

NOVIKOV ALGEBRAS AND NOVIKOV STRUCTURES ON LIE ALGEBRAS

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ABSTRACT. We study ideals of Novikov algebras and Novikov structures on finite-dimensional Lie algebras. We present the first example of a three-step nilpotent Lie algebra which does not admit a Novikov structure. On the other hand we show that any free three-step nilpotent Lie algebra admits a Novikov structure. We study the existence question also for Lie algebras of triangular matrices. Finally we show that there are families of Lie algebras of arbitrary high solvability class which admit Novikov structures.

1. INTRODUCTION

A Novikov algebra is a special kind of a pre-Lie algebra, or left-symmetric algebra, arising in many contexts in mathematics and physics. Pre-Lie algebras already have been introduced by Cayley in 1896 via rooted tree algebras. Vinberg classified convex homogeneous cones using pre-Lie algebras, Milnor and Auslander discovered the connection to affinely flat manifolds and their fundamental groups. Recently Connes, Kreimer and Kontsevich introduced pre-Lie algebras in mathematical physics, for quantum field theory and renormalization theory. Also Bakalov and Kac have used pre-Lie algebras in the study of vertex algebras. For a survey on this topic see [3].

On the other hand, Novikov algebras in particular were introduced in the study of Hamiltonian operators in the context of integrability of certain nonlinear partial differential equations. They also appear in the study of Poisson brackets of hydrodynamic type, see [1], and operator Yang-Baxter equation. Since then the algebraic structure of Novikov algebras has been studied by many authors. One of the first results here has been obtained by Zelmanov, see [6]. It is our aim to continue this study.

Let k be a field of characteristic zero. A Novikov algebra and, more generally, an LSA is defined as follows:

Definition 1.1. An algebra (A, \cdot) over k with product $(x, y) \mapsto x \cdot y$ is called a *left-symmetric algebra (LSA)*, if the product is left-symmetric, i.e., if the identity

$$(1) \quad x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z$$

is satisfied for all $x, y, z \in A$. The algebra is called *Novikov*, if in addition

$$(2) \quad (x \cdot y) \cdot z = (x \cdot z) \cdot y$$

is satisfied.

Denote by $L(x), R(x)$ the left, respectively right multiplication operator in the algebra (A, \cdot) . Then an LSA is a Novikov algebra if the right multiplications commute:

$$[R(x), R(y)] = 0.$$

It is well known that LSAs are Lie-admissible algebras: the commutator

$$(3) \quad [x, y] = x \cdot y - y \cdot x$$

defines a Lie bracket. The associated Lie algebra is denoted by \mathfrak{g}_A . The adjoint operator can be expressed by $\text{ad}(x) = L(x) - R(x)$.

If A is a Novikov algebra, then we obtain, by expanding the condition $0 = [R(x), R(y)] = [L(x) - \text{ad}(x), L(y) - \text{ad}(y)]$, the following operator identity:

$$(4) \quad L([x, y]) + \text{ad}([x, y]) - [\text{ad}(x), L(y)] - [L(x), \text{ad}(y)] = 0.$$

Definition 1.2. An *affine structure* on a Lie algebra \mathfrak{g} over k is a left-symmetric product $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying (3) for all $x, y \in \mathfrak{g}$. If the product is Novikov, we say that \mathfrak{g} admits a *Novikov structure*.

A given Lie algebra need not admit a Novikov structure, or an affine structure. The existence question for affine structures is very hard in general. It is more accessible for Novikov structures. For results, background and references see, for example, [2],[3] and [4].

2. IDEALS IN NOVIKOV ALGEBRAS

In this section we will present some structure theory concerning ideals in Novikov algebras. For related results in this direction see also [1], [6]. We start with two identities which are similar to the Jacobi identity for Lie algebras.

Lemma 2.1. *Let (A, \cdot) be a Novikov algebra. Then we have, for all $x, y, z \in A$:*

$$\begin{aligned} [x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y &= 0, \\ x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] &= 0. \end{aligned}$$

The proof is straightforward.

Next we show that the product of two ideals is again an ideal.

Lemma 2.2. *Let (A, \cdot) be a Novikov algebra and I, J be two-sided ideals of A . Then $I \cdot J$ is also a two-sided ideal of A .*

Proof. Let $a \in A$, $x \in I$ and $y \in J$. Then the identity

$$a \cdot (x \cdot y) = (a \cdot x) \cdot y + x \cdot (a \cdot y) - (x \cdot a) \cdot y$$

shows that $a \cdot (x \cdot y) \in I \cdot J$. Because of $(x \cdot y) \cdot a = (x \cdot a) \cdot y$ we also have $(x \cdot y) \cdot a \in I \cdot J$. \square

We also show that the commutator of two ideals is again an ideal.

Lemma 2.3. *Let (A, \cdot) be a Novikov algebra and assume that I, J are two-sided ideals of A . Then $[I, J]$ is also a two-sided ideal of A .*

Proof. Let $a \in A$, $x \in I$ and $y \in J$. The operator identity (4) implies that

$$\begin{aligned} 0 &= [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + y \cdot [x, a] - x \cdot [y, a] + [y, x \cdot a] \\ &= [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + [y, x \cdot a] + (y \cdot [x, a] + x \cdot [a, y] + a \cdot [y, x]) - a \cdot [y, x] \\ &= [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + [y, x \cdot a] + a \cdot [x, y] \\ &= [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + [y, x \cdot a] + [x, y] \cdot a + [a, [x, y]] \\ &= 2[x, y] \cdot a + [y, x \cdot a] - [x, y \cdot a]. \end{aligned}$$

The term in brackets above vanishes because of lemma 2.1. From this we deduce

$$\begin{aligned} [x, y] \cdot a &= \frac{1}{2}([x, y \cdot a] - [y, x \cdot a]) \in [I, J], \\ a \cdot [x, y] &= [x, y] \cdot a + [a, [x, y]] \in [I, J], \end{aligned}$$

which was to be shown. \square

Let (A, \cdot) be a Novikov algebra. Denote by

$$\begin{aligned} \gamma_1(A) &= \gamma_1(\mathfrak{g}_A) = A \\ \gamma_{i+1}(A) &= \gamma_{i+1}(\mathfrak{g}_A) = [A, \gamma_i(A)] \end{aligned}$$

the terms of the lower central series of A , respectively \mathfrak{g}_A . Furthermore denote by

$$\begin{aligned} A^{(0)} &= \mathfrak{g}_A^{(0)} = A \\ A^{(i+1)} &= \mathfrak{g}_A^{(i+1)} = [A^{(i)}, A^{(i)}] \end{aligned}$$

the terms of the derived series of A , respectively \mathfrak{g}_A . Then the above lemma immediately implies the following result:

Corollary 2.4. *Let (A, \cdot) be a Novikov algebra. Then all $\gamma_i(A)$, and all $A^{(i)}$ are two-sided ideals of A .*

The ideals of the lower central series satisfy the following property.

Lemma 2.5. *Let (A, \cdot) be a Novikov algebra. Then we have*

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+1}(A)$$

for all $i, j \geq 0$.

Proof. We will show this by induction on $i \geq 0$. The case $i = 0$ follows from the fact that $\gamma_{j+1}(A)$ is an ideal in A , see corollary 2.4. Assume now that $\gamma_k(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{k+j}(A)$ for all $k = 1, \dots, i$.

Let $x \in \gamma_1(A)$, $y \in \gamma_i(A)$ and $z \in \gamma_{j+1}(A)$. We have to show that $[x, y] \cdot z \in \gamma_{i+j+1}(A)$. The first identity of lemma 2.1 says, that $[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0$. By (3) we have

$$[z, x] \cdot y = y \cdot [z, x] + [[z, x], y].$$

Then $y \cdot [z, x] \in \gamma_{i+j+1}(A)$ by induction hypothesis, and $[[z, x], y] \in \gamma_{i+j+2}(A)$. It follows that $[z, x] \cdot y \in \gamma_{i+j+1}(A)$. Similarly we obtain $[y, z] \cdot x \in \gamma_{i+j+1}(A)$. Now the first identity of lemma 2.1 implies $[x, y] \cdot z \in \gamma_{i+j+1}(A)$. \square

Denote the center of a Novikov algebra A by $Z(A) = \{x \in A \mid x \cdot y = y \cdot x \text{ for all } y \in A\}$. Note that $Z(A)$ is also the center of the associated Lie algebra \mathfrak{g}_A .

Lemma 2.6. *Let (A, \cdot) be a Novikov algebra. Then $Z(A) \cdot [A, A] = [A, A] \cdot Z(A) = 0$.*

Proof. Let $a, b \in A$ and $z \in Z(A)$. Again by lemma 2.1 we have

$$z \cdot [a, b] + a \cdot [b, z] + b \cdot [z, a] = 0.$$

Since z is also in the center of the associated Lie algebra of A we obtain $z \cdot [a, b] = 0$. Furthermore we have

$$\begin{aligned} 0 &= [z, [b, a]] = z \cdot [b, a] - [b, a] \cdot z \\ &= [a, b] \cdot z. \end{aligned}$$

\square

Lemma 2.7. *Let (A, \cdot) be a Novikov algebra. Then $Z(A)$ is a two-sided ideal of A .*

Proof. Let $z \in Z(A)$. For any $b \in A$ we have the following two identities

$$\begin{aligned} [L(b), L(z)] &= L([b, z]) = 0, \\ [R(b), R(z)] &= 0. \end{aligned}$$

Because $z \in Z(A)$ we have $R(z) = L(z)$. Hence we also have $[L(b), R(z)] = 0$, so that

$$0 = [L(b) - R(b), R(z)] = [\text{ad}(b), R(z)].$$

In particular it follows $[b, a \cdot z] - [b, a] \cdot z = 0$ for all $a \in A$. By Lemma 2.6 we have $[b, a] \cdot z = 0$, hence $[b, a \cdot z] = 0$. Because this is true for every $b \in A$, we can conclude that $a \cdot z \in Z(A)$. Since $z \in Z(A)$ we also have $z \cdot a \in Z(A)$. \square

Let $Z_1(A) = Z(A)$ and define $Z_{i+1}(A)$ by the identity $Z_{i+1}(A)/Z_i(A) = Z(A/Z_i(A))$. Note that the $Z_i(A)$ are the terms of the upper central series of the associated Lie algebra \mathfrak{g}_A . As an immediate consequence of the previous lemma, we obtain

Corollary 2.8. *Let (A, \cdot) be a Novikov algebra. Then all terms $Z_i(A)$ of the upper central series of A are two-sided ideals of A .*

Denote by $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ the associator of three elements in A .

Lemma 2.9. *Let A be a Novikov algebra and one of the elements x, y, z in $Z(A)$. Then $(x, y, z) = 0$.*

Proof. In any LSA we have the identity

$$(x, y, z) = x \cdot [y, z] + [z, x \cdot y] + [x, z] \cdot y.$$

If $z \in Z(A)$, then this implies $(x, y, z) = 0$. If $y \in Z(A)$ then also $x \cdot y \in Z(A)$ by lemma 2.7, and $[A, A] \cdot Z(A) = 0$ by lemma 2.6. Hence the above identity implies $(x, y, z) = 0$. The same argument shows the claim for $x \in Z(A)$. \square

3. NOVIKOV STRUCTURES ON 3-STEP NILPOTENT LIE ALGEBRAS

In [4, Remark 4.11] it was questioned whether or not there exists a 3-step nilpotent Lie algebra not admitting a Novikov structure. In the same paper it was shown that a Novikov structure does exist when the 3-step nilpotent Lie algebra \mathfrak{g} can be generated by at most 3 elements. This result was obtained by first considering a Novikov structure on the free 3-step nilpotent Lie algebra \mathfrak{f} on 3 generators and then it was shown that \mathfrak{g} could be realized as a quotient $\mathfrak{g} = \mathfrak{f}/I$, where I is an ideal of \mathfrak{f} seen as a Novikov algebra.

Having this in mind, we first study the free 3-step nilpotent case.

Proposition 3.1. *Let \mathfrak{g} be a free 3-step nilpotent Lie algebra on n generators x_1, x_2, \dots, x_n . Then \mathfrak{g} admits a Novikov structure.*

Proof. As a vector space, \mathfrak{g} has a basis

$$\begin{aligned} &x_1, x_2, \dots, x_n, \\ &y_{i,j} = [x_i, x_j], \quad (1 \leq i < j \leq n), \\ &z_{i,j,k} = [x_i, y_{j,k}], \quad (1 \leq j < k \leq n, 1 \leq i \leq k \leq n). \end{aligned}$$

Note that in case $i > k$, we have that

$$z_{i,j,k} = [x_i, y_{j,k}] = [x_i, [x_j, x_k]] = -[x_j, [x_k, x_i]] - [x_k, [x_i, x_j]] = -z_{j,k,i} + z_{k,j,i}.$$

A Novikov structure on \mathfrak{g} is defined by

- If $n \geq i > j \geq 1$ then $x_i \cdot x_j = -y_{j,i}$.
- If $1 \leq i \leq j < k \leq n$ then $x_i \cdot y_{j,k} = \frac{z_{i,j,k}}{2}$.
 If $1 \leq j < i < k \leq n$ then $x_i \cdot y_{j,k} = -\frac{z_{j,i,k}}{2} + z_{i,j,k}$.
 If $1 \leq j < k \leq i \leq n$ then $x_i \cdot y_{j,k} = z_{i,j,k}$.
- If $1 \leq i \leq j < k \leq n$, then $y_{j,k} \cdot x_i = -\frac{z_{i,j,k}}{2}$.
 If $1 \leq j < i < k \leq n$, then $y_{j,k} \cdot x_i = -\frac{z_{j,i,k}}{2}$.

All other products are zero.

By considering each of the above cases it is easy to see that the identity $[a, b] = a \cdot b - b \cdot a$ holds for all basis elements a and b . We have to show the following two other identities:

$$\begin{aligned} (a \cdot b) \cdot c - a \cdot (b \cdot c) - (b \cdot a) \cdot c + b \cdot (a \cdot c) &= 0, \\ (a \cdot b) \cdot c - (a \cdot c) \cdot b &= 0, \end{aligned}$$

for all basis elements a, b and c . It is clear that we only have to consider the case where $a = x_i$, $b = x_j$ and $c = x_k$: otherwise the two identities will be trivially satisfied, because any product of the form $y_{i,j} \cdot y_{k,l}$ is zero, and any product that involves an element $z_{i,j,k}$ is also zero. For the first condition we will consider the case $1 \leq k < i < j \leq n$. We have

$$\begin{aligned} (x_i \cdot x_j) \cdot x_k - x_i \cdot (x_j \cdot x_k) - (x_j \cdot x_i) \cdot x_k + x_j \cdot (x_i \cdot x_k) \\ &= -x_i \cdot (-y_{k,j}) - (-y_{i,j}) \cdot x_k + x_j \cdot (-y_{k,i}) \\ &= -\frac{z_{k,i,j}}{2} + z_{i,k,j} - \frac{z_{k,i,j}}{2} - z_{j,k,i} \\ &= -z_{k,i,j} + z_{i,k,j} + z_{k,i,j} - z_{i,k,j} \\ &= 0. \end{aligned}$$

Similarly the other cases can be treated. For the second condition we consider the case $1 \leq j < k < i \leq n$. We have

$$\begin{aligned} (x_i \cdot x_j) \cdot x_k - (x_i \cdot x_k) \cdot x_j &= -y_{j,i} \cdot x_k + y_{k,i} \cdot x_j \\ &= \frac{z_{j,k,i}}{2} - \frac{z_{j,k,i}}{2} \\ &= 0. \end{aligned}$$

Similarly the other cases can be shown. It follows that the product defines a Novikov structure on \mathfrak{g} . \square

As a motivation for what follows, we provide a detailed description in the four generator case.

Example 3.2. *Let $n = 4$. Then $\dim \mathfrak{g} = 30$. The nonzero Lie brackets and Novikov products are given as follows.*

$$\begin{array}{lll} [x_1, x_2] = y_{1,2} & [x_1, y_{1,2}] = z_{1,1,2} & [x_1, y_{2,3}] = z_{1,2,3} \\ [x_1, x_3] = y_{1,3} & [x_1, y_{1,3}] = z_{1,1,3} & [x_1, y_{2,4}] = z_{1,2,4} \\ [x_1, x_4] = y_{1,4} & [x_1, y_{1,4}] = z_{1,1,4} & [x_1, y_{3,4}] = z_{1,3,4} \end{array}$$

$$\begin{array}{lll}
[x_2, x_3] = y_{2,3} & [x_2, y_{3,4}] = z_{2,3,4} & [x_3, y_{3,4}] = z_{3,3,4} \\
[x_2, x_4] = y_{2,4} & [x_3, x_4] = y_{3,4} & [x_4, y_{1,2}] = -z_{1,2,4} + z_{2,1,4} \\
[x_2, y_{1,2}] = z_{2,1,2} & [x_3, y_{1,2}] = -z_{1,2,3} + z_{2,1,3} & [x_4, y_{1,3}] = -z_{1,3,4} + z_{3,1,4} \\
[x_2, y_{1,3}] = z_{2,1,3} & [x_3, y_{1,3}] = z_{3,1,3} & [x_4, y_{1,4}] = z_{4,1,4} \\
[x_2, y_{1,4}] = z_{2,1,4} & [x_3, y_{1,4}] = z_{3,1,4} & [x_4, y_{2,3}] = -z_{2,3,4} + z_{3,2,4} \\
[x_2, y_{2,3}] = z_{2,2,3} & [x_3, y_{2,3}] = z_{3,2,3} & [x_4, y_{2,4}] = z_{4,2,4} \\
[x_2, y_{2,4}] = z_{2,2,4} & [x_3, y_{2,4}] = z_{3,2,4} & [x_4, y_{3,4}] = z_{4,3,4} \\
\\
x_1 \cdot y_{1,2} = \frac{z_{1,1,2}}{2} & x_3 \cdot y_{1,2} = -z_{1,2,3} + z_{2,1,3} & y_{1,2} \cdot x_1 = -\frac{z_{1,1,2}}{2} \\
x_1 \cdot y_{1,3} = \frac{z_{1,1,3}}{2} & x_3 \cdot y_{1,3} = z_{3,1,3} & y_{1,3} \cdot x_1 = -\frac{z_{1,1,3}}{2} \\
x_1 \cdot y_{1,4} = \frac{z_{1,1,4}}{2} & x_3 \cdot y_{1,4} = \frac{-z_{1,3,4}}{2} + z_{3,1,4} & y_{1,3} \cdot x_2 = -\frac{z_{1,2,3}}{2} \\
x_1 \cdot y_{2,3} = \frac{z_{1,2,3}}{2} & x_3 \cdot y_{2,3} = z_{3,2,3} & y_{1,4} \cdot x_1 = -\frac{z_{1,1,4}}{2} \\
x_1 \cdot y_{2,4} = \frac{z_{1,2,4}}{2} & x_3 \cdot y_{2,4} = \frac{-z_{2,3,4}}{2} + z_{3,2,4} & y_{1,4} \cdot x_2 = -\frac{z_{1,2,4}}{2} \\
x_1 \cdot y_{3,4} = \frac{z_{1,3,4}}{2} & x_3 \cdot y_{3,4} = \frac{z_{3,3,4}}{2} & y_{1,4} \cdot x_3 = -\frac{z_{1,3,4}}{2} \\
x_2 \cdot x_1 = -y_{1,2} & x_4 \cdot x_1 = -y_{1,4} & y_{2,3} \cdot x_1 = -\frac{z_{1,2,3}}{2} \\
x_2 \cdot y_{1,2} = z_{2,1,2} & x_4 \cdot x_2 = -y_{2,4} & y_{2,3} \cdot x_2 = -\frac{z_{2,2,3}}{2} \\
x_2 \cdot y_{1,3} = \frac{-z_{1,2,3}}{2} + z_{2,1,3} & x_4 \cdot x_3 = -y_{3,4} & y_{2,4} \cdot x_1 = -\frac{z_{1,2,4}}{2} \\
x_2 \cdot y_{1,4} = \frac{-z_{1,2,4}}{2} + z_{2,1,4} & x_4 \cdot y_{1,2} = -z_{1,2,4} + z_{2,1,4} & y_{2,4} \cdot x_2 = -\frac{z_{2,2,4}}{2} \\
x_2 \cdot y_{2,3} = \frac{z_{2,2,3}}{2} & x_4 \cdot y_{1,3} = -z_{1,3,4} + z_{3,1,4} & y_{2,4} \cdot x_3 = -\frac{z_{2,3,4}}{2} \\
x_2 \cdot y_{2,4} = \frac{z_{2,2,4}}{2} & x_4 \cdot y_{1,4} = z_{4,1,4} & y_{3,4} \cdot x_1 = -\frac{z_{1,3,4}}{2} \\
x_2 \cdot y_{3,4} = \frac{z_{2,3,4}}{2} & x_4 \cdot y_{2,3} = -z_{2,3,4} + z_{3,2,4} & y_{3,4} \cdot x_2 = -\frac{z_{2,3,4}}{2} \\
x_3 \cdot x_1 = -y_{1,3} & x_4 \cdot y_{2,4} = z_{4,2,4} & y_{3,4} \cdot x_3 = -\frac{z_{3,3,4}}{2} \\
x_3 \cdot x_2 = -y_{2,3} & x_4 \cdot y_{3,4} = z_{4,3,4} &
\end{array}$$

When trying to find an example of a 3-step nilpotent Lie algebra without a Novikov structure, we know from [4] that such an example must have at least 4 generators. Any 3-step nilpotent Lie algebra on 4 generators is a quotient \mathfrak{g}/I of the Lie algebra \mathfrak{g} described in example 3.2, where I is an ideal of \mathfrak{g} . So, in order to find such a Lie algebra \mathfrak{g}/I without Novikov structure, we have to choose an I which is certainly not an ideal of \mathfrak{g} , seen as a Novikov algebra. The following proposition uses such an example.

Proposition 3.3. *Consider the following 3-step nilpotent Lie algebra \mathfrak{g} on 4 generators of dimension 13, with basis (x_1, \dots, x_{13}) and non-trivial Lie brackets*

$$\begin{array}{ll}
[x_1, x_2] = x_5, & [x_3, x_4] = -x_5, \\
[x_1, x_4] = x_6, & [x_3, x_5] = -x_{11}, \\
[x_1, x_6] = x_{10}, & [x_3, x_8] = x_9, \\
[x_1, x_7] = x_{11}, & [x_4, x_5] = -x_{12}, \\
[x_1, x_8] = x_{12}, & [x_4, x_6] = x_9, \\
[x_2, x_3] = x_7, & [x_4, x_7] = x_9 + x_{13}. \\
[x_2, x_4] = x_8, & \\
[x_2, x_5] = x_{13}, & \\
[x_2, x_7] = x_{13}, &
\end{array}$$

This Lie algebra does not admit a Novikov structure.

Note that \mathfrak{g} admits an affine structure since it is positively graded.

Proof. We will assume that \mathfrak{g} admits a Novikov structure and show that this leads to a contradiction. We express the adjoint operators $\text{ad}(x_i)$ and the left (resp. right) multiplication operators $L(x_i)$ (resp. $R(x_i)$) as matrices with respect to the basis x_1, x_2, \dots, x_{13} . The adjoint operators $\text{ad}(x_i)$ are given by the Lie brackets of \mathfrak{g} , while the left multiplication operators are unknown. We denote the (j, k) -th entry of $L(x_i)$ by

$$L(x_i)_{j,k} = x_{j,k}^i.$$

We use the convention that the j -th column of $L(x_i)$ gives the coordinates of $L(x_i)(x_j)$. Note that once the entries of the left multiplication operators are chosen, the right multiplication operators are given by $R(x_i)_{j,k} = x_{j,i}^k$. We have to satisfy all relations given by (1), (2) and (3), where x, y and z run over all basis vectors. This leads to a huge system of quadratic equations in the variables $x_{j,k}^i$ for $1 \leq i, j, k \leq 13$, summing up to a total of $13^3 = 2197$ variables. We need to show that these equations are contradictory. At first sight, this seems to be a rather hopeless task. However, we can use our knowledge on ideals in a Novikov algebra. Then we find that a lot of unknowns $x_{j,k}^i$ already have to be zero. In the table below, we list the triples (i, j, k) for which we already know that $x_{j,k}^i = 0$:

$$\begin{array}{llll} 1 \leq i \leq 13, & 1 \leq j \leq 4, & 5 \leq k \leq 13, & \text{because } \gamma_2(\mathfrak{g}) \text{ is an ideal.} \\ 1 \leq i \leq 13, & 5 \leq j \leq 8, & 9 \leq k \leq 13, & \text{because } \gamma_3(\mathfrak{g}) \text{ is an ideal.} \\ 5 \leq i \leq 13, & 1 \leq j \leq 4, & 1 \leq k \leq 4, & \text{because } \gamma_2(\mathfrak{g}) \text{ is an ideal.} \\ 5 \leq i \leq 13, & 5 \leq j \leq 8, & 5 \leq k \leq 8, & \text{because } \gamma_2(\mathfrak{g}) \cdot \gamma_2(\mathfrak{g}) \subseteq \gamma_3(\mathfrak{g}). \\ 5 \leq i \leq 13, & 9 \leq j \leq 13, & 9 \leq k \leq 13, & \text{because } \gamma_2(\mathfrak{g}) \cdot \gamma_3(\mathfrak{g}) = 0. \\ 9 \leq i \leq 13, & 5 \leq j \leq 8, & 1 \leq k \leq 4, & \text{because } \gamma_3(\mathfrak{g}) \text{ is an ideal.} \\ 9 \leq i \leq 13, & 9 \leq j \leq 13, & 5 \leq k \leq 8, & \text{because } \gamma_3(\mathfrak{g}) \cdot \gamma_2(\mathfrak{g}) = 0. \end{array}$$

It follows that 1421 of the $x_{j,k}^i$ have to be zero, leaving us with 776 variables. On the other hand the conditions

$$\text{ad}(x_i) = L(x_i) - R(x_i), \quad 1 \leq i \leq 13.$$

yield a (large but very simple) system of linear equations; allowing us to determine 352 variables $x_{j,k}^i$ in dependence of the remaining $776 - 352 = 424$ ones.

To get a further reduction we use that

$$x_i \cdot [x_j, x_k] + x_j \cdot [x_k, x_i] + x_k \cdot [x_i, x_j] = 0, \quad 1 \leq i < j < k \leq 13,$$

which is the same as

$$L(x_i)(\text{ad}(x_j)x_k) + L(x_j)(\text{ad}(x_k)x_i) + L(x_k)(\text{ad}(x_i)x_j) = 0, \quad 1 \leq i < j < k \leq 13.$$

Again this leads to a system of linear equations, this time specifying 156 unknowns in terms of the other ones, leaving $424 - 156 = 268$ variables.

Now, we consider the operator identity (4), i.e.,

$$L([x_i, x_j]) + \text{ad}([x_i, x_j]) - [\text{ad}(x_i), L(x_j)] - [L(x_i), \text{ad}(x_j)] = 0, \quad 1 \leq i < j \leq 13.$$

Note that for any pair (i, j) , we can write $[x_i, x_j]$ as a linear combination of the x_k , $1 \leq k \leq 13$. Hence we can also write $L([x_i, x_j])$ as the corresponding linear combination of the $L(x_k)$. Doing this, we obtain another system of linear equations, determining 210 extra variables, leaving

268 – 210 = 58 free variables.

Finally we use that the right multiplications have to commute, i.e.,

$$R(x_i)R(x_j) - R(x_j)R(x_i) = 0, \quad 1 \leq i < j \leq 13.$$

This yields a system of quadratic equations, which is immediately contradictory. In fact, when taking $i = 1$ and $j = 2$, one obtains the equation $0 = \frac{1}{8}$, which is the desired contradiction. \square

4. THE (NON) EXISTENCE OF NOVIKOV STRUCTURES ON TRIANGULAR MATRIX ALGEBRAS

One of the most fundamental examples for solvable, resp. nilpotent Lie algebras are the Lie algebras of upper-triangular, resp. strictly upper triangular matrices of size n over a field k , which we denote by $\mathfrak{t}(n, k)$, resp. $\mathfrak{n}(n, k)$. It is therefore natural to ask, which of those Lie algebras admit a Novikov structure. It turns out that such structures exist only in very small dimensions.

Proposition 4.1. *The Lie algebra $\mathfrak{n}(n, k)$ admits a Novikov structure if and only if $n \leq 4$.*

Proof. If $n \leq 4$, the Lie algebra $\mathfrak{n}(n, k)$ is abelian ($n = 2$), 2-step nilpotent ($n = 3$) or 3-step nilpotent and generated by 3 elements ($n = 4$). In any of these cases, we know that a Novikov structure exists.

Now let $n > 4$ and suppose that $\mathfrak{n}(n, k)$ admits a Novikov structure. Denote by $e_{i,j}$ the elementary matrices, which have a 1 on the (i, j) -th position and a zero elsewhere. The $e_{i,j}$ with $1 \leq i < j \leq n$ form a basis of $\mathfrak{n}(n, k)$. The Lie bracket is given by

$$[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l} - \delta_{i,l}e_{k,j}.$$

Assume that (A, \cdot) defines a Novikov structure on $\mathfrak{n}(n, k)$. Then some easy calculations, using lemma 2.1 and identity (3) yield:

$$\begin{aligned} e_{1,2} \cdot [e_{3,4}, e_{4,5}] + e_{3,4} \cdot [e_{4,5}, e_{1,2}] + e_{4,5} \cdot [e_{1,2}, e_{3,4}] &= 0 &\Rightarrow e_{1,2} \cdot e_{3,5} &= 0 \\ e_{3,4} \cdot [e_{4,5}, e_{1,3}] + e_{4,5} \cdot [e_{1,3}, e_{3,4}] + e_{1,3} \cdot [e_{3,4}, e_{4,5}] &= 0 &\Rightarrow e_{1,3} \cdot e_{3,5} &= -e_{4,5} \cdot e_{1,4} \\ e_{3,4} \cdot [e_{4,5}, e_{2,3}] + e_{4,5} \cdot [e_{2,3}, e_{3,4}] + e_{2,3} \cdot [e_{3,4}, e_{4,5}] &= 0 &\Rightarrow e_{2,3} \cdot e_{3,5} &= -e_{4,5} \cdot e_{2,4} \\ e_{1,2} \cdot [e_{4,5}, e_{2,4}] + e_{4,5} \cdot [e_{2,4}, e_{1,2}] + e_{2,4} \cdot [e_{1,2}, e_{4,5}] &= 0 &\Rightarrow e_{1,2} \cdot e_{2,5} &= -e_{4,5} \cdot e_{1,4} = e_{1,3} \cdot e_{3,5} \\ e_{1,2} \cdot [e_{2,3}, e_{3,5}] + e_{2,3} \cdot [e_{3,5}, e_{1,2}] + e_{3,5} \cdot [e_{1,2}, e_{2,3}] &= 0 &\Rightarrow e_{1,2} \cdot e_{2,5} &= -e_{3,5} \cdot e_{1,3} = e_{1,3} \cdot e_{3,5} \\ [e_{1,3}, e_{3,5}] - e_{1,3} \cdot e_{3,5} + e_{3,5} \cdot e_{1,3} &= 0 &\Rightarrow e_{1,3} \cdot e_{3,5} &= e_{1,5}/2 = -e_{3,5} \cdot e_{1,3} \\ [e_{1,4}, e_{4,5}] - e_{1,4} \cdot e_{4,5} + e_{4,5} \cdot e_{1,4} &= 0 &\Rightarrow e_{1,4} \cdot e_{4,5} &= e_{1,5}/2 = -e_{4,5} \cdot e_{1,4}. \end{aligned}$$

Applying the operator identity (4) for $x = e_{1,2}$, $y = e_{2,3}$ to $z = e_{3,5}$, and using the above computations, we find

$$\begin{aligned} 0 &= (L([e_{1,2}, e_{2,3}]) + \text{ad}[e_{1,2}, e_{2,3}] - [L(e_{1,2}), \text{ad}(e_{2,3})] - [\text{ad}(e_{1,2}), L(e_{2,3})])(e_{3,5}) \\ &= e_{1,3} \cdot e_{3,5} + e_{1,5} - e_{1,2} \cdot e_{2,5} - [e_{1,2}, e_{2,3} \cdot e_{3,5}] \\ &= e_{1,5} + [e_{1,2}, e_{4,5} \cdot e_{2,4}] \\ &= e_{1,5} + [e_{1,2}, e_{2,4} \cdot e_{4,5} + [e_{4,5}, e_{2,4}]]. \end{aligned}$$

It follows that $[e_{1,2}, e_{2,4} \cdot e_{4,5}] = 0$. Furthermore it follows that

$$\begin{aligned} 0 &= (L([e_{1,2}, e_{2,4}]) + \text{ad}[e_{1,2}, e_{2,4}] - [L(e_{1,2}), \text{ad}(e_{2,4})] - [\text{ad}(e_{1,2}), L(e_{2,4})])(e_{4,5}) \\ &= e_{1,4} \cdot e_{4,5} + e_{1,5} - e_{1,2} \cdot e_{2,5} + [e_{2,4}, e_{1,2} \cdot e_{4,5}] - [e_{1,2}, e_{2,4} \cdot e_{4,5}] \\ &= e_{1,5} + [e_{2,4}, e_{1,2} \cdot e_{4,5}]. \end{aligned}$$

However, this is impossible, since $e_{1,5} \notin [e_{2,4}, \mathfrak{n}(n, k)]$. This contradiction shows that there is no Novikov structure on $\mathfrak{n}(n, k)$ when $n \geq 5$. \square

As a consequence we can easily prove the following result.

Proposition 4.2. *The Lie algebra $\mathfrak{t}(n, k)$ admits a Novikov structure if and only if $n \leq 2$.*

Proof. It is easy to construct a Novikov structure on $\mathfrak{t}(1, k) \cong k$ and on $\mathfrak{t}(2, k)$. For $n = 3$ and $n = 4$ it is not difficult to see by direct calculations that $\mathfrak{t}(n, k)$ does not admit a Novikov structure. For $n \geq 5$ we can use the above proposition. Assume that $\mathfrak{t}(n, k)$ admits a Novikov structure for $n \geq 5$. Then also $[\mathfrak{t}(n, k), \mathfrak{t}(n, k)] = \mathfrak{n}(n, k)$ admits a Novikov structure, in contradiction to our previous proposition. \square

5. NOVIKOV STRUCTURES ON k -STEP SOLVABLE LIE ALGEBRAS

A natural question is, whether there are families of Lie algebras of solvability class k , which admit Novikov structures for all $k \geq 1$. The same question for nilpotency class has an easy answer. Here the standard filiform nilpotent Lie algebras with basis (e_1, \dots, e_n) and brackets $[e_1, e_i] = e_{i+1}$ for $i = 2, \dots, n-1$ admit Novikov structures. Hence they provide examples of nilpotency class $k = n-1$, see [4]. The following result shows that there are indeed filiform nilpotent Lie algebras of arbitrary solvability class, which admit Novikov structures. Define for every $n \geq 3$ a filiform Lie algebra $\mathfrak{f}_{\frac{9}{10}, n}$ of dimension n by

$$\begin{aligned} [e_1, e_j] &= e_{j+1}, \quad 2 \leq j \leq n-1, \\ [e_i, e_j] &= \frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}} e_{i+j}, \quad 2 \leq i \leq j; \quad i+j \leq n \end{aligned}$$

In particular we have

$$\begin{aligned} [e_2, e_j] &= \frac{6(j-2)}{j(j-1)} e_{j+2}, \quad 3 \leq j \leq n-3, \\ [e_j, e_{j+1}] &= \frac{6(j-1)!(j-2)!}{(2j-1)!} e_{2j+1}, \quad 2 \leq j \leq (n-1)/2. \end{aligned}$$

Then $[e_2, e_3] = e_5$, $[e_2, e_4] = e_6$, $[e_2, e_5] = \frac{9}{10}e_7$, etc. Similar Lie algebras were studied in [2]. To verify the Jacobi identity introduce a new basis (f_1, \dots, f_n) by

$$\begin{aligned} f_1 &= 6e_1, \\ f_j &= \frac{1}{(j-2)!} e_j, \quad 2 \leq j \leq n. \end{aligned}$$

Then the new brackets are given by

$$[f_i, f_j] = 6(j-i)f_{i+j}, \quad 1 \leq i \leq j; \quad i+j \leq n.$$

Here the Jacobi identity is obvious.

Proposition 5.1. *For each $n \geq 3$ the Lie algebra $\mathfrak{f}_{\frac{9}{10},n}$ admits a complete Novikov structure. It is given by the following multiplication:*

$$\begin{aligned} e_1 \cdot e_j &= e_{j+1}, \quad 2 \leq j \leq n-1, \\ e_i \cdot e_j &= \frac{6}{j \binom{j+i-2}{i-2}} e_{i+j}, \quad 2 \leq i, j \leq n, \quad i+j \leq n. \end{aligned}$$

Remark 5.2. Note that $\mathfrak{f}_{\frac{9}{10},n}$ is k -step solvable if $2^k \leq n+1 < 2^{k+1}$. Indeed,

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g}, \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}, \mathfrak{g}] = \langle e_3, \dots, e_n \rangle, \\ \mathfrak{g}^{(i)} &= [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}] = \langle e_{2^{i+1}-1}, \dots, e_n \rangle. \end{aligned}$$

Hence these algebras can have arbitrary high solvability class.

Proof. In the new basis (f_1, \dots, f_n) the Novikov product is given by

$$f_i \cdot f_j = 6(j-1)f_{i+j}, \quad 1 \leq i, j \leq n, \quad i+j \leq n.$$

Now it is easy to verify the required identities. We have (using the convention that $f_m = 0$ when $m > n$)

$$\begin{aligned} f_i \cdot f_j - f_j \cdot f_i &= 6(j-1)f_{i+j} - 6(i-1)f_{i+j} \\ &= 6(j-i)f_{i+j} = [f_i, f_j], \quad i, j \geq 1, \end{aligned}$$

so that (3) is satisfied. We have

$$\begin{aligned} (f_i \cdot f_j) \cdot f_k &= 36(j-1)(k-1)f_{i+j+k}, \\ (f_i \cdot f_k) \cdot f_j &= 36(k-1)(j-1)f_{i+j+k}, \end{aligned}$$

so that (2) is satisfied. Finally,

$$\begin{aligned} f_i \cdot (f_j \cdot f_k) - (f_i \cdot f_j) \cdot f_k &= 36 \cdot k(k-1)f_{i+j+k}, \\ f_j \cdot (f_i \cdot f_k) - (f_j \cdot f_i) \cdot f_k &= 36 \cdot k(k-1)f_{i+j+k}, \end{aligned}$$

so that (1) is satisfied. □

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