\varepsilon\to 0} M_\varepsilon = M_0$ by componentwise convergence of the structure constants, where $\ker(M_0) = \operatorname{Der}_{(\alpha,\beta,\gamma)}(B)$. Let m be the rank of the matrix M. Then every submatrix of size $(m+1)\times(m+1)$ has zero determinant. Because of the convergence the same is true for M_0. It follows that $\operatorname{rank}(M) \ge \operatorname{rank}(M_0)$, or $\dim\ker(M) \le \dim\ker(M_0)$. □

4. Degenerations in dimension 2

In this section, we determine the Hasse diagram of degenerations for 2-dimensional pre-Lie algebras. This is already quite complicated, and we can apply our results in a non-trivial way. In dimension $n = 2$ there are two different complex Lie algebras. Let (e_1, e_2) be a basis. Then either $\mathfrak{g} = \mathbb{C}^2$, or $\mathfrak{g} = \mathfrak{r}_2(\mathbb{C})$, where we can choose $[e_1, e_2] = e_1$. The classification of 2-dimensional complex pre-Lie algebras is well known, see for example [3]:

A	Products	$\mathfrak{g}_A$	$\dim \mathrm{Der}(A)$
A_1	$-$	$\mathbb{C}^2$	4
A_2	$e_1 \cdot e_1 = e_1$	$\mathbb{C}^2$	1
A_3	$e_1 \cdot e_1 = e_1,\ e_2 \cdot e_2 = e_2$	$\mathbb{C}^2$	0
A_4	$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_2$	$\mathbb{C}^2$	1
A_5	$e_2 \cdot e_2 = e_1$	$\mathbb{C}^2$	2
$B_1(\alpha)$	$e_2 \cdot e_1 = -e_1,\ e_2 \cdot e_2 = \alpha e_2$	$\mathfrak{r}_2(\mathbb{C})$	1 if $\alpha \neq -1$ 2 if $\alpha = -1$
$B_2(\beta)$ $\beta \neq 0$	$e_1 \cdot e_2 = \beta e_1,\ e_2 \cdot e_1 = (\beta - 1)e_1,$ $e_2 \cdot e_2 = \beta e_2$	$\mathfrak{r}_2(\mathbb{C})$	1 if $\beta \neq 1$ 2 if $\beta = 1$
B_3	$e_2 \cdot e_1 = -e_1,\ e_2 \cdot e_2 = e_1 - e_2$	$\mathfrak{r}_2(\mathbb{C})$	1
B_4	$e_1 \cdot e_1 = e_2,\ e_2 \cdot e_1 = -e_1$ $e_2 \cdot e_2 = -2e_2$	$\mathfrak{r}_2(\mathbb{C})$	0
B_5	$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_2 = e_1 + e_2$	$\mathfrak{r}_2(\mathbb{C})$	1

Here we have $B_2(0) \simeq B_1(0)$, so that we require $\beta \neq 0$. The pre-Lie algebras $A_1, \ldots, A_5$ are commutative and associative, since their Lie algebra $\mathfrak{g} = \mathbb{C}^2$ is abelian. The following algebras are Novikov algebras:

$$A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ B_2(\beta)_{\beta \in \mathbb{C}},\ B_5.$$

From the list of non-commutative pre-Lie algebras, $B_1(-1)$ and $B_2(1)$ are associative, and B_4 is simple.

For a pre-Lie algebra A we consider the quantities $c_{i,j}(A)$ for $i, j \in \mathbb{N}$. We have $c_{i,j}(A_2) = 1$, $c_{i,j}(A_4) = 2$, $c_{i,j}(B_3) = 2$, $c_{i,j}(B_5) = 1$ for all $i, j \geq 1$, and

$$c_{i,j}(B_1(\alpha)) = \frac{(\alpha^i + (-1)^i)(\alpha^j + (-1)^j)}{\alpha^{i+j} + (-1)^{i+j}},$$

$$c_{i,j}(B_2(\beta)) = \frac{(\beta^i + (\beta - 1)^i)(\beta^j + (\beta - 1)^j)}{\beta^{i+j} + (\beta - 1)^{i+j}},$$

$$c_{i,j}(B_4) = \frac{(2^i + 1)(2^j + 1)}{2^{i+j} + 1}.$$

If $\lambda \to_{\deg} \mu$ properly, then $\dim O(\lambda) > \dim O(\mu)$. Therefore, to determine the degenerations, we can order the algebras by orbit dimension, i.e., by $\dim \mathrm{Der}(\lambda)$, as follows:

$$A_3,\, B_4,\quad A_2,\, A_4,\, B_1(\alpha)_{\alpha\neq -1},\, B_2(\beta)_{\beta\neq 0,1},\, B_3,\, B_5,\quad A_5,\, B_1(-1),\, B_2(1),\quad A_1.$$

Lemma 4.1. *The orbit closure of A_3 in $\mathcal{P}_2(\mathbb{C})$ contains exactly the following algebras:*

$$A_3,\, A_2,\, A_4,\, A_5,\, A_1.$$

In other words, A_3 can only properly degenerate to A_2, A_4, A_5, A_1.

Proof. First of all, A_3 can only degenerate to commutative algebras, see lemma 3.1. The orbit dimension of A_3 is equal to 4. Hence A_3 can only properly degenerate to commutative algebras of lower orbit dimension, which are exactly the above algebras. For these we find the following degenerations: We have $A_3 \to_{\deg} A_2$ via $g_t^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ t^2 & t \end{smallmatrix}\right)$. Also, we have a degeneration $A_3 \to_{\deg} A_4$ via $g_t^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & t \end{smallmatrix}\right)$. Finally, $A_3 \to_{\deg} A_5$ by $g_t^{-1} = \left(\begin{smallmatrix} 2t^2 & 2t \\ 0 & 1 \end{smallmatrix}\right)$. □

Lemma 4.2. *The orbit closure of B_4 in $\mathcal{P}_2(\mathbb{C})$ contains exactly the following algebras:*

$$B_4,\; B_1(-2),\; B_2(-1),\; A_5,\; A_1.$$

Proof. The orbit dimension of B_4 equals 4. Hence B_4 can only properly degenerate to algebras of lower orbit dimension, which are the following ones:

$$A_2,\; A_4,\; B_1(\alpha),\; B_2(\beta),\; B_3,\; B_5,\; A_5,\; A_1.$$

Now B_4 cannot degenerate to A_2 since

$$c_{1,1}(B_4) = \frac{9}{5} \neq 1 = c_{1,1}(A_2).$$

In the same way, B_4 cannot degenerate to A_4.
Assume that $B_4 \to_{\deg} B_1(\alpha)$. Comparing the invariants $c_{i,j}$ for the algebras B_4 and $B_1(\alpha)$ yields that we must have $(\alpha+2)(2\alpha+1) = 0$. For these two values of α all invariants $c_{i,j}$ coincide, so that we cannot exclude that B_4 possibly degenerates to $B_1(-2)$, $B_1(-1/2)$. In fact, there is a degeneration $B_4 \to_{\deg} B_1(-2)$ by $g_t^{-1} = \left(\begin{smallmatrix} t & 0 \\ t & 1 \end{smallmatrix}\right)$. But there is no degeneration of B_4 to any other algebra $B_1(\alpha)$ for $\alpha \neq -2$. To see this we use lemma 3.1. If $x = x_1e_1 + x_2e_2$, $y = y_1e_1 + y_2e_2$ then the left and right multiplications of B_4 are given by

$$L(x) = \begin{pmatrix} -x_2 & 0 \\ x_1 & -2x_2 \end{pmatrix}, \quad R(x) = \begin{pmatrix} 0 & -x_1 \\ x_1 & -2x_2 \end{pmatrix},$$
$$L(y) = \begin{pmatrix} -y_2 & 0 \\ y_1 & -2y_2 \end{pmatrix}, \quad R(y) = \begin{pmatrix} 0 & -y_1 \\ y_1 & -2y_2 \end{pmatrix}.$$

Searching for quadratic operator identities $T(x,y) = 0$ for all $x, y \in B_4$, we find that $T_{r,s}(x,y) = 0$ for all $r, s \in \mathbb{C}$, where

$$\begin{aligned} T_{r,s}(x,y) &= r(L(x)R(y) - L(y)R(x)) + s(R(x)L(y) - R(y)L(x)) \\ &\quad + (s-3r)[L(x), L(y)] + \frac{1}{2}(r-2s)[R(x), R(y)]. \end{aligned}$$

Since $T_{r,s}(x,y)$ is skew-symmetric in x and y, it is enough to check this for $x = e_1$ and $y = e_2$. For $r = s = -2$ we obtain

$$T(x,y) = [2L(x) - R(x), 2L(y) - R(y)] = 0.$$

But for $B_1(\alpha)$ the left and right multiplication operators satisfy this identity if and only if $\alpha = -2$, since in this case

$$T(x,y) = \begin{pmatrix} 0 & (\alpha+2)(x_2y_1 - x_1y_2) \\ 0 & 0 \end{pmatrix}.$$

Hence only $B_4 \to_{\deg} B_1(-2)$ is possible.
Next assume that B_4 degenerates to $B_2(\beta)$. A calculation shows that $B_2(\beta)$ satisfies the above operator identity if and only if $\beta = -1$. We have a degeneration $B_4 \to_{\deg} B_2(-1)$ however by $g_t^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & t \end{pmatrix}$.
The algebra B_4 cannot degenerate to B_3 since $c_{1,1}(B_4) = 9/5 \neq 2 = c_{1,1}(B_3)$, and also not to B_5 since $c_{1,1}(B_5) = 1$. Finally $B_4 \to_{\deg} A_5$ by $g_t^{-1} = \begin{pmatrix} 2t & 0 \\ t & 3t^2 \end{pmatrix}$. □

The classification of degenerations among 2-dimensional pre-Lie algebras is as follows. We restrict ourselfs to proper degenerations, so that we do not list the algebra itself in the orbit closure.

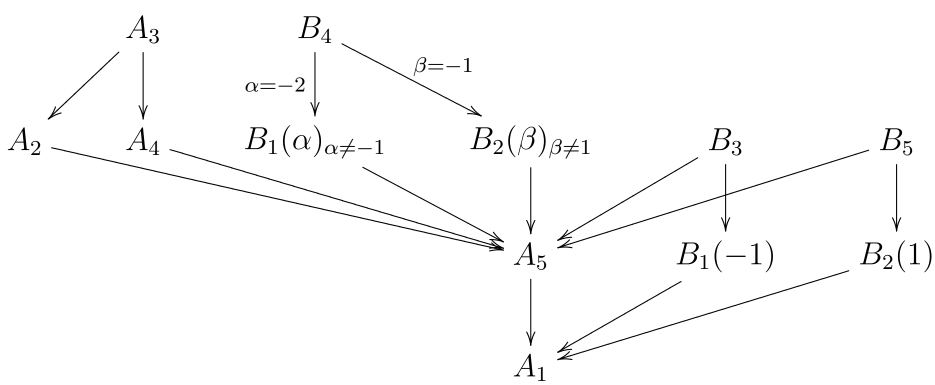
Theorem 4.3. *The orbit closures in $\mathcal{P}_2(\mathbb{C})$ are given as follows:*

A	$\overline{O(A)}$
A_3	$A_2,\ A_4,\ A_5,\ A_1$
B_4	$B_1(-2),\ B_2(-1),\ A_5,\ A_1$
A_2	$A_5,\ A_1$
A_4	$A_5,\ A_1$
$B_1(\alpha)_{\alpha\neq -1}$	$A_5,\ A_1$
$B_2(\beta)_{\beta\neq 1}$	$A_5,\ A_1$
B_3	$A_5,\ B_1(-1),\ A_1$
B_5	$A_5,\ B_2(1),\ A_1$
A_5	A_1
$B_1(-1)$	A_1
$B_2(1)$	A_1

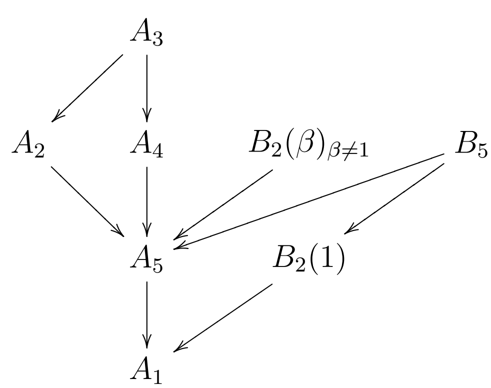
Proof. The classification of the orbit closures for A_3 and B_4 is given in the two lemmas above. A_2 can only degenerate to commutative algebras of orbit dimension smaller than 3, that is to A_5 and A_1. Both is possible, we have $A_2 \to_{\deg} A_5$ by $g_t^{-1} = \begin{pmatrix} t & 0 \\ 1 & -t \end{pmatrix}$. For A_4 the same reasoning applies and we have $A_4 \to_{\deg} A_5$ by $g_t^{-1} = \begin{pmatrix} t & 0 \\ 1 & t \end{pmatrix}$.
The orbit dimension for $B_1(\alpha)$, $\alpha \neq -1$ is 3, hence possible algebras in the orbit closure are A_5, $B_1(-1)$ and $B_2(1)$. We have a degeneration to A_5 by $g_t^{-1} = \begin{pmatrix} 1 & 0 \\ t & t^2(\alpha+1) \end{pmatrix}$, for $\alpha \neq -1$. Comparing $c_{1,1}(B_1(\alpha)) = \frac{(\alpha-1)^2}{\alpha^2+1}$ and $c_{i,j}(B_1(-1)) = 2$ for $\alpha^2+1 \neq 0$, we see that a degeneration to $B_1(-1)$ is only possible, if $\alpha = -1$. But we assumed $\alpha \neq -1$. Similarly we see that $B_1(\alpha)$, $\alpha \neq -1$ does not degenerate to $B_2(1)$.
The only candidates for proper degenerations of the algebras $B_2(\beta)$, $\beta \neq 1$ are again A_5, $B_1(-1)$ and $B_2(1)$. There is a degeneration to A_5 by $g_t^{-1} = \begin{pmatrix} 1 & 0 \\ t & t^2(\beta-1) \end{pmatrix}$, for $\beta \neq 1$. For $\beta \neq 0, 1$ assume that $B_2(\beta) \to_{\deg} B_2(1)$. Comparing $c_{1,1}$ we obtain $\frac{(2\beta-1)^2}{\beta^2+(\beta-1)^2} = 1$, or equivalently $\beta(\beta-1) = 0$ which was excluded. Similarly $B_2(\beta)$, $\beta \neq 1$ cannot degenerate to $B_1(-1)$. Another possibility to show this is to use lemma 3.7.

B_3 degenerates to A_5 by $g_t^{-1} = \left(\begin{smallmatrix} t^{-2} & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, and to $B_1(-1)$ by $g_t^{-1} = \left(\begin{smallmatrix} -t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Because $c_{1,1}(B_3) = 2$ and $c_{1,1}(B_2(1)) = 1$, there is no degeneration from B_3 to $B_2(1)$.
B_5 degenerates to A_5 by $g_t^{-1} = \left(\begin{smallmatrix} t^{-2} & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, and to $B_2(1)$ by $g_t^{-1} = \left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Because $c_{1,1}(B_5) = 1$ and $c_{1,1}(B_1(-1)) = 2$, there is no degeneration from B_5 to $B_1(-1)$. □

Corollary 4.4. *The Hasse diagram of degenerations in $\mathcal{P}_2(\mathbb{C})$ is given as follows:*



Corollary 4.5. *The Hasse diagram for degenerations of Novikov algebra structures in $\mathcal{P}_2(\mathbb{C})$ is given as follows:*



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