# AFFINE ACTIONS ON NILPOTENT LIE GROUPS

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ABSTRACT. To any connected and simply connected nilpotent Lie group $N$, one can associate its group of affine transformations $\mathrm{Aff}(N)$. In this paper, we study simply transitive actions of a given nilpotent Lie group $G$ on another nilpotent Lie group $N$, via such affine transformations.

We succeed in translating the existence question of such a simply transitive affine action to a corresponding question on the Lie algebra level. As an example of the possible use of this translation, we then consider the case where $\dim(G) = \dim(N) \leq 5$.

Finally, we specialize to the case of abelian simply transitive affine actions on a given connected and simply connected nilpotent Lie group. It turns out that such a simply transitive abelian affine action on $N$ corresponds to a particular Lie compatible bilinear product on the Lie algebra $\mathfrak{n}$ of $N$, which we call an LR-structure.

## 1. NIL-affine actions

In 1977 [12], J. Milnor asked whether or not any connected and simply connected solvable Lie group $G$ of dimension $n$ admits a representation $\rho : G \to \mathrm{Aff}(\mathbb{R}^n)$ into the group of invertible affine mappings, letting $G$ operate simply transitively on $\mathbb{R}^n$.

For some time, most people were convinced that the answer to Milnor's question was positive until Y. Benoist ([1], [2]) proved the existence of a simply connected, connected nilpotent Lie group $G$ (of dimension 11) not allowing such a simply transitive affine action. These examples were generalized to a family of examples by D. Burde and F. Grunewald ([6]), also in dimension 10 ([4]).

To be able to construct these counterexamples, both Benoist and Burde–Grunewald used the fact that the notion of a simply transitive affine action can be translated onto the Lie algebra level. In fact, if $G$ is a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, the existence of a simply transitive affine action of $G$ is equivalent to the existence of a certain Lie algebra representation

$$\varphi : \mathfrak{g} \to \mathfrak{aff}(n) = \mathbb{R}^n \rtimes \mathfrak{gl}(n, \mathbb{R}).$$

As the answer to Milnor's question turned out to be negative, one tried to find a more general setting, providing a positive answer to the analogue of Milnor's problem. One such a setting is the setting of NIL-affine actions. To define this setting, we consider a simply

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connected, connected nilpotent Lie group and define the affine group $\mathrm{Aff}(N)$ as being the group $N \rtimes \mathrm{Aut}(N)$, which acts on $N$ via

$$\forall m, n \in N,\ \forall \alpha \in \mathrm{Aut}(N):\ {}^{(m,\alpha)}n = m \cdot \alpha(n).$$

Note that this is really a generalization of the usual affine group $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$, where $\mathrm{GL}(n, \mathbb{R})$ is the group of continuous automorphisms of the abelian Lie group $\mathbb{R}^n$. In [7] it was shown that for any connected and simply connected solvable Lie group $G$, there exists a connected and simply connected nilpotent Lie group $N$ and a representation $\rho : G \to \mathrm{Aff}(N)$ letting $G$ act simply transitively on $N$. This shows that in this new setting, any connected and simply connected solvable Lie group does appear as a simply transitive group of affine motions of a nilpotent Lie group (referred to as NIL-affine actions in the sequel).

Apart from the existence of such a simply transitive NIL-affine action for any simply connected solvable Lie group $G$ not much is known about such actions. As a first approach towards a further study of this topic, we concentrate in this paper on the situation where both $G$ and $N$ are nilpotent.

In the following section we show that in this case any simply transitive action $\rho : G \to \mathrm{Aff}(N)$ is unipotent. This result was known in the usual affine case too. Then we obtain a translation to the Lie algebra level, which is again a very natural generalization of the known result in the usual affine situation.

Thereafter we present some examples of simply transitive NIL-affine actions in low dimensions and finally, we specialize to the situation where $G$ is abelian.

## 2. Nilpotent simply transitive NIL-affine groups are unipotent

Let $N$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. It is well known that $N$ has a unique structure of a real algebraic group (e.g., see [13]) and also $\mathrm{Aut}(N) \cong \mathrm{Aut}(\mathfrak{n})$ carries a natural structure of a real algebraic group. It follows that we can consider $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ as being a real algebraic group. The aim of this section is to show that any nilpotent simply transitive subgroup of $\mathrm{Aff}(N)$ is an algebraic subgroup and in fact unipotent. This generalizes the analogous result for ordinary nilpotent and simply transitive subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ proved by J. Scheuneman in [15, Theorem 1].

Throughout this section we will use $A(G)$ to denote the algebraic closure of a subgroup $G \subseteq \mathrm{Aff}(N)$ and we will refer to the unipotent radical of $A(G)$, by writing $U(G)$. Using these notations, the aim of this section is to prove the following theorem:

**Theorem 2.1.** *Let $N$ be a connected and simply connected nilpotent Lie group and assume that $G \subset \mathrm{Aff}(N)$ is a nilpotent Lie subgroup acting simply transitively on $N$, then $G = A(G) = U(G)$.*

We should mention here that with some effort this theorem can be reduced from the corresponding theorem in the setting of polynomial actions as obtained in [3, Lemma 5]. For the readers convenience we will repeat the necessary steps here and adapt them to our specific situation.

First we recall two basic technical results which will be needed later on:

**Lemma 2.2.** [3, Lemma 3] *Let $T$ be a real algebraic torus acting algebraically on $\mathbb{R}^n$, then the set of fixed points $(\mathbb{R}^n)^T$ is non-empty.*

Any connected and simply connected nilpotent Lie group $N$ can be identified with its Lie algebra $\mathfrak{n}$ using the exponential map exp. We can therefore speak of a polynomial map of $N$, by which we will mean that the corresponding map on the Lie algebra $\mathfrak{n}$ is expressed by polynomials (with respect to coordinates to any given basis of $\mathfrak{n}$). For instance, it is well known that the multiplication map $N \times N \to N : (n_1, n_2) \mapsto n_1 n_2$ is polynomial. We use such polynomial maps in the following lemma.

**Lemma 2.3.** [3, Lemma 2 (a)] *Let $N$ be a connected and simply connected nilpotent Lie group. Assume that $\theta : N \times \mathbb{R}^n \to \mathbb{R}^n$ is an action which is polynomial in both variables. Let $v_0$ be a point in $\mathbb{R}^n$. Then the isotropy group of $v_0$ is a closed connected subgroup of $N$.*

Now we adapt [3, Proposition 1] to our situation:

**Proposition 2.4.** *Let $G \subset \mathrm{Aff}(N)$ be a solvable Lie group, acting simply transitively on $N$. Then the unipotent radical $U(G)$ of $A(G)$ also acts simply transitively on $N$.*

*Proof.* As $G$ is a connected, solvable Lie subgroup, its algebraic closure $A(G)$ is solvable also and Zariski connected. Therefore it splits as a semi-direct product $A(G) = U(G) \rtimes T$ where $T$ is a real algebraic torus. From the fact that $G$ acts simply transitively, we immediately get that $A(G)$ acts transitively on $N$.

By Lemma 2.2 and the fact that $N$ is diffeomorphic to $\mathbb{R}^n$ (for $n = \dim N$), we know that there exists a point $n_0 \in N$ which is fixed under the action of $T$. It follows that $N = (U(G) \rtimes T) \cdot n_0 = U(G) \cdot n_0$, showing that $U(G)$ acts transitively on $N$.

By [14, Lemma 4.36], we know that $\dim U(G) \leq \dim G$. On the other hand $\dim G = \dim N$, as $G$ is acting simply transitively on $N$, from which we deduce that $\dim U(G) = \dim G = \dim N$. It follows that $U(G)$ acts with discrete isotropy groups (because stabilizers have dimension 0). Lemma 2.3 then implies that $U(G)$ acts with trivial isotropy groups, allowing us to conclude that $U(G)$ acts simply transitively on $N$. □

The last step we need before we can prove Theorem 2.1 is the following (compare with [3, Lemma 4])

**Lemma 2.5.** *Let $G \subset \mathrm{Aff}(N)$ be a solvable Lie group acting simply transitively on $N$ and let $U(G)$ be the unipotent radical of $A(G)$. Then the centralizer of $U(G)$ in $A(G)$ coincides with the center of $U(G)$.*

*Proof.* Using Proposition 2.4 we know that $U(G)$ acts simply transitively on $N$, and $A(G)$ splits as a semi-direct product $A(G) = U(G) \rtimes T$ where $T$ is a real algebraic torus.

The centralizer $C$ of $U(G)$ in $A(G)$ is also an algebraic group and therefore, for every element $c \in C$, the unipotent part $c_u$ and the semisimple part $c_s$ of $c$ are also in $C$. Now assume that $C$ does not belong completely to $U(G)$, then there is an element $c$ of $C$ with a nontrivial semisimple part $c_s \neq 1$, and $c_s$ also belongs to $C$. This semisimple part $c_s$, belongs to the algebraic torus $T$. So, as in lemma 2.2, we can conclude that $(N)^{c_s}$ is non-empty.

However, as $c_s$ centralizes $U(G)$, the set $(N)^{c_s}$ is $U(G)$-invariant. This contradicts the transitivity of $U(G)$ on $N$. It follows that $C$ is included in $U(G)$. □

We are now ready to prove the main result of this section.

**Proof of theorem 2.1:** As $G$ is nilpotent, its algebraic closure $A(G)$ is also nilpotent. A nilpotent algebraic group splits as a direct product $A(G) = U(G) \times S(G)$, where $S(G)$ denotes the set of semi-simple elements of $A(G)$. In this case $S(G)$ centralizes $U(G)$ and by

Lemma 2.5, we can conclude that $S(G)$ has to be trivial, so $A(G) = U(G)$. From the fact that both $G$ and $A(G) = U(G)$ are acting simply transitively on $N$ (Proposition 2.4), we can conclude that $G = A(G) = U(G)$. $\square$

## 3. Translation to the Lie algebra level

In this section we will show that we can completely translate the notion of a simply transitive NIL-affine action of a nilpotent Lie group $G$ to a notion on the Lie algebra level.

As before, let $G$ and $N$ be connected, simply connected Lie groups. We will use $\mathfrak{g}$ to denote the Lie algebra of $G$ and $\mathfrak{n}$ to denote the Lie algebra of $N$. Recall that the Lie algebra corresponding to the semi-direct product $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ is equal to the semi-direct product $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$, where $\mathrm{Der}(\mathfrak{n})$ is the set of all derivations of $\mathfrak{n}$. This semi-direct product $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is a Lie algebra via

$$\big[(X,D),(X',D')\big] = \big([X,X'] + DX' - D'X, [D,D']\big)\,.$$

Assume that $\rho : G \to \mathrm{Aff}(N)$ is a representation of Lie groups. Then there exists a unique homomorphism $d\rho$ (differential of $\rho$) of their respective Lie algebras $\mathfrak{g}$ and $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ making the following diagram commutative:

$$\begin{tikzcd}
G \arrow[r, "\rho"] & \mathrm{Aff}(N) \\
\mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\rho"'] & \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \arrow[u, "\exp"']
\end{tikzcd}$$

Conversely, any Lie algebra homomorphism $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ can be seen as the differential of a Lie group homomorphism $\rho : G \to \mathrm{Aff}(N)$.

Let us use the following notation for $d\rho$:

$$d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : X \mapsto d\rho(X) = (t_X, D_X).$$

As $d\rho$ is a Lie algebra homomorphism, $\mathcal{D} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{n}) : X \mapsto D_X$ is also a Lie algebra homomorphism.

**Theorem 3.1.** *Let $G$ and $N$ be simply connected, connected nilpotent Lie groups. Let $\rho : G \to \mathrm{Aff}(N)$ be a representation. Using the notations introduced above, we have the following:*

$\rho : G \to \mathrm{Aff}(N)$ *induces a simply transitive NIL-affine action of $G$ on $N$*

$$\Updownarrow$$

(1) *$t : \mathfrak{g} \to \mathfrak{n} : X \mapsto t_X$ is a bijection.*
(2) *For any $X \in \mathfrak{g}$, $D_X$ is nilpotent.*

This theorem is known to hold in the usual affine case, i.e., with $N = \mathbb{R}^n$ (see for example [9, section 3], [10] and [8, p. 100]), and so we really obtain a very natural generalization.

*Proof.* We first prove the direction from the Lie group level (top statement) to the Lie algebra level (bottom statement). We will proceed by induction on the dimension $n$ of $G$, where the situation $n = 1$ is trivial.

So, we assume that $\rho : G \to \mathrm{Aff}(N)$ induces a simply transitive NIL-affine action of $G$ on $N$. As $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$, we can decompose $\rho$ in two component maps $tr : G \to N$ and $lin : G \to \mathrm{Aut}(N)$ with $\rho(g) = (tr(g), lin(g))$.

The fact that the action is simply transitive, is equivalent to $tr$ being a bijection between $G$ and $N$, so also $\dim N = \dim G = n$.

By Proposition 2.4, we know that $lin(G)$ is a Lie subgroup of $\mathrm{Aut}(N)$ consisting of unipotent elements. We can therefore choose a basis $A_1, \ldots, A_n$ of $\mathfrak{n}$, such that the following two properties hold:

(1) For any $i \in \{1, 2, \ldots, n\}$: $\mathfrak{n}_i = \langle A_i, A_{i+1}, \ldots, A_n \rangle$ is an ideal of $\mathfrak{n}$. The corresponding Lie subgroup $N_i = \exp(\mathfrak{n}_i)$ of $N$ is then a normal subgroup of $N$.
If we set $a_j^{x_j} = \exp(x_j A_j)$, for any $x_j \in \mathbb{R}$, then every element of $N_i$ can be written uniquely in the form $a_i^{x_i} \ldots a_n^{x_n}$, with $x_i, \ldots, x_n \in \mathbb{R}$.

(2) For any $g \in G$: $lin(g)(N_i) = N_i$, and $lin(g)$ induces the identity on each quotient $N_i/N_{i+1}$ (with $N_{n+1} = 1$). Seen as an element of $\mathrm{Aut}(\mathfrak{n})$, the matrix $lin(g)$ is lower triangular unipotent w.r.t. the basis $A_1, A_2 \ldots, A_n$.

Inspired by this last property, we introduce $U(N)$ to denote the Lie subgroup of $\mathrm{Aut}(N)$, consisting of all $\mu \in \mathrm{Aut}(N)$, for which $\mu(N_i) = N_i$ and for which $\mu$ induces the identity on each quotient $N_i/N_{i+1}$. So $lin(G) \subseteq U(N)$.

$U(N)$ consists of unipotent elements, so denote by $O(\mathfrak{n}) = \log(U(N))$ the corresponding Lie subalgebra of $\mathrm{Der}(\mathfrak{n})$. Every element in $O(\mathfrak{n})$ is therefore a derivation $D$ for which $D(\mathfrak{n}_i) \subseteq \mathfrak{n}_{i+1}$ for all $i$.

As $N_2 \rtimes U(N)$ is a closed normal subgroup of the Lie group $N \rtimes U(N)$, we can construct a Lie group homomorphism $\psi : G \to N/N_2 \cong \mathbb{R}$ as the composition

$$G \xrightarrow{\rho} N \rtimes U(N) \longrightarrow \frac{N \rtimes U(N)}{N_2 \rtimes U(N)} \cong \frac{N}{N_2}.$$

Of course, $\psi : G \to N/N_2$ can also be written as the composition of maps

$$G \xrightarrow{tr} N \longrightarrow \frac{N}{N_2}.$$

It follows that $\psi$ is onto and that the kernel of $\psi$ is a Lie subgroup $G_2$ of $G$ of codimension 1. We will now use the induction hypothesis for this Lie subgroup $G_2$. We denote the Lie subalgebra of $\mathfrak{g}$ corresponding to $G_2$ by $\mathfrak{g}_2$.

Note that there is a natural Lie group homomorphism $N_2 \rtimes U(N) \to N_2 \rtimes \mathrm{Aut}(N_2)$ obtained by restricting automorphisms $\mu \in U(N)$ to automorphisms of $N_2$. As a conclusion, we find the following diagram in which each square is commutative:

(1)
$$
\begin{tikzcd}
\mathfrak{g}_2 \arrow[rrrr] \arrow[drr, "\exp"'] \arrow[dd, "d\rho'"'] & & & & \mathfrak{g} \arrow[drr, "\exp"] \arrow[dd, "d\rho"] & & \\
& & G_2 \arrow[rrrr] \arrow[dd, "\rho'"] & & & & G \arrow[dd, "\rho"] \\
\mathfrak{n}_2 \rtimes O(\mathfrak{n}) \arrow[rrrr] \arrow[drr, "\exp"'] \arrow[dd] & & & & \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \arrow[drr, "\exp"] & & \\
& & N_2 \rtimes U(N) \arrow[rrrr] \arrow[dd] & & & & N \rtimes \mathrm{Aut}(N) \\
\mathfrak{n}_2 \rtimes \mathrm{Der}(\mathfrak{n}_2) \arrow[drr, "\exp"'] & & & & & & \\
& & N_2 \rtimes \mathrm{Aut}(N_2) & & & &
\end{tikzcd}
$$

In the above diagram, $\rho'$ (resp. $d\rho'$) is obtained from $\rho$ (resp. $d\rho$) by restricting both the domain and the codomain. Now, the composite map

$$G_2 \xrightarrow{\rho'} N_2 \rtimes U(N) \longrightarrow N_2 \rtimes \mathrm{Aut}(N_2)$$

satisfies the condition from the statement of the theorem. Therefore, we can use the induction hypothesis to conclude that the composite map

$$\mathfrak{g}_2 \xrightarrow{d\rho'} \mathfrak{n}_2 \rtimes O(\mathfrak{n}) \to \mathfrak{n}_2 \rtimes \mathrm{Der}(\mathfrak{n}_2),$$

with

$$X_2 \mapsto (t_{X_2}, D_{X_2}) \mapsto (t_{X_2}, D^{'}_{X_2}),$$

(where $D^{'}_{X_2}$ denotes the restriction of $D_{X_2}$ to $\mathfrak{n}_2$) satisfies the following conditions:

(1) $t : \mathfrak{g}_2 \to \mathfrak{n}_2 \subseteq \mathfrak{n} : X_2 \mapsto t_{X_2}$ is a bijection.
(2) For any $X_2 \in \mathfrak{g}_2$, $D^{'}_{X_2}$ is nilpotent (evidently, as $D_{X_2} \in O(\mathfrak{n})$).

Now, consider the map $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$. As $\rho(G) \subseteq N \rtimes U(N)$, we get that $d\rho(\mathfrak{g}) \subseteq \mathfrak{n} \rtimes O(\mathfrak{n})$, so for any $X \in \mathfrak{g}$, $D_X$ is evidently nilpotent.

Now, fix an element $b_1 \in G$, for which $\psi(b_1) = a_1 \cdot N_2$. Take $B_1 = \log(b_1)$, and we already know that $A_1 = \log(a_1)$. Then we have that

$$\mathfrak{n} = \langle A_1 \rangle + \mathfrak{n}_2 \text{ and } \mathfrak{g} = \langle B_1 \rangle + \mathfrak{g}_2.$$

To show that $t : \mathfrak{g} \to \mathfrak{n}$ is bijective, take an arbitrary element of $\mathfrak{n}$. Such an element is of the form $sA_1 + m_2$, for some $s \in \mathbb{R}$ and some $m_2 \in \mathfrak{n}_2$.

From the fact that $\psi(b_1) = a_1 \cdot N_2$, one deduces that:

$$d\rho(B_1) = (A_1 + m^{'}_2, D), \text{ for some } m^{'}_2 \in \mathfrak{n}_2 \text{ and } D \in O(\mathfrak{n}).$$

It follows that

$$t_{sB_1} = sA_1 + sm^{'}_2$$

As $t : \mathfrak{g}_2 \to \mathfrak{n}_2 \subseteq \mathfrak{n}$ is a bijection, and $m_2 - sm_2' \in \mathfrak{n}_2$, there exists a $g_2 \in \mathfrak{g}_2$ such that

$$t_{g_2} = m_2 - sm_2'.$$

So $t$ maps $sB_1 + g_2$ to $sA_1 + m_2$, showing that $t$ is surjective. In an analogous way, one shows that $t$ is injective, from which we conclude that $t : \mathfrak{g} \to \mathfrak{n}$ is a bijection.

We now prove the direction from the Lie algebra level (bottom statement) to the Lie group level (top statement). We will proceed by induction on the dimension $n$ of $\mathfrak{g}$, where the situation $n = 1$ is trivial.
So, we assume that

(1) $t : \mathfrak{g} \to \mathfrak{n} : X \mapsto t_X$ is a bijection. So $\dim \mathfrak{n} = \dim \mathfrak{g} = n$.
(2) For any $X \in \mathfrak{g}$, $D_X$ is nilpotent.

As $\mathcal{D}(\mathfrak{g})$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{n})$ consisting of nilpotent elements, we can choose a vector space basis $A_1, \ldots, A_n$ of $\mathfrak{n}$, such that

(1) for any $i \in \{1, 2, \ldots, n\}$: $\mathfrak{n}_i = \langle A_i, A_{i+1}, \ldots, A_n \rangle$ is an ideal of $\mathfrak{n}$, and
(2) for any $X \in \mathfrak{n}$: $D_X(\mathfrak{n}_i) \subseteq \mathfrak{n}_{i+1}$, where $\mathfrak{n}_{n+1} = 0$ (i.e., the matrix $D_X$ is strictly lower triangular w.r.t. the basis $A_1, A_2 \ldots, A_n$).

Now we use $O(\mathfrak{n})$ to denote the Lie subalgebra of $\mathrm{Der}(\mathfrak{n})$ consisting of all derivations $D \in \mathrm{Der}(\mathfrak{n})$ for which $D(\mathfrak{n}_i) \subseteq \mathfrak{n}_{i+1}$ for all $i$. So $\mathcal{D}(\mathfrak{g}) \subseteq O(\mathfrak{n})$.

As the $\mathfrak{n}_i$ are ideals in the Lie algebra $\mathfrak{n}$, the corresponding Lie subgroups $N_i = \exp(\mathfrak{n}_i)$ are closed normal subgroups of $N$. They determine a filtration of closed subgroups

$$N = N_1 \supset N_2 \supset \cdots \supset N_n \supset N_{n+1} = 1$$

where $N_i/N_{i+1} \cong \mathbb{R}$.

Let $\exp(O(\mathfrak{n})) = U(N)$ be the Lie subgroup of $\mathrm{Aut}(N)$ corresponding to $O(\mathfrak{n})$, then for any element $\mu \in U(N)$, we have that $\mu(N_i) = N_i$ and $\mu$ induces the identity on each quotient $N_i/N_{i+1}$.

As $\mathfrak{n}_2 \rtimes O(\mathfrak{n})$ is obviously an ideal of the Lie algebra $\mathfrak{n} \rtimes O(\mathfrak{n})$, we can construct a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{n}/\mathfrak{n}_2 \cong \mathbb{R}$ as the composition

$$\mathfrak{g} \xrightarrow{d\rho} \mathfrak{n} \rtimes O(\mathfrak{n}) \longrightarrow \frac{\mathfrak{n} \rtimes O(\mathfrak{n})}{\mathfrak{n}_2 \rtimes O(\mathfrak{n})} \cong \frac{\mathfrak{n}}{\mathfrak{n}_2}.$$

Of course, $\psi : \mathfrak{g} \to \mathfrak{n}/\mathfrak{n}_2$ can also be written as the composition of linear maps

$$\mathfrak{g} \xrightarrow{t} \mathfrak{n} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}_2}.$$

It follows that $\psi$ is onto and that the kernel of $\psi$ is an ideal $\mathfrak{g}_2$ of $\mathfrak{g}$ of codimension 1. We will now use the induction hypothesis for this ideal $\mathfrak{g}_2$. We denote the subgroup of $G$ corresponding to $\mathfrak{g}_2$ by $G_2$.

Note that there is a natural Lie algebra homomorphism $\mathfrak{n}_2 \rtimes O(\mathfrak{n}) \to \mathfrak{n}_2 \rtimes \mathrm{Der}(\mathfrak{n}_2)$ obtained by restricting derivations $D \in O(\mathfrak{n})$ to derivations of $\mathfrak{n}_2$. As a conclusion, we again obtain diagram (1) in which each square is commutative.

Now, the composite map

$$\mathfrak{g}_2 \xrightarrow{d\rho'} \mathfrak{n}_2 \rtimes O(\mathfrak{n}) \longrightarrow \mathfrak{n}_2 \rtimes \mathrm{Der}(\mathfrak{n}_2)$$

satisfies conditions 1. and 2. from the statement of the theorem. Therefore, we can use the induction hypothesis to conclude that the composite map

$$G_2 \xrightarrow{\rho'} N_2 \rtimes U(N) \to N_2 \rtimes \mathrm{Aut}(N_2)$$

determines a simply transitive action of $G_2$ on $N_2$. It follows that the map $\rho : G \to N \rtimes \mathrm{Aut}(N)$ describes an action of $G$ on $N$, for which the subgroup $G_2$ acts on $N$ in such a way that the subgroup $N_2$ is a single orbit for the $G_2$-action and $G_2$ acts simply transitively on $N_2$.

Now, fix an element $B_1 \in \mathfrak{g}$, for which $\psi(B_1) = A_1 + \mathfrak{n}_2$. Then we use the notation $b_1^t = \exp(tB_1) \in G$ (resp. $a_1^t = \exp(tA_1) \in N$) for all $t \in \mathbb{R}$. Now, we can consider the two 1-parameter subgroups

$$B = \{b_1^t \mid t \in \mathbb{R}\} \cong \mathbb{R} \text{ and } A = \{a_1^t \mid t \in \mathbb{R}\} \cong \mathbb{R}$$

of $G$ and $N$ respectively, for which we have that

$$G = G_2 \cdot B \text{ and } N = A \cdot N_2.$$

To show that $G$ acts transitively on $N$, it is enough to show that any element of $N$ is in the same orbit as the identity. An arbitrary element of $N$ is of the form $a_1^t \cdot n_2$, for some $t \in \mathbb{R}$ and some $n_2 \in N_2$. From the fact that $\psi(B_1) = A_1 + \mathfrak{n}_2$, one deduces that for a given $t \in \mathbb{R}$:

$$\rho(b_1^{-t}) = (a_1^{-t} \cdot n_2', \mu), \text{ for some } n_2' \in N_2 \text{ and } \mu \in U(N).$$

It follows that

$$^{\rho(b_1^{-t})}a_1^t \cdot n_2 = n_2'' \text{ for some } n_2'' \in N_2.$$

As $G_2$ acts simply transitively on $N_2$, there exists a $g_2 \in G_2$ such that

$$^{\rho(g_2)}n_2'' = 1$$

and so $g_2 \cdot b_1^{-t}$ maps $a_1^t \cdot n_2$ to the identity. This shows that the action of $G$ is transitive. Analogously, one shows that $g_2 \cdot b_1^{-t}$ is the unique element mapping $a_1^t \cdot n_2$ to the identity, from which we conclude that the action is simply transitive. $\square$

## 4. The situation in low dimensions

Now that we have obtained a description of simply transitive NIL-affine actions on the Lie algebra level, we are able to study the existence of such actions in low dimensions. As a first result, we obtain the following proposition.

**Proposition 4.1.** *Let $G$ and $N$ be connected, simply connected nilpotent Lie groups of dimension $n$ with $1 \leq n \leq 5$. Then there exists a representation $\rho : G \to \mathrm{Aff}(N)$ which induces a simply transitive NIL-affine action of $G$ on $N$.*

*Proof.* Let $\mathfrak{g}$ and $\mathfrak{n}$ be the corresponding Lie algebras. By theorem 3.1 we know that it is enough to show that there exists a Lie algebra homomorphism $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : X \mapsto d\rho(X) = (t_X, D_X)$ , for which the translational part $t : \mathfrak{g} \to \mathfrak{n} : X \mapsto t_X$ is a bijection and all $D_X$ are nilpotent.

For each Lie algebra we will fix a basis $X_1, \ldots, X_n$ of the underlying vector space and denote the coordinate of an element $\sum_{i=1}^n x_i X_i$ by the column vector $(x_1, x_2, \ldots, x_n)^T$. We will sometimes represent a derivation $D_X$ by its matrix with respect to the given basis $X_1, \ldots, X_n$.

Let us first remark that if $\mathfrak{n} = \mathfrak{g}$, with $\dim(\mathfrak{g}) = n$, then there exists a trivial Lie algebra morphism:

$$d\rho_{triv} : \mathfrak{g} \to \mathfrak{g} \rtimes \mathrm{Der}(\mathfrak{g}) : X \mapsto d\rho_{triv}(X) = (X, D_0(n)),$$

where $D_0(n)$ is the trivial nilpotent derivation which maps every element of $\mathfrak{g}$ to 0. Of course, this representation corresponds to the simply transitive action of $G$ on itself, using left translations.

So up to dimension 3 we only have to search for suitable Lie algebra morphisms between $\mathbb{R}^3$ and $\mathfrak{h}_3$, the 3-dimensional Heisenberg Lie algebra. Let $\mathbb{R}^3$ be the Lie algebra with basis vectors $X_1, X_2, X_3$ and all Lie brackets equal to zero. Let $\mathfrak{h}_3$ be the Lie algebra with basis vectors $X_1, X_2, X_3$ and non-zero brackets $[X_1, X_2] = X_3$. Then one can choose the following suitable Lie algebra morphisms (expressed in coordinates):

$$d\rho : \mathbb{R}^3 \to \mathfrak{h}_3 \rtimes \mathrm{Der}(\mathfrak{h}_3) : X = (x_1, x_2, x_3)^T \mapsto \left( (x_1, x_2, x_3)^T, D_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{x_2}{2} & \frac{-x_1}{2} & 0 \end{pmatrix} \right),$$

and

$$d\rho : \mathfrak{h}_3 \to \mathbb{R}^3 \rtimes \mathrm{Der}(\mathbb{R}^3) : X = (x_1, x_2, x_3)^T \mapsto \left( (x_1, x_2, x_3)^T, D_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-x_2}{2} & \frac{x_1}{2} & 0 \end{pmatrix} \right).$$

In dimension 4 there are 3 different nilpotent Lie algebras over $\mathbb{R}$. Denoting the basis by $X_1, X_2, X_3, X_4$, let $\mathbb{R}^4$ be the Lie algebra with all Lie brackets equal to zero, $\mathfrak{h}_3 \oplus \mathbb{R}$ be the Lie algebra with non-zero bracket $[X_1, X_2] = X_3$, and $\mathfrak{f}_4$ be the filiform nilpotent Lie algebra with non-zero brackets $[X_1, X_2] = X_3, [X_1, X_3] = X_4$.

We can find suitable Lie algebra morphisms $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : X = (x_1, x_2, x_3, x_4)^T \mapsto d\rho(X) = (t_X, D_X)$ for all possible combinations of $\mathfrak{g}$ and $\mathfrak{n}$ in dimension 4. In every case we can choose $t_X = (x_1, x_2, x_3, x_4)^T$. The following table shows possible choices of linear parts:

| | $\mathrm{Der}(\mathbb{R}^4)$ | $\mathrm{Der}(\mathfrak{h}_3 \oplus \mathbb{R})$ | $\mathrm{Der}(\mathfrak{f}_4)$ |
|---|---|---|---|
| $\mathbb{R}^4$ | $D_0(4)$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{x_2}{2} & \frac{-x_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -x_1 & 0 & 0 \\ 0 & 0 & -x_1 & 0 \end{pmatrix}$ |
| $\mathfrak{h}_3 \oplus \mathbb{R}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-x_2}{2} & \frac{x_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | $D_0(4)$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_3 & x_2 & 0 & 0 \end{pmatrix}$ |
| $\mathfrak{f}_4$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ x_1 + x_2 & x_1 & 0 & 0 \\ 3x_1 - x_3 & x_2 & 0 & 0 \end{pmatrix}$ | $D_0(4)$ |

Analogously one can show that the theorem also holds in dimension 5. Up to isomorphism there are 9 nilpotent Lie algebras over $\mathbb{R}$, so one has to find in total 81 suitable Lie algebra morphisms satisfying the properties of Theorem 3.1. A full list of those Lie algebra morphisms is available from the authors. As an example we treat here one case. Let $\mathfrak{h}_3 \oplus \mathbb{R}^2$ be the Lie algebra with basis vectors $X_1, \ldots, X_5$ and non-zero brackets $[X_1, X_2] = X_3$ and let $\mathfrak{g}_{5,6}$ (using the notation of [11]) be the Lie algebra with basis vectors $X_1, \ldots, X_5$ and non-zero brackets

$[X_1, X_2] = X_3$; $[X_1, X_3] = X_4$; $[X_1, X_4] = X_5$; $[X_2, X_3] = X_5$. Then a suitable Lie algebra morphism is the following:

$$d\rho : \mathfrak{h}_3 \oplus \mathbb{R}^2 \to \mathfrak{g}_{5,6} \rtimes \mathrm{Der}(\mathfrak{g}_{5,6}) : X = (x_1, x_2, x_3, x_4, x_5)^T \mapsto \left( \left( x_1, x_2, \frac{x_3}{\sqrt{3}}, x_4, x_5 \right)^T, D_X \right),$$

with

$$D_X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{x_1}{-3+\sqrt{3}} & 0 & 0 & 0 & 0 \\ x_2 & \frac{x_1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{3}\left(1+\sqrt{3}\right) x_3 & x_2 - \frac{x_2}{\sqrt{3}} & \frac{x_1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{6}\left(3+\sqrt{3}\right) x_4 & \frac{1}{3}\left(-1+\sqrt{3}\right) x_3 & -\frac{x_2}{\sqrt{3}} & \frac{1}{6}\left(-3+\sqrt{3}\right) x_1 & 0 \end{pmatrix}.$$

We remark here that for most other cases, the representation is of a simpler form than the one above and can be obtained by rather simple computations. □

**Proposition 4.2.** *Consider the nilpotent Lie algebra $\mathfrak{n}$ of dimension 6, with basis $X_1, \ldots, X_6$ and non-trivial Lie brackets*

$$\begin{array}{lll} [X_1, X_2] = X_3, & [X_1, X_3] = X_4, & [X_1, X_4] = X_5, \\ [X_2, X_5] = X_6, & [X_3, X_4] = -X_6. & \end{array}$$

*and let $N$ be the corresponding Lie group. Then $\mathbb{R}^6$ cannot act simply transitive by NIL-affine actions on $N$.*

*Proof.* We will assume that there exists a representation $\rho : \mathbb{R}^6 \to \mathrm{Aff}(N)$ which induces a simply transitive NIL-affine action of $\mathbb{R}^6$ on $N$, and show that this leads to a contradiction. By theorem 3.1 we know that the differential $d\rho : \mathbb{R}^6 \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : X \mapsto d\rho(X) = (t_X, D_X)$ is a Lie algebra morphism, for which the translational part $t : \mathbb{R}^6 \to \mathfrak{n} : X \mapsto t_X$ is a bijection and all $D_X$ are nilpotent. As $t$ is a bijection, there exist $Y_i \in \mathbb{R}^6$ for which $t_{Y_i} = X_i$. So we have $d\rho(Y_i) = (X_i, D_i)$, with $D_i \in \mathrm{Der}(\mathfrak{n})$. One can check (or consult [11], where $\mathfrak{n}$ equals the Lie algebra $\mathfrak{g}_{6,18}$) that the matrix of any derivation $D_i$, $i = 1, \ldots, 6$, with respect to the basis $X_1, \ldots, X_6$ is of the form:

$$D_i = \begin{pmatrix} \alpha_i & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_i & 0 & 0 & 0 & 0 \\ \gamma_{i1} & \gamma_{i2} & \alpha_i + \beta_i & 0 & 0 & 0 \\ \delta_i & 0 & \gamma_{i2} & 2\alpha_i + \beta_i & 0 & 0 \\ \epsilon_{i1} & \epsilon_{i2} & 0 & \gamma_{i2} & 3\alpha_i + \beta_i & 0 \\ \varphi_{i1} & \varphi_{i2} & -\epsilon_{i1} & \delta_i & -\gamma_{i1} & 3\alpha_i + 2\beta_i \end{pmatrix}.$$

As $d\rho$ is a Lie algebra morphism, we know that $d\rho([Y_i, Y_j]) = d\rho(0) =$

$$(0,0) = [(X_i, D_i), (X_j, D_j)] = ([X_i, X_j] + D_i(X_j) - D_j(X_i), [D_i, D_j]). \tag{2}$$

It follows that

$$0 = [X_i, X_j] + D_i(X_j) - D_j(X_i), \text{ for all } (i,j) \text{ with } 1 \le i < j \le 6.$$

Considering the above for the pairs $(i, j)$ equal to $(1, 3)$, $(1, 4)$, $(2, 3)$ and $(3, 4)$ we obtain a system of linear equations in the entries of the $D_i$ implying that

$$\epsilon_{41} = \frac{1}{2},\ \gamma_{12} = \frac{-1}{2},\ \delta_3 = \frac{1}{2},\ \alpha_3 = \beta_3 = \gamma_{32} = \epsilon_{31} = 0.$$

From these conditions it follows that $[D_1, D_2] \neq 0$, which contradicts (2). $\square$

**Remark 4.3.** *We will later on, based on Theorem 5.1, obtain a more conceptual proof of this result. In fact, such an action can only exist if the Lie algebra $\mathfrak{n}$ is two-step solvable, see* [5], *section* 2.

## 5. Abelian simply transitive groups and LR-structures

Thus far we have been looking to simply transitive actions $\rho : G \to \mathrm{Aff}(N)$, where both $G$ and $N$ are arbitrary real connected and simply connected nilpotent Lie groups. As already pointed out in the introduction, the case where $N \cong \mathbb{R}^n$ has been well studied and is equivalent to the study of (complete) left symmetric structures on the Lie algebra $\mathfrak{g}$, corresponding to the Lie group $G$.

In this section, we deal with the case that $N$ is arbitrary and $G$ is abelian. It turns out that this case is equivalent to the study of complete LR-structures (see below for a definition) on the Lie algebra $\mathfrak{n}$.

In fact, the main result of this section is the following

**Theorem 5.1.** *Let $N$ be a connected and simply connected nilpotent Lie group of dimension $n$. Then there exists a simply transitive NIL-affine action of $\mathbb{R}^n$ on $N$ via a representation $\rho\colon \mathbb{R}^n \to \mathrm{Aff}(N)$ if and only if the Lie algebra $\mathfrak{n}$ of $N$ admits a complete LR-structure.*

Before giving the proof let us define the notion of an LR-algebra and an LR-structure.

**Definition 5.2.** *An algebra $(A, \cdot)$ over a field $k$ with product $(X, Y) \mapsto X \cdot Y$ is called LR-algebra, if the product satisfies the identities*

$$X \cdot (Y \cdot Z) = Y \cdot (X \cdot Z) \tag{3}$$

$$(X \cdot Y) \cdot Z = (X \cdot Z) \cdot Y \tag{4}$$

*for all $X, Y, Z \in A$.*

If we denote by $L(X)$, $R(X)$ the left, respectively right multiplication operator in the algebra $(A, \cdot)$, then the above conditions say that all left-multiplications and all right multiplications commute:

$$[L(X), L(Y)] = 0,$$
$$[R(X), R(Y)] = 0.$$

Note that LR-algebras are Lie-admissible algebras: the commutator defines a Lie bracket. The associated Lie algebra then is said to admit an LR-structure:

**Definition 5.3.** *An LR-structure on a Lie algebra $\mathfrak{g}$ over $k$ is an LR-product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying*

$$[X, Y] = X \cdot Y - Y \cdot X \tag{5}$$

*for all $X, Y \in \mathfrak{g}$. It is said to be complete, if all left multiplications $L(X)$ are nilpotent.*

We now come to the proof of Theorem 5.1.

*Proof.* Suppose that $\rho\colon \mathbb{R}^n \to \mathrm{Aff}(N)$ induces a simply transitive NIL-affine action of $\mathbb{R}^n$ on $N$. Theorem 3.1 says that this is equivalent to the existence of a Lie algebra homomorphism

$$d\rho : \mathbb{R}^n \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : X \mapsto (t_X, D_X),$$

where $t_X : \mathbb{R}^n \to \mathfrak{n}$ is a linear isomorphism and each $D_X$ is nilpotent. As $t_X$ is bijective, we identify $\mathbb{R}^n$ and $\mathfrak{n}$ as vector spaces and hence we can write $d\rho(X) = (X, D_X)$. The fact that $d\rho$ is a homomorphism from the abelian Lie algebra to the Lie algebra $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is equivalent to the requirement that for all $X, Y \in \mathfrak{n}$

$$\begin{aligned} 0 &= [(X, D_X), (Y, D_Y)] \\ &= ([X, Y] + D_X(Y) - D_Y(X), [D_X, D_Y]). \end{aligned}$$

Define a bilinear product on $\mathfrak{n}$ by $X \cdot Y = -D_X(Y)$. We will show that this defines a complete LR-structure on $\mathfrak{n}$. In fact, the above condition is equivalent to the conditions

$$\begin{aligned} [X, Y] &= X \cdot Y - Y \cdot X, \\ X \cdot (Y \cdot Z) &= Y \cdot (X \cdot Z). \end{aligned}$$

We have to show that also the right multiplications commute. Since $D_X$ is a derivation of $\mathfrak{n}$ we have

$$\begin{aligned} X \cdot (Z \cdot Y) - X \cdot (Y \cdot Z) &= X \cdot [Z, Y] \\ &= -D_X([Y, Z]) \\ &= -[D_X(Y), Z] - [Y, D_X(Z)] \\ &= (X \cdot Y) \cdot Z - Z \cdot (X \cdot Y) + Y \cdot (X \cdot Z) - (X \cdot Z) \cdot Z \end{aligned}$$

for all $X, Y, Z \in \mathfrak{n}$. Since the left multiplications commute, this is equivalent to $(X \cdot Y) \cdot Z = (X \cdot Z) \cdot Y$, showing that also the right multiplications commute, and hence this bilinear product determines an LR-structure on $\mathfrak{n}$.

By Theorem 3.1 we know that all derivations $D_X$, and hence all left multiplications $L(X)$ are nilpotent. This proves that the LR-structure is also complete and so we have obtained the first direction of our result. The converse direction follows in a similar way. $\square$

Theorem 5.1 shows that a deeper study of (complete) LR-algebras is required in order to obtain a good understanding of simply transitive abelian and NIL-affine actions on nilpotent Lie groups. The first steps in this direction are taken in [5].

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