COMPUTING FAITHFUL REPRESENTATIONS FOR NILPOTENT LIE ALGEBRAS

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Abstract. We describe three methods to determine a faithful representation of small dimension for a finite-dimensional nilpotent Lie algebra over an arbitrary field. We apply our methods in finding bounds for the smallest dimension $\mu(\mathfrak{g})$ of a faithful $\mathfrak{g}$-module for some nilpotent Lie algebras $\mathfrak{g}$. In particular, we describe an infinite family of filiform nilpotent Lie algebras $\mathfrak{f}_n$ of dimension n over $\mathbb{Q}$ and conjecture that $\mu(\mathfrak{f}_n) > n+1$. Experiments with our algorithms suggest that $\mu(\mathfrak{f}_n)$ is polynomial in n.

1. Introduction

The Ado-Iwasawa theorem asserts that every finite-dimensional Lie algebra over an arbitrary field has a faithful finite-dimensional representation. A constructive proof for this theorem in characteristic 0 has been given in [5]. It has been implemented as an algorithm in the computer algebra systems GAP and Magma.

Here we consider a variation on this theme: we introduce three algorithms for computing a faithful finite-dimensional representation for a finite-dimensional nilpotent Lie algebra over an arbitrary field. They take as input a nilpotent Lie algebra $\mathfrak{g}$ given by a structure constants table and can be briefly summarized as follows.

- Our first algorithm uses an action of $\mathfrak{g}$ on a quotient of the universal enveloping algebra of $\mathfrak{g}$. It is based on ideas in [2].
- The second algorithm constructs a finite-dimensional faithful submodule of the dual of the universal enveloping algebra of $\mathfrak{g}$. The resulting module is minimal in the sense that it has no faithful quotients or submodules.
- The third algorithm uses a randomised method to try to construct a faithful representation in dimension $\dim(\mathfrak{g}) + 1$. It uses induction on a central series of $\mathfrak{g}$, in each step extending the representation by means of a cohomological construction.

The methods described here are practical. In particular, all three methods are usually more efficient than the general algorithm given in [5]. We include a report on various applications of our GAP implementation of our algorithms below.

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A central aim in the design of our algorithms was the idea to try to construct a faithful representation of possibly small dimension of a given nilpotent Lie algebra. In particular, our three algorithms usually determine representations which are a lot smaller than the representation obtained by the algorithm from [5]. Our algorithms can therefore be used to determine good upper bounds for the minimal possible dimension $\mu(\mathfrak{g})$ of a faithful module of the nilpotent Lie algebra $\mathfrak{g}$.

The interest in the invariant $\mu(\mathfrak{g})$ is motivated, among other things, by problems from geometry and topology. For example, Milnor and Auslander studied generalizations of crystallographic groups; these are related to $\mu(\mathfrak{g})$ as follows: let Γ be a finitely-generated, torsion-free nilpotent group of rank n with real Malcev completion G_Γ. Then, if Γ is the fundamental group of a compact complete affinely-flat manifold, it follows that $\mu(\mathfrak{g}_\Gamma) \leq n+1$ for the Lie algebra $\mathfrak{g}_\Gamma$ of G_Γ. Another motivation for studying $\mu(\mathfrak{g})$ is based on the result that the Lie algebra $\mathfrak{g}$ of a Lie group G admitting a left-invariant affine structure satisfies $\mu(\mathfrak{g}) \leq n+1$. In the context of these two results Milnor conjectured that any solvable Lie group should admit a left-invariant affine structure. The algebraic formulation of this conjecture implies that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$ should hold for the Lie algebra $\mathfrak{g}$ of G.

There are counter-examples known to Milnor's conjecture. Indeed, there are infinitely many filiform nilpotent Lie algebras of dimension 10 which do not have any faithful module of dimension $n+1$. We refer to [3] for details and background.

Here we describe an infinite family of Lie algebras $\mathfrak{f}_n$ of dimension n for $n \geq 13$ and we use our algorithms to study the invariant $\mu(\mathfrak{f}_n)$ for these Lie algebras. We conjecture that these Lie algebras do not have a faithful representation of dimension $n+1$. But our experiments suggest that $\mu(\mathfrak{f}_n)$ is polynomial in n for these Lie algebras.

2. Using quotients of the universal enveloping algebra

Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over an arbitrary field. By $\mathfrak{g}^m$, $m \geq 1$, we denote the terms of the lower central series of $\mathfrak{g}$. If $x_1, \ldots, x_d$ is a basis of $\mathfrak{g}$, then the formal products $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ with $\alpha_i \in \mathbb{N}$ form a basis of the universal enveloping algebra $U(\mathfrak{g})$.

We define the *weight* $wgt(x)$ of an element $x \in \mathfrak{g}$ as the maximal m with $x \in \mathfrak{g}^m$. The *weight* of a basis element of $U(\mathfrak{g})$ is then defined by

$$wgt(x_1^{\alpha_1} \cdots x_d^{\alpha_d}) = \sum_{i=1}^{d} \alpha_i \, wgt(x_i)$$

Let $U^m(\mathfrak{g}) = \langle x^\alpha \mid wgt(x^\alpha) \geq m \rangle$ the ideal in $U(\mathfrak{g})$ generated by all basis elements of weight at least m for some $m \geq 1$. The following theorem is proved in [1],[2].

Theorem 2.1. *If $\mathfrak{g}$ is nilpotent of class c, then $\mathfrak{g}$ acts faithfully on $U(\mathfrak{g})/U^{c+1}(\mathfrak{g})$ by multiplication from the left. If the considered basis of $\mathfrak{g}$ contains bases for $\mathfrak{g}^m$ for* *every $m \geq 1$, then the resulting representation has the dimension*

$$\nu(d, c) = \sum_{j=0}^{c} \binom{d-j}{c-j} p(j),$$

where $p(j)$ is the number of partitions of j with $p(0) = 1$.

This theorem yields a straightforward algorithm to construct a faithful module for $\mathfrak{g}$. We consider a basis of $\mathfrak{g}$ which contains bases of $\mathfrak{g}^m$ for every m. We form the space V spanned by all basis elements of weight at most c in $U(\mathfrak{g})$. An element $x \in \mathfrak{g}$ acts on V by left multiplication, where we treat any element of weight at least $c+1$ as zero. We demonstrate this algorithm in the following example.

Example 2.2. *Let $\mathfrak{g}$ be the 3-dimensional Heisenberg Lie algebra spanned by x, y, z with non-zero bracket $[x, y] = z$. The nilpotency class of $\mathfrak{g}$ is 2. We form the space spanned by the basis elements of $U(\mathfrak{g})$ of weight at most 2. These are*

$$1, x, y, z, x^2, xy, y^2.$$

Thus we obtain a 7-dimensional representation for $\mathfrak{g}$. It is straightforward to determine this representation explicitly by computing actions. For example, $y \cdot x = yx = xy - z$ and $x \cdot xy = x^2y = 0$.

The modules resulting from this simple and very efficient algorithm still have rather large dimension. In the remainder of this section, we describe two methods to determine a module of smaller dimension from this given module. As a first step, we note that a nilpotent Lie algebra $\mathfrak{g}$ acts faithfully on a module if and only if its center $Z(\mathfrak{g})$ acts faithfully. Thus if I is an ideal in $U(\mathfrak{g})$ such that $I \cap Z(\mathfrak{g}) = 0$, then L acts faithfully on $U(\mathfrak{g})/I$.

For our first method we assume that the considered basis of $\mathfrak{g}$ additionally contains a basis for the center $Z(\mathfrak{g})$. We wish to determine an ideal I in $U(\mathfrak{g})$ which has possibly small codimension and satisfies that $I \cap Z(\mathfrak{g}) = 0$. Let B be the set of all basis elements of weight at least $c+1$ in $U(\mathfrak{g})$ and initialise $I = \langle B \rangle$. We now iterate the following procedure: let a be one of the finitely many basis elements of $U(\mathfrak{g})$ not contained in I and not contained in $Z(\mathfrak{g})$. If $xa \in I$ for all $x \in \mathfrak{g}$, then we add a to B and thus enlarge I without destroying the property $I \cap Z(\mathfrak{g}) = 0$. This approach usually yields a rather small dimensional faithful representation of $\mathfrak{g}$. We demonstrate this in the following example.

Example 2.3. *We continue Example 2.2. We initialise B as the basis elements of weight at least $c+1$. Note that $Z(\mathfrak{g}) = \langle z \rangle$. Thus we consider for a the elements x, y, x^2, xy, y^2. Of those, the elements x^2, xy and y^2 satisfy the condition of the algorithm and thus we move these into B. Now also y satisfies the condition and we also move y into B. Now I is an ideal of codimension 3 in $U(\mathfrak{g})$ and hence we obtain a 3-dimensional faithful representation of $\mathfrak{g}$.*

So we now have a procedure to construct a possibly small set of monomials, that yields a faithful $\mathfrak{g}$-module: we first carry out the procedure of Theorem 2.1, and then the procedure outlined above, to make the set of monomials smaller. In the sequel this algorithm will be called *Regular*.

A second method, that can be tried for any faithful $\mathfrak{g}$-module V, is to perform the following algorithm on V.

(1) Compute the space $S = \{v \in V \mid x \cdot v = 0 \text{ for all } x \in \mathfrak{g}\}$.
(2) Compute the space $C = \{x \cdot v \mid v \in V, x \in Z(\mathfrak{g})\}$.
(3) Set $M = S \cap C$ and let W be a complement to M in S.
(4) If $W = 0$ then the algorithm stops, and the output is V. Otherwise, set $V := V/W$, and return to (1).

We note that any subspace of S is a $\mathfrak{g}$-submodule of V. Therefore the quotient V/W is a $\mathfrak{g}$-module. Let U be a complement to W in V such that $C \subset U$. Then $x \cdot V \subset U$ for all $x \in Z(\mathfrak{g})$. So since $Z(\mathfrak{g})$ acts faithfully on V it acts faithfully on V/W. Hence V/W is a faithful $\mathfrak{g}$-module.

Example 2.4. *We consider the module of Example 2.2. Here we get*

$$S = \langle z, x^2, xy, y^2\rangle$$
$$C = \langle z\rangle$$
$$W = \langle x^2, xy, y^2\rangle.$$

After taking the quotient we get a module spanned by (the images of) $1, x, y, z$. *For this module we can perform the algorithm again. We get* $S = \langle y, z\rangle$, $C = \langle z\rangle$, $W = \langle y\rangle$. *So we end up with a faithful module of dimension* 3.

Now the complete algorithm to construct a small-dimensional faithful $\mathfrak{g}$-module consists of first performing the algorithm Regular, followed by the quotient procedure described above. This algorithm will be called *Quotient*.

3. Using the dual of the universal enveloping algebra

Now $\mathfrak{g}$ acts on the dual $U(\mathfrak{g})^*$ by $x \cdot f(a) = f(-xa)$. Let $z_1, \dots, z_r$ be a basis of the center of $\mathfrak{g}$, which we assume to be a subset of the basis $x_1, \dots, x_n$. Let $\psi_i \in U(\mathfrak{g})^*$ for $1 \le i \le r$ be defined by $\psi_i(z_i) = 1$ and $\psi_i(a) = 0$ for any PBW-monomial not equal to z_i (note that this definition depends on the choice of basis of $\mathfrak{g}$).

Let $^- : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the antiautomorphism induced by $\bar{x} = -x$ for $x \in \mathfrak{g}$. Then for $a, b \in U(\mathfrak{g})$, $f \in U(\mathfrak{g})^*$ we get $a \cdot f(b) = f(\bar{a}b)$. (In other words, $^-$ is the antipode of $U(\mathfrak{g})$.) Note that $\bar{\bar{a}} = a$.

Theorem 3.1. *Let V be the $\mathfrak{g}$-submodule of $U(\mathfrak{g})^*$ generated by $\psi_1, \dots, \psi_r$. Then V is a faithful finite-dimensional $\mathfrak{g}$-module. Moreover, V has no faithful $\mathfrak{g}$-submodules, nor has it faithful quotients.*

Proof. For $k \ge 1$ let $U^k(\mathfrak{g})$ be as in Theorem 2.1. Let $W = \{f \in U(\mathfrak{g})^* \mid f(U^{c+1}(\mathfrak{g})) = 0\}$. Then W is finite-dimensional (since $U^{c+1}(\mathfrak{g})$ has finite codimension), and a $\mathfrak{g}$-submodule of $U(\mathfrak{g})^*$ (since $U^{c+1}(\mathfrak{g})$ is an ideal). Now $V \subset W$; hence V is finite-dimensional. Let $z = \sum_i \mu_i z_i$ be an element of the center of $\mathfrak{g}$. Then

$z \cdot \psi_i(1) = -\psi_i(z) = -\mu_i$. So z acts as zero if and only if all μ_i are zero. So V is a faithful module.

Let $\psi_0 \in U(\mathfrak{g})^*$ be defined by $\psi_0(1) = 1$ and $\psi_0(a) = 0$ for PBW-monomials $a \neq 1$. Then $\psi_0 = -z_1 \cdot \psi_1$, so $\psi_0 \in V$. Let $f \in V$, and suppose there is a PBW-monomial $a \neq 1$ such that $f(a) \neq 0$. Then $\bar{a} \cdot f(1) = f(a) \neq 0$. The conclusion is that ψ_0 spans the space of elements that are killed by $\mathfrak{g}$. Set $M = V/\langle \psi_0 \rangle$. Then M is not a faithful $\mathfrak{g}$-module. Indeed, V has a basis consisting of $a \cdot \psi_i$ for $1 \leq i \leq r$, and various PBW-monomials a. Let z lie in the center of $\mathfrak{g}$; then for all PBW-monomials b we get $z \cdot (a \cdot \psi_i)(b) = \psi_i(\bar{a}b\bar{z})$ which is zero unless $a = b = 1$ (note that $\bar{z} = -z$ also lies in the center of $\mathfrak{g}$). Furthermore, $z \cdot \psi_i = -\mu_i \psi_0$ (where $z = \sum_j \mu_j z_j$). It follows that the center of $\mathfrak{g}$ acts trivially on $V/\langle \psi_0 \rangle$. In particular it is not a faithful $\mathfrak{g}$-module. Now, since every $\mathfrak{g}$-submodule of V must contain ψ_0, it follows that V has no faithful quotients.

Let $M \subset V$ be a faithful $\mathfrak{g}$-submodule. Let $b_{ij} \in U(\mathfrak{g})$ be such that $\{b_{ij}\psi_i\}$ is a basis of V. We assume that $b_{i1} = 1$, and that the $b_{ij} \in \mathfrak{g}U(\mathfrak{g})$ if $j > 1$ (i.e., they have no constant term). Then $z_i \cdot b_{i1}\psi_i = -\psi_0$ and $z_i \cdot b_{kj}\psi_k = 0$ if $k \neq i$ or $j > 1$. So since the center acts faithfully on M it follows that M contains elements of the form

$$\varphi_i = \psi_i + \sum_{\substack{j>1 \\ 1 \leq k \leq r}} c_{kj} b_{kj} \psi_k,$$

for $1 \leq i \leq r$. (Here c_{kj} are coefficients in the ground field.) Now we introduce a weight function on $U(\mathfrak{g})^*$. For $k \geq 0$ set $F_k = \{f \in U(\mathfrak{g})^* \mid f(U^k(\mathfrak{g})) = 0\}$, where $U^k(\mathfrak{g})$ is as in Theorem 2.1. Then $0 = F_0 \subset F_1 \subset \cdots$. We set $wgt(f) = k$ if $f \in F_k$ but $f \notin F_{k-1}$. (For example: $wgt(\psi_0) = 1$.) Let $f \in U(\mathfrak{g})^*$ have weight k, and let $a \in U(\mathfrak{g})$ with $wgt(a) = t$; then a calculation shows that $wgt(a \cdot f) \leq k - t$. Hence $b_{ij}\varphi_i$ is equal to $b_{ij}\psi_i$ plus a sum of functions of smaller weight. So if we order the $b_{ij}\psi_i$ according to weight, and express the $b_{ij}\varphi_i$ on the basis $b_{ij}\psi_i$ we get a triangular system. We conclude that the $b_{ij}\varphi_i$ are linearly independent. Hence $\dim M = \dim V$ and $M = V$. □

The algorithm based on this theorem is straightforward. We illustrate it with an example.

Example 3.2. *Let $\mathfrak{g}$ be the Lie algebra of Example 2.2. For a monomial $a \in U(\mathfrak{g})$ we denote by ψ_a the element of $U(\mathfrak{g})^*$ that takes the value 1 on a, and zero on all other monomials. We compute a basis of the submodule of $U(\mathfrak{g})^*$ generated by ψ_z. We have $x \cdot \psi_z(a) = -\psi_z(xa) = 0$ for all monomials a. Secondly, $y \cdot \psi_z(x) = \psi_z(-yx) = \psi_z(-xy + z) = 1$. So we get $y \cdot \psi_z = \psi_x$. Furthermore, $z \cdot \psi_z = -\psi_1$ and $\mathfrak{g} \cdot \psi_1 = 0$, $x \cdot \psi_x = -\psi_1$, $y \cdot \psi_x = z \cdot \psi_x = 0$. So the result is a 3-dimensional $\mathfrak{g}$-module.*

Remark 3.3. Since we are working in the dual of an infinite-dimensional space it is not immediately clear how to implement this algorithm. We proceed as follows. Let V be as in Theorem 3.1. From the proof of Theorem 3.1 it follows that $f(U(\mathfrak{g})^{c+1}) =$

0 for all $f \in V$. In other words, for all monomials a with $wgt(a) \geq c+1$ and all $f \in V$ we have $f(a) = 0$. It follows that we can represent an $f \in V$ by the vector containing the values $f(a)$, where a runs through the monomials of weight $\leq c$. This enables us to perform the operations of linear algebra (testing linear dependence, constructing bases of subspaces and so on) with the elements of V. Furthermore, we can compute the action of elements of $\mathfrak{g}$ on V.

The disadvantage of this approach is that the number of monomials that have to be considered can be very large. So in the same way as in the previous section we try to throw some monomials away. Let A be the set of monomials relative to which we represent the elements of $U(\mathfrak{g})^*$. At the start this will be the set of monomials of weight $\leq c$. Let B be the set of all other monomials. So at the outset B spans a left ideal of $U(\mathfrak{g})$ and $f(b) = 0$ for all $f \in V$ and $b \in B$. We move elements from A to B, without changing this last property. Let $a \in A$ be such that $a \notin Z(\mathfrak{g})$ and xa is a linear combination of elements of B for all $x \in \mathfrak{g}$. Then we claim that $f(a) = 0$ for all $f \in V$. In order to see this we use the basis $\{b_{ij}\psi_i\}$ used in the proof of Theorem 3.1. If $j = 1$ then $b_{ij}\psi_i(a) = \psi_i(a) = 0$ as $a \notin Z(\mathfrak{g})$. If $j > 1$ then $b_{ij} \in \mathfrak{g}U(\mathfrak{g})$ and hence $\bar{b}_{ij}a$ is a linear combination of elements in B. Hence $b_{ij}\psi_i(a) = 0$. Also the span of B along with a continues to be a left ideal. We conclude that we can move a from A to B. We continue this process until we we do not find such monomials any more. The resulting set is usually a lot smaller than the initial one.

We note that the procedure described in the previous remark is exactly the same as the second phase of the algorithm Regular (see Section 2). So we first perform the algorithm Regular, and use the resulting set of monomials to represent elements of the dual of $U(\mathfrak{g})$. The resulting algorithm is called *Dual*.

4. Affine representations at random

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension d. A homomorphism $\rho : \mathfrak{g} \to \mathfrak{aff}(K^d) \subseteq \mathfrak{gl}_{d+1}(K)$ into the Lie algebra of affine transformations

$$\mathfrak{aff}(K^d) \simeq \mathfrak{gl}(K^d) \ltimes K^d$$

is called an *affine representation* of $\mathfrak{g}$. In this section we describe a method that tries to determine a faithful affine representation of $\mathfrak{g}$ of dimension $d+1$. If the method succeeds, then it returns such a faithul representation of dimension $d+1$. However, it may also happen that the method fails and does not return a representation. Also, it is worth noting that the algorithm uses random methods and hence different runs of the algorithm may produce different results.

The method uses induction on a central series in $\mathfrak{g}$. Thus we assume by induction that we have given a central ideal I in $\mathfrak{g}$ with $\dim(I) = 1$ and a faithful affine representation

$$\rho : \mathfrak{g}/I \to M_d(K).$$

Let $\{a_1, \ldots, a_d\}$ be a basis of $\mathfrak{g}$ with $I = \langle a_d \rangle$ and let $M_i = \rho(a_i + I)$ for $1 \leq i \leq d-1$. We assume that every M_i is a lower triangular matrix. Clearly, we can readily extend

ρ to an affine representation of $\mathfrak{g}$ with $\rho(a_i) = M_i$ for $1 \leq i \leq d$ where we set $M_d = 0$ (so that $\rho(a_d) = 0$). This extended representation has kernel I.

Our aim is to extend ρ to a faithful affine representation

$$\psi : \mathfrak{g} \to M_{d+1}(K)$$

such that

$$\psi(a_i) = \begin{pmatrix} M_i & v_i \\ 0 & 0 \end{pmatrix} \text{ for } 1 \leq i \leq d,$$

for certain vectors $v_i \in K^d$. The following lemma shows that the possible values for v_i can be determined using a cohomology computation. Recall that

$$Z^1(\mathfrak{g}, K^d) = \{\nu : \mathfrak{g} \to K^d \text{ linear} \mid \nu([x,y]) = \rho(x)\nu(y) - \rho(y)\nu(x)\}$$

is the space of 1-cocycles with values in the $\rho(\mathfrak{g})$-module K^d.

Lemma 4.1. *ψ is a representation of $\mathfrak{g}$ if and only if $v_i = \delta(a_i)$ for $1 \leq i \leq d$ for some $\delta \in Z^1(\mathfrak{g}, K^d)$.*

Proof. Let $\delta \in Z^1(\mathfrak{g}, K^d)$ with $\delta(a_i) = v_i$. The linearity of δ implies that ψ is linear. The defining condition for maps in $Z^1(\mathfrak{g}, K^d)$ implies that ψ is a Lie algebra representation. The converse follows with similar arguments. □

Note that $Z^1(\mathfrak{g}, K^d)$ is a vector space over K and can be computed readily using linear algebra methods. The computation of $Z^1(\mathfrak{g}, K^d)$ allows to describe all affine representations of $\mathfrak{g}$ extending ρ. It remains to determine the faithful representation among these.

Lemma 4.2. *ψ is faithful if and only if $v_{d+1} \neq 0$.*

Proof. If ψ is faithful, then $v_{d+1} \neq 0$. Conversely, suppose that $v_{d+1} \neq 0$. As ρ is faithful, it follows that $ker(\psi) \subseteq I$. As $v_{d+1} \neq 0$, we find that $ker(\psi) = 0$. □

These ideas can be combined to the following algorithm.

(1) Choose a central series $\mathfrak{g} = \mathfrak{g}_0 > \mathfrak{g}_1 > \ldots > \mathfrak{g}_d > \mathfrak{g}_{d+1} = 0$ of ideals in $\mathfrak{g}$ such that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$.
(2) by induction, extend a faithful affine representation from $\mathfrak{g}/\mathfrak{g}_i$ to $\mathfrak{g}/\mathfrak{g}_{i+1}$:
  - Compute $Z^1(\mathfrak{g}/\mathfrak{g}_{i+1}, K^i)$.
  - Choose a $\delta \in Z^1(\mathfrak{g}/\mathfrak{g}_{i+1}, K^i)$ with $\delta(a_i) \neq 0$.
  - If no such δ exists, then return fail.
  - If δ exists, then extend ρ to $\mathfrak{g}/\mathfrak{g}_{i+1}$.

If $\mathfrak{g}$ has a faithful affine representation of dimension $d+1$, then this algorithm can in principle find it. However, it may be that a "wrong" choice of a δ at a certain step may cause the algorithm to fail at a later step.

The algorithm is based on linear algebra only and hence is very effective. It often suceeds in finding a faithful representation in dimension $d+1$ if it exists.

5. A series of filiform nilpotent Lie algebras

Let K be a field of characteristic zero. In this section we define a filiform Lie algeba $\mathfrak{f}_n$ in each dimension $n \geq 13$ having interesting properties concerning Lie algebra cohomology, affine structures and faithful representations. In fact, we believe that the algebras $\mathfrak{f}_n$ are counter examples to the conjecture of Milnor mentioned in the introduction, i.e., that $\mu(\mathfrak{f}_n) \geq n+2$ holds. Hence it is interesting to compute the invariants $\mu(\mathfrak{f}_n)$.

Define an index set $\mathcal{I}_n$ by

$$\mathcal{I}_n^0 = \{(k,s) \in \mathbb{N}\times\mathbb{N} \mid 2 \leq k \leq [n/2],\ 2k+1 \leq s \leq n\},$$
$$\mathcal{I}_n = \begin{cases} \mathcal{I}_n^0 & \text{if } n \text{ is odd,} \\ \mathcal{I}_n^0 \cup \{(\frac{n}{2}, n)\} & \text{if } n \text{ is even.} \end{cases}$$

Now fix $n \geq 13$. We define a filiform Lie algebra $\mathfrak{f}_n$ of dimension n over K as follows. For $(k,s) \in \mathcal{I}_n$ let $\alpha_{k,s}$ be a set of parameters, subject to the following conditions: all $\alpha_{k,s}$ are zero, except for the following ones:

$$\alpha_{\ell,2\ell+1} = \frac{3}{\binom{\ell}{2}\binom{2\ell-1}{\ell-1}}, \quad \ell = 2,3,\dots,\lfloor\tfrac{n-1}{2}\rfloor,$$
$$\alpha_{3,n-4} = 1,$$
$$\alpha_{4,n-2} = \frac{1}{7} + \frac{10}{21}\frac{(n-7)(n-8)}{(n-4)(n-5)},$$
$$\alpha_{4,n} = \begin{cases} \frac{22105}{15246}, & \text{if } n = 13, \\ 0 & \text{if } n \geq 14, \end{cases}$$

and

$$\alpha_{5,n} = \frac{1}{42} - \frac{70(n-8)}{11(n-2)(n-3)(n-4)(n-5)} + \frac{25}{99}\frac{(n-6)(n-7)(n-8)}{(n-2)(n-3)(n-4)}$$
$$+ \frac{5}{66}\frac{(n-5)(n-6)}{(n-2)(n-3)} - \frac{65}{1386}\frac{(n-7)(n-8)}{(n-4)(n-5)}.$$

Let $(e_1,\dots,e_n)$ be a basis of $\mathfrak{f}_n$ and define the Lie brackets as follows:

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, n-1$$

$$[e_i, e_j] = \sum_{r=1}^{n} \left(\sum_{\ell=0}^{\lfloor \frac{j-i-1}{2} \rfloor} (-1)^{\ell} \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,\, r-j+i+2\ell+1} \right) e_r, \quad 2 \leq i < j \leq n.$$

In order to show that this defines a Lie bracket we need the following lemma which follows from the Pfaff–Saalschütz sum formula:

Lemma 5.1. *We have the following identities for all $n \geq 13$:*

$$\sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell-1} \binom{n-\ell-5}{\ell-2} \alpha_{\ell,2\ell+1} = \frac{(n-7)(n-8)}{(n-4)(n-5)},$$

$$\sum_{\ell=5}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \binom{n-\ell-5}{\ell-4} \alpha_{\ell,2\ell+1} = -\frac{1}{70} + \frac{12(n-8)}{(n-2)(n-3)(n-4)(n-5)},$$

$$\sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \binom{n-\ell-3}{\ell-2} \alpha_{\ell,2\ell+1} = -\frac{(n-5)(n-6)}{(n-2)(n-3)}.$$

Proposition 5.2. *The Jacobi identity is satisfied, so that $\mathfrak{f}_n$ is a Lie algebra for any $n \geq 13$.*

Proof. Let $n \geq 14$ and choose the parameters $\alpha_{k,s}$ as follows. Consider $\alpha_{k,2k+1}$, $k = 3, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and $\alpha_{4,n-2}, \alpha_{5,n}$ as free variables. Let the remaining parameters be zero, except for $\alpha_{2,5} = 1$, $\alpha_{3,7} \neq 0$ and $\alpha_{3,n-4} = 1$. The Jacobi identity is equivalent to a system of polynomial equations in the free parameters. First we obtain the equation $\alpha_{3,7}(10\alpha_{3,7} - \alpha_{2,5}) = 0$, so that $\alpha_{3,7} = \frac{1}{10}$. More generally we see that

$$(\ell - 1) \cdot \alpha_{\ell,2\ell+1} = (4\ell + 2) \cdot \alpha_{\ell+1,2\ell+3}, \quad \ell = 2, 3, \ldots, \lfloor \tfrac{n-1}{2} \rfloor.$$

This implies the given explicit formula for all $\alpha_{\ell,2\ell+1}$. Secondly we obtain

$$\alpha_{4,n-2} = \frac{\alpha_{4,9}}{\alpha_{3,7}} + \frac{\alpha_{4,9}}{3\alpha_{3,7}^2} \sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell-1} \binom{n-\ell-5}{\ell-2} \alpha_{\ell,2\ell+1},$$

$$\alpha_{5,n} = \frac{1}{\alpha_{4,9}+\alpha_{3,7}-2\alpha_{2,5}} \left(-4\alpha_{4,9} + \sum_{\ell=5}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^\ell \binom{n-\ell-5}{\ell-4} \alpha_{\ell,2\ell+1} \right)$$

$$+ \frac{1}{\alpha_{4,9}+\alpha_{3,7}-2\alpha_{2,5}} \left(\alpha_{4,n-2} \left(13\alpha_{4,9} + \sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^\ell \binom{n-\ell-3}{\ell-2} \alpha_{\ell,2\ell+1} \right) \right).$$

This amounts to the given formulas in the definition of $\mathfrak{f}_n$, if we substitute the identities from Lemma 5.1. Conversely this also shows that the Jacobi identity is satisfied if the free parameters are given in this way.
For $n = 13$ there is one difference. The parameter $\alpha_{4,n}$ coincides with the parameter $\alpha_{4,13}$, which is given by

$$\alpha_{4,13} = \frac{\alpha_{3,9}(-\alpha_{5,13} + 6\alpha_{4,11} - 5\alpha_{3,9})}{\alpha_{3,7} + 2\alpha_{2,5}},$$

and cannot be chosen to be zero. For $n \geq 14$ the choice $\alpha_{4,n} = 0$ is consistent with the Jacobi identity. □

Example 5.3. *The parameters for $\mathfrak{f}_{13}$ are given as follows:*

$$\alpha_{2,5} = 1,\ \alpha_{3,7} = \frac{1}{10},\ \alpha_{4,9} = \frac{1}{70},\ \alpha_{5,11} = \frac{1}{420},\ \alpha_{6,13} = \frac{1}{2310},$$

$$\alpha_{3,9} = 1,\ \alpha_{4,11} = \frac{43}{126},\ \alpha_{4,13} = \frac{22105}{15246}, \alpha_{5,13} = \frac{313}{3388}.$$

The algebras $\mathfrak{f}_n$ belong to the family of filiform Lie algebras $\mathfrak{A}_n^2(K)$ defined in [3]. Let us recall the following definition.

Definition 5.4. Let $\mathfrak{g}$ be a filiform nilpotent Lie algebra of dimension n. A 2–cocycle $\omega \in Z^2(\mathfrak{g}, K)$ is called *affine*, if $\omega\colon \mathfrak{g} \wedge \mathfrak{g} \to K$ does not vanish on $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}$. A class $[\omega] \in H^2(\mathfrak{g}, K)$ is called affine if every representative is affine.

The cohomology class $[\omega] \in H^2(\mathfrak{g}, K)$ of an affine 2-cocycle ω is affine and nonzero. If a filiform Lie algebra $\mathfrak{g}$ of dimension $n \geq 6$ has second Betti number $b_2(\mathfrak{g}) = 2$, then there exists no affine cohomology class.
We have shown in [3] that a filiform Lie algebra $\mathfrak{g}$ which has an affine cohomology class, admits a central extension

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with some Lie algebra $\mathfrak{h}$ and $\iota(\mathfrak{a}) = \mathfrak{z}(\mathfrak{h})$, and has an affine structure. In particular, such a Lie algebra has a faithful representation of dimension $n+1$.

We can conclude from the results in [3] that the Lie algebras $\mathfrak{f}_n$ do *not* have an affine structure arising this way.

Proposition 5.5. *The algebras $\mathfrak{f}_n$, $n \geq 13$ have second Betti number $b_2(\mathfrak{f}_n) = 2$. Hence there exists no affine cohomology class $[\omega] \in H^2(\mathfrak{g}, K)$. In particular there is no central Lie algebra extension as above.*

For Lie algebras in $\mathfrak{A}_n^2(K)$ the second Betti number is 3 or 2, depending on whether a certain polynomial identity $\alpha_{3,n-4} = P$ in the free parameters does hold or does not hold. For $\mathfrak{f}_n$ we have chosen the parameters in such a way that $P \equiv 0$ and $\alpha_{3,n-4} = 1$. This implies that $b_2(\mathfrak{f}_n) = 2$.

It follows that a very natural way to obtain a faithful representation of dimension $n+1$ does not work. In fact, we believe that there is no such representation at all for these algebras:

Conjecture 5.6. *The Lie algebras $\mathfrak{f}_n$, $n \geq 13$ do not have any faithful representation of dimension $n+1$, i.e., $\mu(\mathfrak{f}_n) \geq n+2$.*

For $n = 13$ a very complicated analysis of possible faithful representations seems to confirm this conjecture. In general our methods are not sufficient to prove this for all $n \geq 14$. Even more difficult of course is the determination of $\mu(\mathfrak{f}_n)$.

6. Practical experiences

We implemented all the algorithms described above in the computer algebra system GAP. In this section we report on the application of these implementations to various examples. From Section 2 we have the algorithms Regular and Quotient. From Section 3 we have the algorithm Dual. Finally the algorithm of Section 4 is called *Affine*.

In all our experiments Quotient and Dual returned faithful representations of the same dimension (with Dual being slightly faster). This is illustrated in Table 3. We believe that there must be an intrinsic reason for this to happen, such as one module being the dual of the other. But we have no proof of that. We only exhibit the results of Dual in Tables 1 and 2, noting that the results for Quotient are similar in all cases.

All computations were done on a 2GHz processor with 1GB of memory for GAP.

6.1. **Upper triangular matrix Lie algebras.** The upper triangular matrices in $M_n(\mathbb{F})$ form a nilpotent Lie algebra $U_n(\mathbb{F})$ with $n-1$ generators and class $n-1$. We applied our algorithms to some Lie algebras of this type. The results are recorded in Table 1.

Table 1 exhibits that the underlying field does not have much impact on the runtime or the result. The larger the dimension of the considered Lie algebra is, the more superior is Affine. It yields small dimensional representations and is the fastest of all methods.

n	$\mathbb{F}$	$\dim(U_n(\mathbb{F}))$	Regular		Dual		Affine	
			time	dim	time	dim	time	dim
4	$\mathbb{F}_2$	6	0.0	7	0.1	5	0.0	7
5	$\mathbb{F}_2$	10	0.25	15	0.3	11	0.3	11
6	$\mathbb{F}_2$	15	3.4	35	3.6	17	3.5	16
7	$\mathbb{F}_2$	21	65	79	66	35	45	22
4	$\mathbb{F}_3$	6	0.0	7	0.0	5	0.0	7
5	$\mathbb{F}_3$	10	0.2	15	0.3	11	0.3	11
6	$\mathbb{F}_3$	15	3.4	35	3.6	17	3.7	16
7	$\mathbb{F}_3$	21	65	79	67	35	46	22
4	$\mathbb{Q}$	6	0.0	7	0.0	5	0.0	7
5	$\mathbb{Q}$	10	0.2	15	0.3	11	0.3	11
6	$\mathbb{Q}$	15	3.0	35	3.2	17	3.6	16
7	$\mathbb{Q}$	21	66	79	67	35	45	22

TABLE 1. Running times (in seconds) for $U_n(\mathbb{F})$.

6.2. **Free nilpotent Lie algebras.** Next we consider the free nilpotent Lie algebras with n generators of class c over the field $\mathbb{F}$, denoted $N_{n,c}(\mathbb{F})$.

n	c	$\mathbb{F}$	$\dim(N_{n,c}(\mathbb{F}))$	Regular		Dual		Affine	
				time	dim	time	dim	time	dim
2	5	$\mathbb{Q}$	14	0.2	20	0.3	20	0.5	15
2	6	$\mathbb{Q}$	23	0.9	34	1.3	34	8.4	24
2	7	$\mathbb{Q}$	41	3.2	65	4.8	65	☹	☹
2	8	$\mathbb{Q}$	71	14	117	21	117	☹	☹
3	4	$\mathbb{Q}$	32	0.8	41	1.7	41	54	33
3	5	$\mathbb{Q}$	80	11.5	113	17.5	113	☹	☹
4	3	$\mathbb{Q}$	30	0.9	36	1.3	36	37	31
4	4	$\mathbb{Q}$	90	13	113	19.7	113	☹	☹

TABLE 2. Running times (in seconds) for $N_{n,c}(\mathbb{Q})$.

Table 2 displays the time in seconds for the three algorithms, with input $N_{n,c}$. The ☹ in the last two columns indicates that the algorithm Affine did not succeed, either because it made the "wrong" choice at some stage, or due to Memory problems: for its cohomology computation it has to solve a system of linear equations which is of the size $O(\dim(\mathfrak{g})^2)$ and this can be time-and space consuming.

6.3. **The Lie algebras $\mathfrak{f}_n$.** Finally, we consider the Lie algebras $\mathfrak{f}_n$ of the previous section. The results of that are contained in Table 3.

n	Regular		Quotient		Dual		Affine
	time	dim	time	dim	time	dim	
13	8.6	85	14	43	12.3	43	☹
14	17	105	28	53	24.7	53	☹
15	33	145	63	64	50	64	☹
16	64	185	125	77	102	77	☹
17	123	256	323	94	218	94	☹
18	234	316	731	111	461	111	☹
19	487	433	1844	134	1162	134	☹
20	920	538	4009	158	3039	158	☹

TABLE 3. Running time (in seconds) for the Lie algebras $\mathfrak{f}_n$.

Table 3 displays the time in seconds for the algorithms Quotient and Dual, with input $\mathfrak{f}_n$. The ☹ in the last column indicates that the algorithm Affine did not succeed. In this case, this was due to the fact that Affine did not find any possible faithful representation of dimension $n+1$. Of course, if our conjecture on $\mathfrak{f}_n$ holds, then it cannot succeed.

Note that the dimensionals of the determined modules for $\mathfrak{f}_n$ are significantly larger than $n+1$. However, they do not seem to grow very fast. Some naive tests with least squares fits seem to suggest that the dimensions grow quadratically or cubically.

6.4. **Some comments.** From the above tables we conclude that if Affine succeeds, then it usually finds a module of significantly smaller dimension than Regular, Quotient or Dual. This supports the suggested strategy to try this algorithm first.

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