LR-ALGEBRAS

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ABSTRACT. In the study of NIL-affine actions on nilpotent Lie groups we introduced so called LR-structures on Lie algebras. The aim of this paper is to consider the existence question of LR-structures, and to start a structure theory of LR-algebras. We show that any Lie algebra admitting an LR-structure is 2-step solvable. Conversely we find several classes of 2-step solvable Lie algebras admitting an LR-structure, but also classes not admitting such a structure. We study also ideals in LR-algebras, and classify low-dimensional LR-algebras over $\mathbb{R}$.

1. INTRODUCTION

LR-algebras and LR-structures on Lie algebras arise in the study of affine actions on nilpotent Lie groups as follows. Let N be a real, connected and simply connected nilpotent Lie group. Denote by $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ the group of affine transformations of N, acting on N via

$$\forall m, n \in N, \ \forall \alpha \in \mathrm{Aut}(N) : \ ^{(m,\alpha)}n = m \cdot \alpha(n).$$

Note that for the special case where $N = \mathbb{R}^n$, we obtain the usual group of affine transformations $\mathrm{Aff}(\mathbb{R}^n)$ of n-dimensional space. When N is not abelian, we sometimes talk about the NIL-affine group $\mathrm{Aff}(N)$, or NIL-affine motions. Recently, there has been a growing interest in those subgroups $G \subseteq \mathrm{Aff}(N)$ which act either properly discontinuously (in case G is discrete) or simply transitively (in case G is a Lie group) on N (see for example [1], [6]). It is known that all groups which appear as such a simply transitive NIL-affine group have to be solvable. Conversely for any connected and simply connected solvable Lie group G, there exists a nilpotent Lie group N for which one can find an embedding $\rho : G \to \mathrm{Aff}(N)$ realizing G as a subgroup of $\mathrm{Aff}(N)$ acting simply transitively on N (see [6]).

Nevertheless, it is still an open problem to determine for a given G all connected, simply connected nilpotent Lie groups N, on which G acts simply transitively via NIL-affine motions. Even for the case $G = \mathbb{R}^n$ the problem is non-trivial and interesting. For this case we were able to translate this question in [4] to the existence problem of an LR-structure on the Lie algebra $\mathfrak{n}$ of N. Indeed, we showed the following result (for the definition of a complete LR-structure see 1.2).

Theorem 1.1. [4, Theorem 5.1] *Let N be a connected and simply connected nilpotent Lie group of dimension n. Then there exists a simply transitive NIL-affine action of $\mathbb{R}^n$ on N if and only if the Lie algebra $\mathfrak{n}$ of N admits a complete LR-structure.*

The aim of this paper is to begin a study of LR-algebras and LR-structures on Lie algebras. Although LR-algebras arose, as we just explained, in the context of Lie algebras over the field $\mathbb{R}$, we will work over an arbitrary field k of characteristic zero.

Definition 1.2. An algebra $(A, \cdot)$ over k with product $(x, y) \mapsto x \cdot y$ is called an *LR-algebra*, if the product satisfies the identities

$$(1) \qquad\qquad\qquad x \cdot (y \cdot z) = y \cdot (x \cdot z)$$

$$(2) \qquad\qquad\qquad (x \cdot y) \cdot z = (x \cdot z) \cdot y$$

for all $x, y, z \in A$.

Denote by $L(x), R(x)$ the left respectively right multiplication operator in the algebra $(A, \cdot)$. The letters LR stand for "left and right", indicating that in an LR-algebra the left and right multiplication operators commute:

$$(3) \qquad\qquad\qquad [L(x), L(y)] = 0,$$

$$(4) \qquad\qquad\qquad [R(x), R(y)] = 0.$$

LR-algebras are Lie-admissible algebras:

Lemma 1.3. *The commutator $[x, y] = x \cdot y - y \cdot x$ in an LR-algebra $(A, \cdot)$ defines a Lie bracket.*

Proof. We have, using the above identities for all $x, y, z \in A$,

$$\begin{aligned}
[[x, y], z] + [[y, z], x] + [[z, x], y] &= [x, y] \cdot z - z \cdot [x, y] + [y, z] \cdot x - x \cdot [y, z] \\
&\quad + [z, x] \cdot y - y \cdot [z, x] \\
&= 0.
\end{aligned}$$

This shows that the Jacobi identity is indeed satisfied. $\qquad\qquad\qquad\square$

The associated Lie algebra $\mathfrak{g}$ then is said to admit an LR-structure:

Definition 1.4. An *LR-structure* on a Lie algebra $\mathfrak{g}$ over k is an LR-algebra product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$(5) \qquad\qquad\qquad [x, y] = x \cdot y - y \cdot x$$

for all $x, y, z \in \mathfrak{g}$. The LR-structure, resp. the LR-algebra is said to be *complete*, if all left multiplications $L(x)$ are nilpotent.

Remark 1.5. If $\mathfrak{g}$ is abelian, then an LR-structure on $\mathfrak{g}$ is commutative and associative. Indeed, then we have $R(x) = L(x)$. Conversely, commutative, associative algebras form a subclass of LR-algebras with abelian associated Lie algebra.

To conclude this introduction, let us present some easy examples of LR-algebras. Denote by $\mathfrak{r}_2(k)$ the 2-dimensional non-abelian Lie algebra over k with basis (e_1, e_2), and $[e_1, e_2] = e_1$.

Example 1.6. *The classification of non-isomorphic LR-algebras A with associated Lie algebra $\mathfrak{r}_2(k)$ is given as follows:*

A	*Products*
A_1	$e_1 \cdot e_1 = e_1, \ e_2 \cdot e_1 = -e_1.$
A_2	$e_1 \cdot e_2 = e_1.$
A_3	$e_2 \cdot e_1 = -e_1.$

The proof consists of an easy computation. The left multiplications defining an LR-algebra with associated Lie algebra $\mathfrak{r}_2(k)$ are of the following form:

$$L(e_1) = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \; L(e_2) = \begin{pmatrix} \beta - 1 & \gamma \\ 0 & 0 \end{pmatrix},$$

where $\alpha\gamma = \beta(\beta - 1)$. All these algebras are isomorphic to one of the algebras A_1, A_2, A_3. Note that the algebra A_2 is complete, whereas the algebras A_1 and A_3 are incomplete.

2. Structural properties of LR-algebras

We just saw examples of LR-structures on a 2-step solvable Lie algebra, i.e., on $\mathfrak{r}_2(k)$. It turns out that *all* Lie algebras admitting an LR-structure are two-step solvable.

Proposition 2.1. *Any Lie algebra over k admitting an LR-structure is two-step solvable.*

Proof. For any $x, y, u, v \in A$ we have the following symmetry relation:

$$\begin{aligned}
(x \cdot y) \cdot (u \cdot v) & = (x \cdot (u \cdot v)) \cdot y \\
& = (u \cdot (x \cdot v)) \cdot y \\
& = (u \cdot y) \cdot (x \cdot v) \\
& = x \cdot ((u \cdot y) \cdot v) \\
& = x \cdot ((u \cdot v) \cdot y) \\
& = (u \cdot v) \cdot (x \cdot y).
\end{aligned}$$

Using this we obtain

$$\begin{aligned}
[[x, y], [u, v]] & = [x \cdot y - y \cdot x, u \cdot v - v \cdot u] \\
& = (x \cdot y - y \cdot x) \cdot (u \cdot v - v \cdot u) - (u \cdot v - v \cdot u) \cdot (x \cdot y - y \cdot x) \\
& = (x \cdot y) \cdot (u \cdot v) - (x \cdot y) \cdot (v \cdot u) - (y \cdot x) \cdot (u \cdot v) + (y \cdot x) \cdot (v \cdot u) \\
& \quad - (u \cdot v) \cdot (x \cdot y) + (v \cdot u) \cdot (x \cdot y) + (u \cdot v) \cdot (y \cdot x) - (v \cdot u) \cdot (y \cdot x) \\
& = 0.
\end{aligned}$$

This shows that the associated Lie algebra is two-step solvable. $\qquad\square$

When we translate this result, using Theorem 1.1, in terms of NIL-affine actions, we find the following:

Theorem 2.2. *Let N be a connected and simply connected nilpotent Lie group for which $\mathrm{Aff}(N)$ contains an abelian Lie subgroup acting simply transitively on N, then N is two-step solvable.*

Remark 2.3. This result also explains Proposition 4.2 of [4] in a much more conceptual way.

We now present some identities, which are useful when constructing LR-structures on a given Lie algebra. The first pair of identities remind one of the Jacobi identity for Lie algebras:

Lemma 2.4. *Let $(A, \cdot)$ be an LR-algebra. For all $x, y, z \in A$ we have:*

(6) $$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0,$$

(7) $$x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] = 0.$$

Proof. The first identity holds because we have

$$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = (x \cdot y - y \cdot x) \cdot z + (y \cdot z - z \cdot y) \cdot x + (z \cdot x - x \cdot z) \cdot y$$
$$= ((x \cdot y) \cdot z - (x \cdot z) \cdot y) + ((y \cdot z) \cdot x - (y \cdot x) \cdot z)$$
$$+ ((z \cdot x) \cdot y - (z \cdot y) \cdot x)$$
$$= 0.$$

The second identity follows similarly. ☐

We also have the following operator identities:

Lemma 2.5. *In an LR-algebra we have the following operator identities:*

$$(8) \qquad \mathrm{ad}([x, y]) - [\mathrm{ad}(x), L(y)] - [L(x), \mathrm{ad}(y)] = 0.$$
$$(9) \qquad \mathrm{ad}([x, y]) + [\mathrm{ad}(x), R(y)] + [R(x), \mathrm{ad}(y)] = 0.$$

Proof. Using $\mathrm{ad}(x) = L(x) - R(x)$ and (3) and (4) we obtain

$$\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$$
$$= [L(x) - R(x), L(y) - R(y)]$$
$$= [L(x), L(y)] - [R(x), L(y)] - [L(x), R(y)] + [R(x), R(y)]$$
$$= [-R(x), L(y)] + [L(x), -R(y)]$$
$$= [\mathrm{ad}(x), L(y)] + [L(x), \mathrm{ad}(y)]$$

This shows the first identity. The second identity follows similarly. ☐

We now study ideals of LR–algebras.

Lemma 2.6. *Let $(A, \cdot)$ be an LR-algebra and I, J be two-sided ideals of A. Then $I \cdot J$ is also a two-sided ideal of A.*

Proof. It is enough to show that for all $a \in A$, $x \in I$ and $y \in J$, both $a \cdot (x \cdot y)$ and $(x \cdot y) \cdot a$ belong to $I \cdot J$. But this is easy to see:

$$a \cdot (x \cdot y) = x \cdot (a \cdot y) \in I \cdot J,$$
$$(x \cdot y) \cdot a = (x \cdot a) \cdot y \in I \cdot J.$$

☐

Before continuing the study of ideals let us note the following:

Lemma 2.7. *Let $(A, \cdot)$ be an LR-algebra with associated Lie algebra $\mathfrak{g}$, and $a \in A$. Then all operators $L(a)$ and $R(a)$ are Lie derivations of $\mathfrak{g}$, i.e., for any $x, y \in A$, the following identities hold:*

$$a \cdot [x, y] = [a \cdot x, y] + [x, a \cdot y],$$
$$[x, y] \cdot a = [x \cdot a, y] + [x, y \cdot a].$$

Proof. We have

$$a \cdot [x, y] = a \cdot (x \cdot y) - a \cdot (y \cdot x)$$
$$= x \cdot (a \cdot y) - y \cdot (a \cdot x) - (a \cdot y) \cdot x + (a \cdot x) \cdot y$$
$$= [a \cdot x, y] + [x, a \cdot y].$$

The second identity follows similarly. ☐

The above lemma implies the following result:

Corollary 2.8. *Let $(A, \cdot)$ be an LR-algebra and assume that I, J are two-sided ideals of A. Then $[I, J]$ is also a two-sided ideal of A.*

In particular, $[A, A]$ is a two-sided ideal in A. Let $\gamma_1(A) = A$ and $\gamma_{i+1}(A) = [A, \gamma_i(A)]$ for all $i \geq 1$.

Corollary 2.9. *Let A be an LR-algebra. Then all $\gamma_i(A)$ are two-sided ideals of A.*

Lemma 2.10. *Let A be an LR-algebra. Then we have*

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+1}(A)$$

for all $i, j \geq 0$.

Proof. We will use induction on $i \geq 0$. The case $i = 0$ follows from the fact that $\gamma_{j+1}(A)$ is an ideal of A. Now assume $i \geq 1$ and $\gamma_k(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{k+j}(A)$ for all $k = 1, \ldots, i$.
Let $x \in \gamma_1(A)$, $y \in \gamma_i(A)$ and $z \in \gamma_{j+1}(A)$. We have to show that $[x, y] \cdot z \in \gamma_{i+j+1}(A)$. Using Lemma 2.7 and the induction hypothesis, we see that

$$[x, y] \cdot z = [x \cdot z, y] + [x, y \cdot z] \in \gamma_{i+j+1}(A).$$

$\square$

It is natural to introduce the center of an LR-algebra $(A, \cdot)$ by

$$Z(A) = \{x \in A \mid x \cdot y = y \cdot x \text{ for all } y \in A\}.$$

Clearly $Z(A)$ coincides with $Z(\mathfrak{g})$, the center of the associated Lie algebra $\mathfrak{g}$.

Lemma 2.11. *Let $(A, \cdot)$ be an LR-algebra. Then $Z(A) \cdot [A, A] = [A, A] \cdot Z(A) = 0$.*

Proof. Let $a, b \in A$ and $z \in Z(A)$. By (7) we have

$$z \cdot [a, b] + a \cdot [b, z] + b \cdot [z, a] = 0.$$

Since $z \in Z(\mathfrak{g})$, where $\mathfrak{g}$ is the associated Lie algebra of A, we obtain $z \cdot [a, b] = 0$. Analogously one shows that $[a, b] \cdot z = 0$. $\square$

Lemma 2.12. *Let A be an LR-algebra. Then $Z(A)$ is a two-sided ideal of A.*

Proof. Let $z \in Z(A)$. We have to show that $[a \cdot z, b] = [z \cdot a, b] = 0$ for all $a, b \in A$. Using Lemma 2.7 we see that

$$a \cdot [z, b] = [a \cdot z, b] + [z, a \cdot b]$$
$$[z, b] \cdot a = [z \cdot a, b] + [z, b \cdot a].$$

Since $z \in Z(A)$ the claim follows. $\square$

Let $Z_1(A) = Z(A)$ and define $Z_{i+1}(A)$ by the identity $Z_{i+1}(A)/Z_i(A) = Z(A/Z_i(A))$. Note that the $Z_i(A)$ are the terms of the upper central series of the associated Lie algebra $\mathfrak{g}$. As an immediate consequence of the previous lemma, we obtain

Corollary 2.13. *Let A be an LR-algebra. Then all $Z_i(A)$ are two-sided ideals of A.*

3. Classification of LR-structures

A classification of LR-structures in general is as hopeless as a classification of Lie algebras. However, one can study such structures in low dimensions. We will give here a classification of complete LR-structures on real nilpotent Lie algebras of dimension $n \leq 4$. The restriction to complete structures reduces the computations a lot, in particular for abelian Lie algebras. Nevertheless, we have classified also incomplete LR-structures in some cases.

If the Lie algebra is abelian then LR-structures are just given by commutative and associative algebras. Here a classification in terms of polynomial rings and their quotients is well known for $n \leq 6$, see [10] and the references cited therein. We would like, however, to have explicit lists in terms of algebra products. This seems only available in dimension $n \leq 3$ over $\mathbb{R}$ and $\mathbb{C}$, see [7]. For $n = 4$, there is an explicit list of *nilpotent* commutative, associative algebras (see the references in [10]), but not for all algebras. Such nilpotent, commutative, associative algebras correspond exactly to what are called complete left-symmetric algebras with abelian associated Lie algebra (see [3], [7] and [8] for more details). As [8] gives a list of all complete left-symmetric algebras with a nilpotent associated Lie algebra in dimension 4, one can easily extract those with an abelian associated Lie algebra from that list and so one obtains the complete abelian LR-structures in dimension 4.

It remains to classify all complete LR-structures on a non-abelian nilpotent Lie algebra of dimension $n \leq 4$ over $\mathbb{R}$, which is one of the following:

$\mathfrak{g}$	Lie brackets
$\mathfrak{n}_3(\mathbb{R})$	$[e_1, e_2] = e_3$
$\mathfrak{n}_3(\mathbb{R}) \oplus \mathbb{R}$	$[e_1, e_2] = e_3$
$\mathfrak{n}_4(\mathbb{R})$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$

Proposition 3.1. *The classification of LR-algebra structures on the Heisenberg Lie algebra* $\mathfrak{n}_3(\mathbb{R})$ *is given as follows:*

A	*Products*
$A_1(\alpha), \ \alpha \in \mathbb{R}$	$e_1 \cdot e_1 = e_3, \ e_1 \cdot e_2 = e_3, \ e_2 \cdot e_2 = \alpha e_3.$
$A_2(\beta), \ \beta \in \mathbb{R}$	$e_1 \cdot e_2 = \beta e_3, \ e_2 \cdot e_1 = (\beta - 1)e_3, \ e_2 \cdot e_2 = e_1.$
A_3	$e_1 \cdot e_2 = \frac{1}{2}e_3, \ e_2 \cdot e_1 = -\frac{1}{2}e_3.$
A_4	$e_2 \cdot e_1 = -e_3, \ e_2 \cdot e_2 = e_2,$
	$e_2 \cdot e_3 = e_3, \ e_3 \cdot e_2 = e_3$

All LR-algebras are complete, except for A_4.

Proof. Any LR-algebra structure on $\mathfrak{n}_3(\mathbb{R})$ is isomorphic to one of the following, written down with left multiplication operators for the basis (e_1, e_2, e_3):

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & \gamma & 0 \\ \beta & \delta & \gamma \end{pmatrix}, \ L(e_2) = \begin{pmatrix} 0 & \lambda & 0 \\ \gamma & \mu & 0 \\ \delta - 1 & \nu & \mu \end{pmatrix}, \ L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma & \mu & 0 \end{pmatrix},$$

satisfying the following polynomial equations:

$$\alpha\lambda = 0$$
$$\gamma\lambda = 0$$
$$\gamma^2 - \alpha\mu = 0$$
$$\gamma(2\delta - 1) - \alpha\nu - \beta\mu = 0$$
$$\beta\lambda = 0$$

A straightforward case by case analysis yields the result. We have $A_1(\alpha) \simeq A_1(\alpha')$ if and only $\alpha' = \alpha$, and the same result for $A_2(\beta)$. $\square$

Similarly we obtain the following result.

Proposition 3.2. *The classification of LR-algebra structures on $\mathfrak{g} = \mathfrak{n}_4(\mathbb{R})$ is given as follows:*

A	Products
$A_1(\alpha)$ $\alpha \in \mathbb{R}$	$e_1 \cdot e_1 = \alpha(\alpha - 1)e_2, \ e_1 \cdot e_2 = \alpha e_3, \ e_1 \cdot e_3 = \alpha e_4,$ $e_2 \cdot e_1 = (\alpha - 1)e_3, \ e_2 \cdot e_2 = e_4, \ e_3 \cdot e_1 = (\alpha - 1)e_4.$
A_2	$e_1 \cdot e_1 = e_3, \ e_2 \cdot e_1 = -e_3,$ $e_2 \cdot e_2 = e_4, \ e_3 \cdot e_1 = -e_4.$
A_3	$e_1 \cdot e_1 = e_3, \ e_1 \cdot e_2 = e_3,$ $e_1 \cdot e_3 = e_4, \ e_2 \cdot e_2 = e_4.$
$A_4(\alpha, \beta, \gamma)$ $\alpha, \beta, \gamma \in \{0, 1\}$	$e_1 \cdot e_1 = \alpha e_2, \ e_1 \cdot e_2 = \beta e_3 + \gamma e_4, \ e_1 \cdot e_3 = \beta e_4,$ $e_2 \cdot e_1 = (\beta - 1)e_3 + \gamma e_4, \ e_3 \cdot e_1 = (\beta - 1)e_4.$
$A_5(\alpha)$ $\alpha \in \{0, 1\}$	$e_1 \cdot e_1 = \alpha e_4, \ e_2 \cdot e_1 = -e_3, \ e_2 \cdot e_2 = e_3,$ $e_2 \cdot e_3 = e_4, \ e_3 \cdot e_1 = -e_4, \ e_3 \cdot e_2 = e_4.$
A_6	$e_2 \cdot e_1 = -e_3, \ e_2 \cdot e_2 = e_2, \ e_2 \cdot e_3 = e_3, \ e_2 \cdot e_4 = e_4,$ $e_3 \cdot e_1 = -e_4, \ e_3 \cdot e_2 = e_3, \ e_3 \cdot e_3 = e_4, \ e_4 \cdot e_2 = e_4.$

The algebra A_6 is not complete. All the other ones are complete.

The family $A_4(\alpha, \beta, \gamma)$ consists of 8 different algebras.

Proposition 3.3. *The classification of complete LR-algebra structures on $\mathfrak{g} = \mathfrak{n}_3(\mathbb{R}) \oplus \mathbb{R}$ is given as follows:*

A	Products
$A_1(\alpha)$ $\alpha \in \mathbb{R}$	$e_1 \cdot e_2 = \alpha e_3, \ e_2 \cdot e_1 = (\alpha - 1)e_3,$ $e_2 \cdot e_2 = e_1, \ e_4 \cdot e_4 = e_3.$
$A_2(\alpha)$ $\alpha \in \{0, 1\}$	$e_1 \cdot e_1 = \alpha e_3, \ e_1 \cdot e_2 = e_4, \ e_2 \cdot e_1 = -e_3 + e_4,$ $e_2 \cdot e_2 = e_1, \ e_2 \cdot e_4 = \alpha e_3, \ e_4 \cdot e_2 = \alpha e_3.$
$A_3(\alpha, \beta)$ $\alpha \in \mathbb{R}, \ \beta \in \{0, 1\}$	$e_1 \cdot e_2 = \alpha e_3, \ e_2 \cdot e_1 = (\alpha - 1)e_3, \ e_2 \cdot e_2 = e_1,$ $e_2 \cdot e_4 = \beta e_3, \ e_4 \cdot e_2 = \beta e_3.$
$A_4(\alpha)$ $\alpha \in \{0, 1\}$	$e_1 \cdot e_1 = e_4, \ e_1 \cdot e_4 = e_3, \ e_2 \cdot e_1 = -e_3,$ $e_2 \cdot e_2 = \alpha e_3, \ e_4 \cdot e_1 = e_3.$
$A_5(\alpha)$ $\alpha \in \{0, 1\}$	$e_1 \cdot e_4 = e_3, \ e_2 \cdot e_1 = -e_3,$ $e_2 \cdot e_2 = \alpha e_3, \ e_4 \cdot e_1 = e_3.$

A	$Products$
$A_6(\alpha)$	$e_1 \cdot e_1 = \alpha e_3,\ e_2 \cdot e_1 = -e_3,$
$\alpha \in \mathbb{R}$	$e_2 \cdot e_2 = e_3,\ e_4 \cdot e_4 = e_3.$
$A_7(\alpha)$	$e_1 \cdot e_1 = \alpha e_3,\ e_2 \cdot e_1 = -e_3,$
$\alpha \leq \frac{-1}{4}$	$e_2 \cdot e_2 = -e_3,\ e_4 \cdot e_4 = e_3.$
A_8	$e_1 \cdot e_2 = \frac{1}{2}e_3,\ e_2 \cdot e_1 = -\frac{1}{2}e_3,$
	$e_4 \cdot e_4 = e_3.$
$A_9(\alpha)$	$e_1 \cdot e_1 = e_4,\ e_1 \cdot e_2 = \alpha e_3,$
$\alpha \geq \frac{1}{2}$	$e_2 \cdot e_1 = (\alpha - 1)e_3,\ e_2 \cdot e_2 = e_4.$
$A_{10}(\alpha)$	$e_1 \cdot e_1 = e_4,\ e_1 \cdot e_2 = \alpha e_3,$
$\alpha \geq \frac{1}{2}$	$e_2 \cdot e_1 = (\alpha - 1)e_3,\ e_2 \cdot e_2 = -e_4.$
$A_{11}(\alpha)$	$e_1 \cdot e_1 = e_4,\ e_1 \cdot e_2 = \alpha e_3,$
$\alpha \in \mathbb{R}$	$e_2 \cdot e_1 = (\alpha - 1)e_3.$
A_{12}	$e_1 \cdot e_1 = e_4,\ e_2 \cdot e_1 = -e_3,$
	$e_2 \cdot e_2 = e_3.$
$A_{13}(\alpha)$	$e_1 \cdot e_1 = e_3,\ e_2 \cdot e_1 = -e_3,$
$\alpha \in \mathbb{R}$	$e_2 \cdot e_2 = \alpha e_3.$
A_{14}	$e_1 \cdot e_2 = \frac{1}{2}e_3,\ e_2 \cdot e_1 = -\frac{1}{2}e_3.$
$A_{15}(\alpha)$	$e_1 \cdot e_1 = e_4,\ e_2 \cdot e_1 = -e_3,$
$\alpha \geq 1$	$e_2 \cdot e_2 = \alpha e_3 - e_4.$

Remark 3.4. The computations for the above result are quite complicated, but do not give much insight. Therefore we have omitted them here. However, we did the computations independently to be sure that they are correct.

4. LR-STRUCTURES ON NILPOTENT LIE ALGEBRAS

We know that any Lie algebra admitting an LR-structure must be 2-step solvable. Conversely we can ask which 2-step solvable Lie algebras admit an LR-structure. We start with 2-step solvable, filiform nilpotent Lie algebras $\mathfrak{f}_n$ of dimension n. There exists a so called adapted basis $(e_1, \ldots, e_n)$ of $\mathfrak{f}_n$ such that the Lie brackets are given as follows:

$$[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1,$$

$$[e_2, e_i] = \sum_{k=i+2}^{n} c_{i,k} e_k, \quad 3 \leq i \leq n-2,$$

$$[e_i, e_j] = 0, \quad 3 \leq i \leq j.$$

The Jacobi identity is satisfied if and only if $c_{i+1,k} = c_{i,k-1}$ for all $6 \leq i+3 \leq k \leq n$. For details, see for example [2].

Lemma 4.1. *Let $\mathfrak{f}_n$ be given as above. Then the identities*

$$(10) \qquad\qquad \mathrm{ad}(e_2)\,\mathrm{ad}(e_1)^2 = \mathrm{ad}(e_1)\,\mathrm{ad}(e_2)\,\mathrm{ad}(e_1),$$

$$(11) \qquad\qquad \mathrm{ad}(e_1)\,\mathrm{ad}(e_2)^2 = \mathrm{ad}(e_2)\,\mathrm{ad}(e_1)\,\mathrm{ad}(e_2),$$

$$(12) \qquad\qquad \mathrm{ad}(e_{i+2}) = \mathrm{ad}(e_1)^i\,\mathrm{ad}(e_2) - \mathrm{ad}(e_2)\,\mathrm{ad}(e_1)^i, \quad i \geq 1.$$

hold.

Proof. The identity (10) is equivalent to

$$0 = [\mathrm{ad}(e_1), \mathrm{ad}(e_2)] \, \mathrm{ad}(e_1) = \mathrm{ad}(e_3) \, \mathrm{ad}(e_1).$$

But this follows from $[e_3, [e_1, e_k]] = 0$ for all $k \geq 1$. Similarly, (11) is equivalent to

$$\mathrm{ad}(e_3) \, \mathrm{ad}(e_2) = 0,$$

which follows again by definition. The identity (12) is proved by induction on $i \geq 1$. For $i = 1$ we have

$$\mathrm{ad}(e_3) = [\mathrm{ad}(e_1), \mathrm{ad}(e_2)] = \mathrm{ad}(e_1) \, \mathrm{ad}(e_2) - \mathrm{ad}(e_2) \, \mathrm{ad}(e_1).$$

By induction hypothesis, $\mathrm{ad}(e_{i+1}) = \mathrm{ad}(e_1)^{i-1} \, \mathrm{ad}(e_2) - \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^{i-1}$. Then, using (10) repeatedly, we obtain for $i \geq 2$

$$\begin{aligned}
\mathrm{ad}(e_{i+2}) &= \mathrm{ad}(e_1) \, \mathrm{ad}(e_{i+1}) - \mathrm{ad}(e_{i+1}) \, \mathrm{ad}(e_1) \\
&= \mathrm{ad}(e_1)^i \, \mathrm{ad}(e_2) - \mathrm{ad}(e_1) \, \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^{i-1} - \mathrm{ad}(e_1)^{i-1} \, \mathrm{ad}(e_2) \, \mathrm{ad}(e_1) + \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^i \\
&= \mathrm{ad}(e_1)^i \, \mathrm{ad}(e_2) - \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^i - \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^i + \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^i \\
&= \mathrm{ad}(e_1)^i \, \mathrm{ad}(e_2) - \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^i.
\end{aligned}$$

$\square$

Proposition 4.2. *Any 2-step solvable filiform nilpotent Lie algebra $\mathfrak{f}_n$ admits a complete LR-structure.*

Proof. Define an LR-structure on $\mathfrak{f}_n$ as follows:

$$\begin{aligned}
L(e_1) &= 0, \\
L(e_i) &= \mathrm{ad}(e_1)^{i-2} \, \mathrm{ad}(e_2), \quad 2 \leq i \leq n.
\end{aligned}$$

In particular, this means

$$\begin{aligned}
e_1 \cdot e_j &= 0, \quad e_2 \cdot e_j = [e_2, e_j], \quad 1 \leq j \leq n, \\
e_j \cdot e_1 &= [e_j, e_1], \quad e_j \cdot e_2 = 0, \quad 1 \leq j \leq n,
\end{aligned}$$

so that $R(e_1) = -\mathrm{ad}(e_1)$ and $R(e_2) = 0$. Furthermore, we have

$$e_i \cdot e_j = [e_2, e_{i+j-2}], \quad 3 \leq i, j \leq n.$$

To see this, note that $e_3 \cdot e_j = \mathrm{ad}(e_1) \, \mathrm{ad}(e_2)(e_j) = [e_2, e_{j+1}]$ for $j \geq 3$. Then the result for $i \geq 3$ follows inductively.

Now let us prove that

$$e_i \cdot e_j - e_j \cdot e_i = [e_i, e_j], \quad 1 \leq i \leq j \leq n.$$

The cases $i = 1$ and $i = 2$ are obvious. For $j \geq i \geq 3$ we have

$$e_i \cdot e_j - e_j \cdot e_i = 0 = [e_i, e_j].$$

In particular it follows $R(e_i) = L(e_i) - \mathrm{ad}(e_i)$. The formula (12) then implies

(13) $$R(e_i) = \mathrm{ad}(e_2) \, \mathrm{ad}(e_1)^{i-2}, \quad i \geq 3.$$

It remains to show that all operators $L(e_i)$ commute, and all $R(e_i)$ commute, i.e.,

$$\begin{aligned}
L(e_i)L(e_j) &= L(e_j)L(e_i), \quad 1 \leq i < j \leq n, \\
R(e_i)R(e_j) &= R(e_j)R(e_i), \quad 1 \leq i < j \leq n.
\end{aligned}$$

The first identity is obvious for $i = 1$. For $2 \leq i < j \leq n$ use (10) and (11) repeatedly to obtain

$$\begin{aligned}
L(e_i)L(e_j) &= \operatorname{ad}(e_1)^{i-2} \operatorname{ad}(e_2) \operatorname{ad}(e_1)^{j-2} \operatorname{ad}(e_2) \\
&= \operatorname{ad}(e_1)^{j-2} \operatorname{ad}(e_2) \operatorname{ad}(e_1)^{i-2} \operatorname{ad}(e_2) \\
&= L(e_j)L(e_i).
\end{aligned}$$

This argument also shows $R(e_i)R(e_j) = R(e_j)R(e_i)$ for $3 \leq i < j \leq n$, because of (13). For $i = 2$ this is trivially true since $R(e_2) = 0$. For $i = 1$ and $j \geq 3$ we have to show that $\operatorname{ad}(e_1)R(e_j) = R(e_j)\operatorname{ad}(e_1)$. This follows again from (10). It is obvious that all $L(e_i)$ are nilpotent, hence the LR-structure is complete. $\qquad\square$

It is natural to ask which other nilpotent Lie algebras admit LR-structures. We first observe the following fact.

Proposition 4.3. *Every 2-step nilpotent Lie algebra $\mathfrak{g}$ admits a complete LR-structure.*

Proof. For $x \in \mathfrak{g}$ define an LR-structure by

$$L(x) = \frac{1}{2} \operatorname{ad}(x).$$

Indeed, for all $x, y, z \in \mathfrak{g}$ we have

$$\begin{aligned}
x \cdot y - y \cdot x &= \frac{1}{2}[x, y] - \frac{1}{2}[y, x] = [x, y], \\
x \cdot (y \cdot z) &= 0 = y \cdot (x \cdot z), \\
(x \cdot y) \cdot z &= 0 = (x \cdot z) \cdot y.
\end{aligned}$$

Finally $L(x)$ is a nilpotent derivation for all $x \in \mathfrak{g}$, since $\mathfrak{g}$ is nilpotent. $\qquad\square$

Proposition 4.4. *Every free 3-step nilpotent Lie algebra $\mathfrak{g}$ on n generators $x_1, \ldots, x_n$ admits a complete LR-structure.*

Proof. A vector space basis of $\mathfrak{g}$ is given by

$$\begin{aligned}
x_1, x_2, \cdots, x_n \\
y_{i,j} = [x_i, x_j], \quad 1 \leq i < j \leq n, \\
z_{i,j,k} = [x_i, y_{j,k}].
\end{aligned}$$

An LR-structure on $\mathfrak{g}$ is defined as follows:

$$\begin{aligned}
x_j \cdot x_i &= -y_{i,j}, \quad 1 \leq i < j \leq n \\
x_i \cdot y_{j,k} &= z_{i,j,k}, \quad 1 \leq j < k \leq i \\
&= z_{k,j,i}, \quad j < i < k \\
y_{j,k} \cdot x_i &= z_{k,j,i} - z_{i,j,k}, \quad j < i < k \\
&= -z_{i,j,k}, \quad i \leq j < k
\end{aligned}$$

and all other products equal to zero. $\qquad\square$

Example 4.5. *Let $\mathfrak{f}$ be the free 3-step nilpotent Lie algebra with 3 generators. Then there is a basis $(x_1, \ldots, x_{14})$ of $\mathfrak{f}$ with generators (x_1, x_2, x_3) and Lie brackets*

$$\begin{aligned}
x_4 &= [x_1, x_2] & x_{10} &= [x_1, [x_1, x_3]] = [x_1, x_5] \\
x_5 &= [x_1, x_3] & x_{11} &= [x_2, [x_1, x_3]] = [x_2, x_5] \\
x_6 &= [x_2, x_3] & x_{12} &= [x_3, [x_1, x_3]] = [x_3, x_5] \\
x_7 &= [x_1, [x_1, x_2]] = [x_1, x_4] & x_{11} - x_9 &= [x_1, [x_2, x_3]] = [x_1, x_6] \\
x_8 &= [x_2, [x_1, x_2]] = [x_2, x_4] & x_{13} &= [x_2, [x_2, x_3]] = [x_2, x_6] \\
x_9 &= [x_3, [x_1, x_2]] = [x_3, x_4] & x_{14} &= [x_3, [x_2, x_3]] = [x_3, x_6]
\end{aligned}$$

An LR-structure is given by

$$\begin{aligned}
x_2.x_1 &= -x_4, & x_3.x_6 &= x_{14}, \\
x_2.x_4 &= x_8, & x_4.x_1 &= -x_7, \\
x_2.x_5 &= x_9, & x_5.x_1 &= -x_{10}, \\
x_3.x_1 &= -x_5, & x_5.x_2 &= x_9 - x_{11}, \\
x_3.x_2 &= -x_6, & x_6.x_1 &= x_9 - x_{11}, \\
x_3.x_4 &= x_9, & x_6.x_2 &= -x_{13}. \\
x_3.x_5 &= x_{12},
\end{aligned}$$

Proposition 4.4 implies, in the same way as for Novikov structures (see [3]), the following corollary.

Corollary 4.6. *Any 3-generated 3-step nilpotent Lie algebra admits a complete LR-structure.*

One might ask whether or not all 3-step nilpotent Lie algebras admit an LR-structure. This turns out to be not the case. To find a counterexample we have to look at Lie algebras with at least 4 generators.

Proposition 4.7. *Let $\mathfrak{g}$ be the following 3-step nilpotent Lie algebra on 4 generators of dimension 13, with basis $(x_1, \ldots, x_{13})$ and non-trivial Lie brackets*

$$\begin{aligned}
[x_1, x_2] &= x_5, & [x_3, x_4] &= -x_5, \\
[x_1, x_4] &= x_6, & [x_3, x_5] &= -x_{11}, \\
[x_1, x_6] &= x_{10}, & [x_3, x_8] &= x_9, \\
[x_1, x_7] &= x_{11}, & [x_4, x_5] &= -x_{12}, \\
[x_1, x_8] &= x_{12}, & [x_4, x_6] &= x_9, \\
[x_2, x_3] &= x_7, & [x_4, x_7] &= x_9 + x_{13}. \\
[x_2, x_4] &= x_8, \\
[x_2, x_5] &= x_{13}, \\
[x_2, x_7] &= x_{13},
\end{aligned}$$

This 2-step solvable Lie algebra does not admit an LR-structure.

Proof. We will assume that $\mathfrak{g}$ admits an LR-structure and show that this leads to a contradiction. We denote by $\operatorname{ad}(x_i)$ the adjoint operators, by $L(x_i)$ the left multiplications, and by $R(x_i)$ the right multiplication operators with respect to the basis $(x_1, x_2, \ldots, x_{13})$. The operators $\operatorname{ad}(x_i)$ are given by the Lie brackets of $\mathfrak{g}$, while the left multiplication operators are

unknown. We denote the (j, k)-th entry of $L(x_i)$ by

$$L(x_i)_{j,k} = x^i_{j,k}.$$

The j-th column of $L(x_i)$ gives the coordinates of $L(x_i)(x_j)$. We have to satisfy the identities (1), (2) and (5), where x, y and z run over all basis vectors. This leads to a huge system of quadratic equations in the variables $x^i_{j,k}$ for $1 \le i, j, k \le 13$, summing up to a total of $13^3 = 2197$ variables. It is quite impossible to solve these equations without further information. However, we can use our knowledge on ideals in LR-algebras to conclude that a lot of unknowns $x^i_{j,k}$ already have to be zero. This, together with Lemmas 2.4 and 2.5, simplifies the system of equations considerably. In this way we can show that the equations are contradictory. This works exactly as in the proof of proposition 3.3 of our paper [5], where we proved that the above Lie algebra does not admit a Novikov structure. $\square$

In [3] we showed that the 2-generated, free 4-step nilpotent Lie algebra (which is 2-step solvable) does not admit any Novikov structure. It turns out however, that this example does admit an LR-structure:

Let $\mathfrak{g}$ be the free 4-step nilpotent Lie algebra on 2 generators x_1 and x_2. Let $(x_1, \ldots, x_8)$ be a basis of $\mathfrak{g}$ with the following Lie brackets:

$$
\begin{aligned}
x_3 &= [x_1, x_2] & x_6 &= [x_1, [x_1, [x_1, x_2]]] = [x_1, x_4] \\
x_4 &= [x_1, [x_1, x_2]] = [x_1, x_3] & x_7 &= [x_2, [x_1, [x_1, x_2]]] = [x_2, x_4] \\
x_5 &= [x_2, [x_1, x_2]] = [x_2, x_3] & &= [x_1, [x_2, [x_1, x_2]]] = [x_1, x_5] \\
& & x_8 &= [x_2, [x_2, [x_1, x_2]]] = [x_2, x_5]
\end{aligned}
$$

Proposition 4.8. *The* 2-*step solvable Lie algebra from the above example does admit an LR-structure.*

Proof. We define the left multilications by $L(x_1) = 0$, $L(x_2) = \operatorname{ad}(x_2)$ and, if x_i, $i \ge 3$ is a bracket of x_1 and x_2, then $L(x_i)$ is the corresponding composition of $\operatorname{ad}(x_1)$ and $\operatorname{ad}(x_2)$:

$$
\begin{aligned}
L(x_3) &= L([x_1, x_2]) = \operatorname{ad}(x_1) \operatorname{ad}(x_2), \\
L(x_4) &= L([x_1, [x_1, x_2]]) = \operatorname{ad}(x_1)^2 \operatorname{ad}(x_2) \\
L(x_5) &= L([x_2, [x_1, x_2]]) = \operatorname{ad}(x_2) \operatorname{ad}(x_1) \operatorname{ad}(x_2) \\
L(x_6) &= L([x_1, [x_1, [x_1, x_2]]]) = \operatorname{ad}(x_1)^3 \operatorname{ad}(x_2) \\
L(x_7) &= L([x_2, [x_1, [x_1, x_2]]]) = \operatorname{ad}(x_2) \operatorname{ad}(x_1)^2 \operatorname{ad}(x_2) \\
&= L([x_1, [x_2, [x_1, x_2]]]) = \operatorname{ad}(x_1) \operatorname{ad}(x_2) \operatorname{ad}(x_1) \operatorname{ad}(x_2) \\
L(x_8) &= L([x_2, [x_2, [x_1, x_2]]]) = \operatorname{ad}(x_2)^2 \operatorname{ad}(x_1) \operatorname{ad}(x_2).
\end{aligned}
$$

In fact this really defines an LR-structure. Note that $L(x_6) = L(x_7) = L(x_8) = 0$. $\square$

5. Construction of LR-structures via extensions

In the following we will consider Lie algebras $\mathfrak{g}$ which are an extension of a Lie algebra $\mathfrak{b}$ by an abelian Lie algebra $\mathfrak{a}$. Hence we have a short exact sequence of Lie algebras

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0.$$

Since $\mathfrak{a}$ is abelian, there exists a natural $\mathfrak{b}$-module structure on $\mathfrak{a}$. We denote the action of $\mathfrak{b}$ on $\mathfrak{a}$ by $(x, a) \mapsto \varphi(x)a$, where $\varphi \colon \mathfrak{b} \to \operatorname{End}(\mathfrak{a})$ is the corresponding Lie algebra representation.

We have

(14)
$$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$$

for all $x, y \in \mathfrak{b}$. The extension $\mathfrak{g}$ is determined by a two-cohomology class (see [9] for more information). Let $\Omega \in Z^2(\mathfrak{b}, \mathfrak{a})$ be a 2-cocycle describing the extension $\mathfrak{g}$. This implies that $\Omega : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}$ is a skew-symmetric bilinear map satisfying

(15)
$$\varphi(x)\Omega(y,z) - \varphi(y)\Omega(x,z) + \varphi(z)\Omega(x,y) = \Omega([x,y],z) - \Omega([x,z],y) + \Omega([y,z],x),$$

such that the Lie algebra with underlying vector space $\mathfrak{a} \times \mathfrak{b}$ and Lie bracket given by

(16)
$$[(a,x),(b,y)] := (\varphi(x)b - \varphi(y)a + \Omega(x,y), [x,y])$$

for $a, b \in \mathfrak{a}$ and $x, y \in \mathfrak{b}$, is isomorphic to $\mathfrak{g}$. As a shorthand, we will use $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ to say that $\mathfrak{g}$ is the extension determined by this specific data.

Note that the Lie algebras $\mathfrak{g}$ we are interested in, are all 2-step solvable Lie algebras and hence can be obtained as extensions of two abelian Lie algebras $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{b} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. So, although we will treat extensions in the general case, we pay special attention to this specific situation where both $\mathfrak{a}$ and $\mathfrak{b}$ are abelian. In this specific case, the Lie bracket of $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ is given by

$$[(a,x),(b,y)] := (\varphi(x)b - \varphi(y)a + \Omega(x,y), 0)$$

and the conditions on φ and Ω are now given as follows: since $\mathfrak{a}$ and $\mathfrak{b}$ are abelian, φ is just a linear map satisfying

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$$

for all $x, y \in \mathfrak{b}$. On the other hand, $\Omega : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}$ is a skew-symmetric bilinear map satisfying

$$\varphi(x)\Omega(y,z) - \varphi(y)\Omega(x,z) + \varphi(z)\Omega(x,y) = 0.$$

Now, let us return to the more general case (i.e., $\mathfrak{b}$ does not have to be abelian), and try to construct LR-structures on Lie algebras $\mathfrak{g}$ which are given as extension $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ of a Lie algebra $\mathfrak{b}$ by an abelian Lie algebra $\mathfrak{a}$. Suppose that we have already an LR-product $(a,b) \mapsto a \cdot b$ on $\mathfrak{a}$ and an LR-product $(x,y) \mapsto x \cdot y$ on $\mathfrak{b}$. In other words, we have

$$x \cdot y - y \cdot x = [x,y]$$
$$x \cdot (y \cdot z) = y \cdot (x \cdot z)$$
$$(x \cdot y) \cdot z = (x \cdot z) \cdot y$$
$$a \cdot b = b \cdot a$$
$$a \cdot (b \cdot c) = b \cdot (a \cdot c)$$
$$(a \cdot b) \cdot c = (a \cdot c) \cdot b$$

for all $x, y, z \in \mathfrak{b}$ and for all $a, b, c \in \mathfrak{a}$. (In fact the product on $\mathfrak{a}$ has to be commutative and associative). We want to lift these LR-products to $\mathfrak{g}$. Consider

$$\omega : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}$$
$$\varphi_1, \varphi_2 : \mathfrak{b} \to \operatorname{End}(\mathfrak{a})$$

where ω is a bilinear map and φ_1, φ_2 are linear maps. We will define a bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

(17)
$$(a,x) \circ (b,y) := (a \cdot b + \varphi_1(y)a + \varphi_2(x)b + \omega(x,y), x \cdot y)$$

Proposition 5.1. *The above product defines an LR-structure on $\mathfrak{g}$ if and only if the following conditions hold:*

$$(18) \qquad\qquad \omega(x,y) - \omega(y,x) = \Omega(x,y)$$

$$(19) \qquad\qquad \varphi_2(x) - \varphi_1(x) = \varphi(x)$$

$$(20) \qquad \varphi_2(x)\omega(y,z) - \varphi_2(y)\omega(x,z) = \omega(y, x \cdot z) - \omega(x, y \cdot z)$$

$$(21) \qquad a \cdot \omega(y,z) + \varphi_1(y \cdot z)a = \varphi_2(y)\varphi_1(z)a$$

$$(22) \qquad\qquad [\varphi_2(x), \varphi_2(y)] = 0$$

$$(23) \qquad\qquad \varphi_2(y)(a \cdot c) = a \cdot (\varphi_2(y)c)$$

$$(24) \qquad\qquad a \cdot (\varphi_1(z)b) = b \cdot (\varphi_1(z)a)$$

$$(25) \qquad \varphi_1(z)\omega(x,y) - \varphi_1(y)\omega(x,z) = \omega(x \cdot z, y) - \omega(x \cdot y, z)$$

$$(26) \qquad \omega(x,y) \cdot c + \varphi_2(x \cdot y)c = \varphi_1(y)\varphi_2(x)c$$

$$(27) \qquad\qquad [\varphi_1(x), \varphi_1(y)] = 0$$

$$(28) \qquad\qquad \varphi_1(z)(a \cdot b) = (\varphi_1(z)a) \cdot b$$

$$(29) \qquad\qquad (\varphi_2(x)c) \cdot b = (\varphi_2(x)b) \cdot c$$

for all $a, b, c \in \mathfrak{a}$ and $x, y, z \in \mathfrak{b}$.

Proof. Let $u = (a, x), v = (b, y), w = (c, z)$ denote three arbitrary elements of $\mathfrak{g}$. Let us first consider the equation (5) for the product, i.e., $[u, v] = u \circ v - v \circ u$. Using (16), (17) and the commutativity of the LR-product in $\mathfrak{a}$ we obtain

$$[u, v] = (\varphi(x)b - \varphi(y)a + \Omega(x,y), [x,y])$$
$$u \circ v - v \circ u = ((\varphi_2(x) - \varphi_1(x))b - (\varphi_2(y) - \varphi_1(y))a + \omega(x,y) - \omega(y,x), [x,y]).$$

Suppose that the two expressions are equal for all $a, b \in \mathfrak{a}$ and $x, y \in \mathfrak{b}$. For $a = b = 0$ we obtain $\omega(x,y) - \omega(y,x) = \Omega(x,y)$. Taking this into account, $a = 0$ implies $\varphi_2(x) - \varphi_1(x) = \varphi(x)$. Conversely, equations (18) and (19) imply (5).

A similar computations shows that (1) corresponds to the equations $20, \dots, 24$, and (2) corresponds to $25, \dots, 29$. $\qquad\qquad\square$

Corollary 5.2. *Assume that $\mathfrak{g} = \mathfrak{a} \rtimes_\varphi \mathfrak{b}$ is a semidirect product of an abelian Lie algebra $\mathfrak{a}$ and a Lie algebra $\mathfrak{b}$ by a representation $\varphi\colon \mathfrak{b} \to \operatorname{End}(\mathfrak{a}) = \operatorname{Der}(\mathfrak{a})$, i.e., we have a split exact sequence*

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0.$$

If $\mathfrak{b}$ admits an LR-structure $(x, y) \mapsto x \cdot y$ such that $\varphi(x \cdot y) = 0$ for all $x, y \in \mathfrak{b}$, then also $\mathfrak{g}$ admits an LR-structure.

Proof. Because the short exact sequence is split, the 2-cocycle Ω in the Lie bracket of $\mathfrak{g}$ is trivial, i.e., $\Omega(x,y) = 0$. Let $a \cdot b = 0$ be the trivial product on $\mathfrak{a}$ and take $\varphi_1 = 0$, $\varphi_2 = \varphi$ and $\omega(x,y) = 0$. Assume that $(x, y) \mapsto x \cdot y$ is an LR-product. Then all conditions of Proposition 5.1 are satisfied except for (26) and (22) requiring

$$\varphi(x \cdot y) = 0$$
$$[\varphi(x), \varphi(y)] = 0.$$

But we have (26) by assumption, and since φ is a representation it follows

$$0 = \varphi(x \cdot y - y \cdot x) = \varphi([x,y]) = [\varphi(x), \varphi(y)].$$

Hence (17) defines an LR-structure on $\mathfrak{g}$, given by

$$(a, x) \circ (b, y) = (\varphi(x)b, x \cdot y).$$

$\square$

Corollary 5.3. *Suppose that the LR-products on $\mathfrak{a}$ and $\mathfrak{b}$ are trivial. Hence $\mathfrak{b}$ is also abelian. Then* (17) *defines an LR-structure on $\mathfrak{g}$ if and only if the following conditions hold:*

$$(30) \qquad\qquad \omega(x, y) - \omega(y, x) = \Omega(x, y)$$
$$(31) \qquad\qquad \varphi_2(x) - \varphi_1(x) = \varphi(x)$$
$$(32) \qquad\qquad \varphi_2(x)\omega(y, z) = \varphi_2(y)\omega(x, z)$$
$$(33) \qquad\qquad \varphi_2(x)\varphi_1(y) = 0$$
$$(34) \qquad\qquad [\varphi_2(x), \varphi_2(y)] = 0$$
$$(35) \qquad\qquad \varphi_1(z)\omega(x, y) = \varphi_1(y)\omega(x, z)$$
$$(36) \qquad\qquad \varphi_1(x)\varphi_2(y) = 0$$
$$(37) \qquad\qquad [\varphi_1(x), \varphi_1(y)] = 0$$

We can apply this corollary as follows:

Proposition 5.4. *Let $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ be an extension in which both $\mathfrak{a}$ and $\mathfrak{b}$ are abelian. If there exists an $e \in \mathfrak{b}$ such that $\varphi(e) \in \mathrm{End}(\mathfrak{a})$ is an isomorphism, then $\mathfrak{g}$ admits an LR-structure. In fact, in that case* (17) *defines an LR-product, where $\varphi_1 = 0$, $\varphi_2 = \varphi$, the product on $\mathfrak{a}$ and $\mathfrak{b}$ is trivial, and*

$$\omega(x, y) = \varphi(e)^{-1}\varphi(x)\Omega(e, y).$$

Proof. We have to show that the above conditions of corollary 5.3 are satisfied. Applying $\varphi(e)^{-1}$ to (15) with $z = e$ it follows $\Omega(x, y) - \varphi(e)^{-1}\varphi(x)\Omega(e, y) + \varphi(e)^{-1}\varphi(y)\Omega(e, x) = 0$. This just means that $\Omega(x, y) = \omega(x, y) - \omega(y, x)$. Furthermore we have, since $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$ for all $x, y \in \mathfrak{b}$,

$$\begin{aligned}
\varphi(x)\omega(y, z) - \varphi(y)\omega(x, z) &= \varphi(x)\varphi(e)^{-1}\varphi(y)\Omega(e, z) - \varphi(y)\varphi(e)^{-1}\varphi(x)\Omega(e, z) \\
&= \varphi(e)^{-1}(\varphi(x)\varphi(y) - \varphi(y)\varphi(x))\Omega(e, z) \\
&= 0.
\end{aligned}$$

All the other conditions follow trivially from $\varphi_1 = 0$. Hence the product defines an LR-structure. $\square$

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