MINIMAL FAITHFUL REPRESENTATIONS OF REDUCTIVE LIE ALGEBRAS

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Abstract. We prove an explicit formula for the invariant $\mu(\mathfrak{g})$ for finite-dimensional semisimple, and reductive Lie algebras $\mathfrak{g}$ over $\mathbb{C}$. Here $\mu(\mathfrak{g})$ is the minimal dimension of a faithful linear representation of $\mathfrak{g}$. The result can be used to study Dynkin's classification of maximal reductive subalgebras of semisimple Lie algebras.

1. Introduction

Let $\mathfrak{g}$ be an n-dimensional Lie algebra over a field K. Denote by $\mu(\mathfrak{g})$ the minimal dimension of a faithful linear representation of $\mathfrak{g}$. By Ado's theorem this is an integer valued invariant of $\mathfrak{g}$, which has been introduced in [1]. We consider K as given by $\mathfrak{g}$, so that we need not refer to K in the notation $\mu(\mathfrak{g})$. In general it is not known how to determine this invariant. In particular, this seems to be very hard in general for a given solvable Lie algebra.
The invariant $\mu(\mathfrak{g})$ plays an important role in the theory of affinely flat manifolds and affine crystallographic groups, see [2]. In particular, the following two results are known:

Proposition 1.1. *Let G be an n-dimensional Lie group with Lie algebra $\mathfrak{g}$. If G admits a left-invariant affine structure then $\mu(\mathfrak{g}) \leq n+1$.*

Proposition 1.2. *Let Γ be a torsionfree finitely generated nilpotent group of rank n and G_Γ its real Malcev-completion with Lie algebra $\mathfrak{g}_\Gamma$. If Γ is the fundamental group of a compact complete affine manifold then $\mu(\mathfrak{g}_\Gamma) \leq n+1$.*

A semisimple Lie group G does not admit any left-invariant affine structures. If G is reductive, the existence problem of such structures has not been solved in general. The problem is even harder for solvable and nilpotent Lie groups. For details and references see [2].
If $\mathfrak{g}$ has trivial center $Z(\mathfrak{g})$, the adjoint representation is faithful and hence we have $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) = n$. If $\mathfrak{g}$ is nilpotent, the adjoint representation is not faithful, and such a result is not even true in general. Since the classification of representations of nilpotent Lie algebras is a wild problem, it seems reasonable to expect difficulties in determining $\mu(\mathfrak{g})$. In this case one tries to obtain good upper and lower bounds for $\mu(\mathfrak{g})$. There is the following result, see [2]. Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n and nilpotency class k. Denote by $p(j)$ the number of partitions of j and let

$$p(n,k) = \sum_{j=0}^{k} \binom{n-j}{k-j} p(j).$$

Then $\mu(\mathfrak{g}) \leq p(n,k)$. In particular, with $\alpha = \frac{113}{40}$, we have $\mu(\mathfrak{g}) < \frac{\alpha}{\sqrt{n}} 2^n$. If $\mathfrak{g}$ is reductive, however, the situation is much better. There are explicit formulas for $\mu(\mathfrak{g})$, in case $\mathfrak{g}$ is abelian

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or $\mathfrak{g}$ is simple. We will always assume that $K = \mathbb{C}$, unless specified otherwise. The aim of this paper is to show the following result:

Theorem 1.3. *Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell \oplus \mathbb{C}^k$ its decomposition into simple ideals $\mathfrak{s}_i$ and center $\mathbb{C}^k$. Then the following formula holds:*

$$\mu(\mathfrak{g}) = \mu(\mathfrak{s}_1) + \ldots + \mu(\mathfrak{s}_\ell) + \mu(\mathbb{C}^{k-\ell}),$$

where the $\mu(\mathfrak{s}_i)$ are listed in section 2.2, $\mu(\mathbb{C}^{k-\ell}) = 0$ for $k - \ell \leq 0$, $\mu(\mathbb{C}^{k-\ell}) = \lceil 2\sqrt{k-\ell-1}\rceil$ for $k - \ell > 1$, and $\mu(\mathbb{C}) = 1$.

2. Faithful representations

We start with two simple lemmas.

Lemma 2.1. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $\mu(\mathfrak{h}) \leq \mu(\mathfrak{g})$. Furthermore, if $\mathfrak{a}$ and $\mathfrak{b}$ are two Lie algebras, then $\mu(\mathfrak{a} \oplus \mathfrak{b}) \leq \mu(\mathfrak{a}) + \mu(\mathfrak{b})$.*

Proof. The composition of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and a faithful representation $\mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ is a faithful representation of $\mathfrak{h}$ of degree n. If φ and ψ are faithful representations of $\mathfrak{a}$ respectively $\mathfrak{b}$, then $\varphi \oplus \psi$ is a faithful representation of $\mathfrak{a} \oplus \mathfrak{b}$. □

Lemma 2.2. *Let $\mathfrak{g}$ be a Lie algebra with trivial center. Then $\mu(\mathfrak{g} \oplus \mathbb{C}) = \mu(\mathfrak{g})$.*

Proof. We have $\mu(\mathfrak{g}) \leq \mu(\mathfrak{g} \oplus \mathbb{C})$. Conversely, let $\rho\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ be a faithful representation of minimal dimension $n = \mu(\mathfrak{g})$. Suppose that there is an $x \in \mathfrak{g}$ such that $\rho(x) = I_n$ is the identity. Then, for all $y \in \mathfrak{g}$,

$$\rho([x, y]) = [\rho(x), \rho(y)] = [I_n, \rho(y)] = 0.$$

Since ρ is faithful, we have $x \in Z(\mathfrak{g}) = 0$, which is a contradiction. It follows that the identity I_n is not in $\rho(\mathfrak{g})$, and hence $\rho(\mathfrak{g}) \oplus \mathbb{C} \cdot I_n$ yields a faithful representation of $\mathfrak{g} \oplus \mathbb{C}$ of dimension n. □

2.1. Faithful representations of abelian Lie algebras.

If $\mathfrak{g}$ is abelian then there exists an explicit formula for $\mu(\mathfrak{g})$, which only depends on the dimension of $\mathfrak{g}$. If V is a d-dimensional vector space, then any faithful representation $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ turns $\varphi(\mathfrak{g})$ into an n-dimensional commutative subalgebra of the matrix algebra $M_d(K)$. Jacobson [8] proved:

Proposition 2.3. *Let M be a commutative subalgebra of $M_d(K)$ over an arbitrary field K. Then $\dim M \leq \lfloor d^2/4 \rfloor + 1$ and the bound is attained.*

For $K = \mathbb{C}$ the result was first proved by I. Schur. The proposition implies the following result, see [1]:

Proposition 2.4. *Let $\mathfrak{g}$ be an abelian Lie algebra of dimension $n > 1$ over a field K. Then $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1}\rceil$.*

For $n = 1$ we have $\mu(\mathfrak{g}) = 1$.

2.2. **Faithful representations of simple Lie algebras.** Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Then every non-trivial representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ is faithful, since $\ker(\rho)$ is an ideal in $\mathfrak{g}$. Hence any simple non-trivial $\mathfrak{g}$-module is faithful. Then $\mu(\mathfrak{g})$ is the smallest dimension of a non-trivial simple module. Note that such a module is not unique in general. For example, for type A_n we have two fundamental simple modules of minimal dimension: $L(\omega_1)$ and $L(\omega_n) \simeq L(\omega_1)^*$. With some exceptions the natural module is of minimal dimension. The dimensions are well known, see for example [4], [9]:

$\mathfrak{g}$	$\dim(\mathfrak{g})$	$\mu(\mathfrak{g})$
$A_n,\ n \geq 1$	$(n+1)^2 - 1$	$n+1$
B_2	10	4
$B_n,\ n \geq 3$	$2n^2 + n$	$2n+1$
$C_n,\ n \geq 3$	$2n^2 + n$	$2n$
$D_n,\ n \geq 4$	$2n^2 - n$	$2n$
E_6	78	27
E_7	133	56
E_8	248	248
F_4	52	26
G_2	14	7

2.3. **Faithful representations of semisimple Lie algebras.** The result for semisimple Lie algebras is as follows.

Proposition 2.5. *Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell$ be a semisimple Lie algebra and $\mathfrak{s}_i$ simple ideals of $\mathfrak{g}$. Then*

$$\mu(\mathfrak{g}) = \mu(\mathfrak{s}_1) + \ldots + \mu(\mathfrak{s}_\ell).$$

Assume first that $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ with Lie algebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$. Let (ρ_i, V_i) be a representation of $\mathfrak{s}_i$ for $i = 1, 2$. Then $\rho_i \circ \pi_i$ is a representation of $\mathfrak{g}$, where π_i is the projection from $\mathfrak{g}$ to $\mathfrak{s}_i$. Let $(x, y) \in \mathfrak{s}_1 \oplus \mathfrak{s}_2$. Recall that the tensor product $\rho_1 \circ \pi_1 \otimes \rho_2 \circ \pi_2$, which we denote also simply by $\rho_1 \otimes \rho_2$, for $v_i \in V_i$ is defined by

$$(\rho_1 \otimes \rho_2)(x, y)(v_1, v_2) = \rho_1(x)(v_1) \otimes v_2 + v_1 \otimes \rho_2(y)(v_2).$$

Lemma 2.6. *Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ with $\mathfrak{s}_1$ simple. Then* $\ker(\rho_1 \otimes \rho_2) = \ker(\rho_1) \oplus \ker(\rho_2)$.

Proof. Obviously $\ker(\rho_1) \oplus \ker(\rho_2) \subseteq \ker(\rho_1 \otimes \rho_2)$. Conversely choose an element $(x, y) \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 = \mathfrak{g}$ which lies in the kernel of $\rho_1 \otimes \rho_2$, i.e., $\rho_1(x)(v_1) \otimes v_2 + v_1 \otimes \rho_2(y)(v_2) = 0$ for all $v_i \in V_i$. Using explicit bases for V_1, V_2 and $V_1 \otimes V_2$ one easily obtains

$$\rho_1(x) = \alpha\, \mathrm{id}_{|V_1}, \quad \rho_2(y) = -\alpha\, \mathrm{id}_{|V_2}$$

with a constant $\alpha \in \mathbb{C}$. Since $\mathfrak{s}_1$ is simple and $\rho_1(x)$ is a traceless linear operator, it follows $\alpha = 0$ and $(x, y) \in \ker(\rho_1) \oplus \ker(\rho_2)$. □

We can extend the above easily to the case $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell$ with representations (ρ_i, V_i) for $i = 1, 2, \ldots, \ell$. We have the following result, see [7]:

Theorem 2.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell$ be a decomposition of $\mathfrak{g}$ into ideals of $\mathfrak{g}$. Then every irreducible representation (ρ, V) of $\mathfrak{g}$ is equivalent to the tensor* *product of ℓ irreducible representations $(\rho_1 \circ \pi_1, V_1), \dots, (\rho_\ell \circ \pi_\ell, V_\ell)$. Conversely, if (ρ_i, U_i) are arbitrary irreducible representations of $\mathfrak{s}_i$ for $i = 1, \dots \ell$, then*

$$(\rho_1 \circ \pi_1 \otimes \cdots \otimes \rho_\ell \circ \pi_\ell, U_1 \otimes \cdots \otimes U_\ell)$$

is an irreducible representation of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell$ be semisimple and ρ be a representation of $\mathfrak{g}$. Then, by Weyl's theorem, $\rho = \rho_1 \oplus \cdots \oplus \rho_n$, with irreducible representations ρ_i of $\mathfrak{g}$. Each of the ρ_i is the tensor product $\rho_i = \rho_{i,1} \otimes \cdots \otimes \rho_{i,\ell}$ where $\rho_{i,j}$ is an irreducible representation of $\mathfrak{s}_j$. This gives

$$\rho = \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{\ell} \rho_{i,j} \tag{1}$$

For the dimension of ρ we obtain

$$\dim \rho = \sum_{i=1}^{n} \prod_{j=1}^{\ell} \dim \rho_{i,j}. \tag{2}$$

To ρ we associate a matrix "of dimensions" $\Phi_\rho = (\dim \rho_{i,j})_{i,j} \in M_{n,\ell}(\mathbb{N})$.

Lemma 2.8. *Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell$ be a complex semisimple Lie algebra. A finite-dimensional representation ρ of $\mathfrak{g}$ is faithful iff the matrix Φ_ρ has no column consisting only of 1's.*

Proof. As before, write $\rho = \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{\ell} \rho_{i,j}$. By lemma 2.6 we have $\ker(\rho_i) = \bigoplus_{j=1}^{\ell} \ker(\rho_{i,j})$ for $i = 1, \dots n$. Furthermore $\ker(\rho) = \cap_{i=1}^{n} \ker(\rho_i)$. If there is a column consisting only of 1's, say colum j, then $\mathfrak{s}_j \subset \mathfrak{g}$ is contained in $\ker(\rho)$, so that ρ is not faithful. Conversely, suppose that there is no column with only 1's. Choose an element $z = \oplus_i z_i \in \ker(\rho)$. Fix a coordinate, say z_j. Because there is no 1-column there must be an i such that $\rho_{i,j}$ is faithful. By assumption we have $0 = \rho_i(z) = \otimes_k \rho_{i,k}(z_j)$. Again by lemma 2.6 we have $\rho_{i,j}(z_j) = 0$, and hence $z_j = 0$. This follows for all j, hence $z = 0$. $\square$

We will use the above lemma to prove proposition 2.5:

Proof. Let $\mathcal{M}$ be the space of all dimension matrices Φ_ρ for faithful representations ρ of a fixed semisimple Lie algebra $\mathfrak{g}$. According to (1) and (2) let $d_{ij} = \dim \rho_{i,j}$. The determination of $\mu(\mathfrak{g})$ is equivalent to minimizing the function

$$f \colon \mathcal{M} \mapsto \mathbb{N}, \quad \Phi_\rho \mapsto \sum_{i=1}^{n} \prod_{j=1}^{\ell} d_{ij}.$$

By lemma 2.8 no column of a matrix $\Phi_\rho \in \mathcal{M}$ contains only 1's. Denote by P the matrix in $\mathcal{M}$, which has diagonal elements $d_{ii} = \mu(\mathfrak{s}_i)$ and all other elements equal to 1. Then

$$f(P) = \sum_{i=1}^{\ell} \mu(\mathfrak{s}_i).$$

We will show that this is the minimal value of f, i.e., $\mu(\mathfrak{g}) = f(P)$. Suppose $D = (d_{ij}) \in \mathcal{M}$ is a matrix with minimal value $f(D)$. If there is a row, say row i, with more than one element unequal to 1, say d_{ij} and d_{ik}, then construct a new matrix C, by replacing the ith row

$$(d_1, \dots, d_{ij}, d_{j+1}, \dots, d_{ik}, d_{k+1}, \dots, d_\ell)$$

of D by two new rows

$$\begin{pmatrix} d_1, \dots, d_{ij}, d_{j+1}, \dots, 1, d_{k+1}, \dots, d_\ell \\ d_1, \dots, 1, d_{j+1}, \dots, d_{ik}, d_{k+1}, \dots, d_\ell \end{pmatrix}.$$

Note that the new matrix C really is in $\mathcal{M}$. It has one more row than D and satisfies $f(C) \leq f(D)$ since $a + b \leq ab$ for integers $a, b \geq 2$. By assumption $f(C) = f(D)$. After repeating this finitely many times we arrive at a matrix $B \in \mathcal{M}$ where every row has at most one element different from 1. In fact, a row $(1, \dots, 1)$ is impossible, because otherwise we remove this row and still obtain a matrix in $A \in \mathcal{M}$ with $f(A) < f(D)$, which is a contradiction. Thus every row of B has a unique entry different from 1. Similarly it is impossible that a column of B contains more than one of these unique entries. This implies that the number of rows and columns of B coincides. Now $f(B)$ is just the sum of these unique entries. Because the value $f(B)$ is minimal, the unique entries must correspond to the numbers $\mu(\mathfrak{s}_i)$. Hence $f(B) = \sum_{i=1}^{\ell} \mu(\mathfrak{s}_i) = f(P)$. □

2.4. **Faithful representations of reductive Lie algebras.** Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell \oplus \mathbb{C}^k$ be a reductive Lie algebra over $\mathbb{C}$. Denote by ℓ the *length* of $\mathfrak{g}$, i.e., the number of simple ideals $\mathfrak{s}_i$.

Lemma 2.9. *Let $\mathfrak{g}$ be a reductive Lie algebra of length ℓ and center $\mathbb{C}^k$. Then*

$$\mu(\mathfrak{g}) \leq \mu([\mathfrak{g}, \mathfrak{g}]) + \mu(\mathbb{C}^{k-\ell}).$$

Proof. If $\ell \leq k$ then $\mathfrak{g} = (\mathfrak{s}_1 \oplus \mathbb{C}) \oplus \cdots \oplus (\mathfrak{s}_\ell \oplus \mathbb{C}) \oplus \mathbb{C}^{k-\ell}$ and we obtain

$$\begin{aligned} \mu(\mathfrak{g}) &\leq \sum_{i=1}^{\ell} \mu(\mathfrak{s}_i \oplus \mathbb{C}) + \mu(\mathbb{C}^{k-\ell}) = \sum_{i=1}^{\ell} \mu(\mathfrak{s}_i) + \mu(\mathbb{C}^{k-\ell}) \\ &= \mu(\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell) + \mu(\mathbb{C}^{k-\ell}) = \mu([\mathfrak{g}, \mathfrak{g}]) + \mu(\mathbb{C}^{k-\ell}). \end{aligned}$$

by lemma 2.1, lemma 2.2 and proposition 2.5. If $k \leq \ell$ then $\mu(\mathbb{C}^{k-\ell}) = 0$ and $\mathfrak{g}$ can be embedded in $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell \oplus \mathbb{C}^\ell$. Then we have, using the above argument for $k = \ell$,

$$\mu(\mathfrak{g}) \leq \mu(\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell \oplus \mathbb{C}^\ell) \leq \mu([\mathfrak{g}, \mathfrak{g}]) + \mu(\mathbb{C}^{k-\ell}).$$

□

The statement of theorem 1.3 is that the inequality of the above lemma is in fact an equality.

Definition 2.10. Denote by $C_\varphi = \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid [A, \varphi(x)] = 0 \ \forall\, x \in \mathfrak{g}\}$ the *centralizer* of a Lie algebra representation $\varphi\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$.

Note that C_φ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

Definition 2.11. A pair of two Lie algebra representations $\varphi\colon \mathfrak{g}_1 \to \mathfrak{gl}_n(\mathbb{C})$ and $\psi\colon \mathfrak{g}_2 \to \mathfrak{gl}_n(\mathbb{C})$ is said to *commute*, if $[\varphi(x), \psi(y)] = 0 \ \forall\, x \in \mathfrak{g}_1, y \in \mathfrak{g}_2$.

Lemma 2.12. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras and suppose that $\mathfrak{g}_1$ has trivial center. There is a bijective correspondence between representations as follows:*

(1) *A faithful representation $\varphi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{gl}_n(\mathbb{C})$ induces a pair of commuting representations (φ_1, φ_2) by inclusion, given by $\varphi_j = \varphi \circ \iota_j : \mathfrak{g}_j \to \mathfrak{gl}_n(\mathbb{C})$ for $j = 1, 2$, where ι_j is the natural inclusion of $\mathfrak{g}_j$ into $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.*
(2) *Conversely a pair of commuting faithful representations $\varphi_j : \mathfrak{g}_j \to \mathfrak{gl}_n(\mathbb{C})$ induces a faithful representation $\varphi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{gl}_n(\mathbb{C})$ by $\varphi = \varphi_1 \circ \pi_1 + \varphi_2 \circ \pi_2$, where π_j is the natural projection of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ onto $\mathfrak{g}_j$.*

Proof. It is clear that φ_1, φ_2 are faithful representations. We have

$$[\varphi_1(x), \varphi_2(y)] = [\varphi(x,0), \varphi(0,y)] = \varphi([(x,0),(0,y)]) = 0.$$

This shows (1). For (2), note that φ is a representation. Let $(x,y) \in \ker(\varphi)$. This means $\varphi_1(x) + \varphi_2(y) = 0$, so that

$$\varphi_1(x) = -\varphi_2(y) \in \varphi_1(\mathfrak{g}_1) \cap \varphi_2(\mathfrak{g}_2) \subseteq Z(\varphi_1(\mathfrak{g}_1)) = Z(\mathfrak{g}_1) = 0.$$

Since $\ker(\varphi_1) = \ker(\varphi_2) = 0$ we have $(x,y) = (0,0)$, and φ is faithful. □

Fix a semisimple Lie algebra $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \mathfrak{s}_\ell$ of length ℓ, and an integer $n \geq \mu(\mathfrak{g})$. We will construct a certain faithful representation $\varphi \colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ for each $n \geq \mu(\mathfrak{g})$. Let $\sigma_i \colon \mathfrak{s}_i \to \mathfrak{gl}_{\mu(\mathfrak{s}_i)}(\mathbb{C})$ be faithful representations of minimal dimension $\mu(\mathfrak{s}_i)$ for $i = 1, \dots \ell$. Denote by φ_0 the one-dimensional trivial representation of $\mathfrak{g}$, and let $m\varphi_0 = \varphi_0 \oplus \cdots \oplus \varphi_0$.

Definition 2.13. Let $\mathfrak{g}$ be as above and $m = n - \mu(\mathfrak{g})$. Define a representation $\sigma \colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ by

$$\sigma = m\varphi_0 \oplus \sigma_1 \oplus \cdots \oplus \sigma_\ell.$$

Then σ is called a *standard block representation of degree* n for $\mathfrak{g}$.

We are interested in determining the centralizer of a faithful representation of $\mathfrak{g}$. We have the following result.

Proposition 2.14. *Let $\mathfrak{g}$ be as above and fix an integer $n \geq \mu(\mathfrak{g})$. The centralizer of any faithful representation $\varphi \colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ can be embedded into the centralizer of a standard block representation of degree n for $\mathfrak{g}$.*

The proof is split up into three lemmas. Let $\varphi \colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ be a faithful representation. Since centralizers of equivalent representations are isomorphic we may assume, by Weyl's theorem, that $\varphi = \oplus_{j=0}^k m_j \varphi_j$ for irreducible, inequivalent representations φ_j of $\mathfrak{g}$, and some $m_j \in \mathbb{N}$. Again let φ_0 denote the 1-dimensional trivial representation.
The following lemma is well known, and follows easily from Schur's lemma.

Lemma 2.15. *Let $\varphi = \oplus_{j=0}^k m_j \varphi_j$ as above. Then, as Lie algebras,*

$$C_\varphi \cong \bigoplus_{j=0}^k \mathfrak{gl}_{m_j}(\mathbb{C}).$$

Corollary 2.16. *The centralizer of a standard block representation σ of degree $n = m + \mu(\mathfrak{g})$ is isomorphic to $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathbb{C}^\ell$.*

Denote by d_j the degree of the representation φ_j. Now associate to $\varphi = \oplus_{j=0}^k m_j \varphi_j$ the representation

$$\tilde{\varphi} = m\varphi_0 \oplus \left(\bigoplus_{j=1}^k \varphi_j \right), \tag{3}$$

so that φ and $\tilde{\varphi}$ have the same degree, equal to n. This means, that $m = m_0 + \sum_{j=1}^k (m_j - 1) d_j$. Note that $\tilde{\varphi}$ is again faithful by lemma 2.8.

Lemma 2.17. *Let $\varphi = \oplus_{j=0}^k m_j \varphi_j$ as above. Then the centralizer C_φ can be embedded into the centralizer $C_{\tilde{\varphi}}$.*

Proof. By permuting the summands in φ we may assume that $m_1 = \ldots = m_r = 1$ for some $0 \le r \le k$, and $m_j \ge 2$ for $j > r$. By lemma 2.15, and using $\mathfrak{gl}_1(\mathbb{C}) \cong \mathbb{C}$, the centralizer C_φ is isomorphic to $\mathfrak{gl}_{m_0}(\mathbb{C}) \oplus \mathbb{C}^r \oplus \left(\oplus_{j=r+1}^k \mathfrak{gl}_{m_j}(\mathbb{C})\right)$, which can be embedded into $\mathfrak{gl}_p(\mathbb{C}) \oplus \mathbb{C}^r$, where $p = m_0 + \sum_{j=r+1}^k m_j$. On the other hand, $C_{\tilde\varphi} \cong \mathfrak{gl}_m(\mathbb{C}) \oplus \mathbb{C}^k$, where $m = m_0 + \sum_{j=r+1}^k (m_j - 1) d_j$. Certainly $\mathbb{C}^r$ can be embedded into $\mathbb{C}^k$ since $r \le k$. To prove the claim of the lemma it remains to show that $p \le m$. This is true because of $m_j, d_j \ge 2$ for $j \ge r+1$, so that $d_j \ge \frac{m_j}{m_j - 1}$. ☐

Lemma 2.18. *Let φ and $\tilde\varphi$ be as above. Then $C_{\tilde\varphi}$ can be embedded into the centralizer of a standard block representation of degree n.*

Proof. Consider the decomposition (3) of $\tilde\varphi$. Then $\rho = \oplus_{j=1}^k \varphi_j$ is a faithful representation of $\mathfrak{g}$. We claim that we can choose r representations φ_j, denoted again by $\varphi_1, \ldots, \varphi_r$, such that their direct sum $\rho' = \oplus_{j=1}^r \varphi_j$ is still a faithful representation of $\mathfrak{g}$, where r is at most ℓ, the length of $\mathfrak{g}$. Since ρ is faithful its dimension matrix has no columns consisting only of 1's. Hence for every column j of our ℓ columns we may choose a row i such that the entry (i,j) is different from 1. To every such row i corresponds a representation φ_i. Then we have chosen ℓ rows, but not necessarily distinct ones. Pick out the φ_i for the distinct rows. Their direct sum is a faithful representation of $\mathfrak{g}$, since its dimension matrix again has no columns consisting only of 1's. Now rewrite $\tilde\varphi$, using ρ', as $\tilde\varphi = m\varphi_0 \oplus \rho' \oplus \left(\oplus_{j=r+1}^k \varphi_j\right)$. Comparing dimensions, we have $n = m + \dim(\rho') + \sum_{j=r+1}^k d_j$. Since ρ' is a faithful representation of $\mathfrak{g}$ we have $\dim(\rho') \ge \mu(\mathfrak{g})$. Using $\sum_{j=r+1}^k d_j \ge k - r$ we obtain

$$\begin{aligned} \mu(\mathfrak{g}) \le \dim(\rho') &= n - m - \sum_{j=r+1}^k d_j \\ &\le n - m - k + r. \end{aligned}$$

Note that $r \le k$, so that $\mathbb{C}^{k-r}$ makes sense. By lemma 2.15 it follows, that

$$\begin{aligned} C_{\tilde\varphi} &\cong \mathfrak{gl}_m(\mathbb{C}) \oplus \mathbb{C}^{k-r} \oplus \mathbb{C}^r \\ &\hookrightarrow \mathfrak{gl}_{m+k-r}(\mathbb{C}) \oplus \mathbb{C}^r \\ &\hookrightarrow \mathfrak{gl}_{n-\mu(\mathfrak{g})}(\mathbb{C}) \oplus \mathbb{C}^\ell \\ &\cong C_\sigma. \end{aligned}$$

☐

Now we can prove proposition 2.14: we have $C_\varphi \hookrightarrow C_{\tilde\varphi} \hookrightarrow C_\sigma$ by the two preceding lemmas.

Corollary 2.19. *Let $\mathfrak{g}$ be a semisimple Lie algebra as above, and $\mathfrak{a}$ be a Lie algebra. Then $\mathfrak{g} \oplus \mathfrak{a}$ can be embedded into $\mathfrak{gl}_n(\mathbb{C})$ if and only if $\mathfrak{a}$ can be embedded into $\mathfrak{gl}_{n-\mu(\mathfrak{g})}(\mathbb{C}) \oplus \mathbb{C}^\ell$.*

Proof. Suppose $\mathfrak{a}$ can be embedded into $\mathfrak{gl}_{n-\mu(\mathfrak{g})}(\mathbb{C}) \oplus \mathbb{C}^\ell \cong C_\sigma$. Then we have a pair of commuting embeddings $\sigma \colon \mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$ and $\tau \colon \mathfrak{a} \hookrightarrow C_\sigma \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$. Lemma 2.12, (2) gives an embedding $\mathfrak{g} \oplus \mathfrak{a} \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$. The converse direction follows from part (1) of lemma 2.12. ☐

Now we turn to the proof of theorem 1.3. Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_\ell \oplus \mathbb{C}^k$ be a complex reductive Lie algebra. We write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{a} = \mathbb{C}^k$. Given any embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$, the above corollary implies that there is an embedding $\mathfrak{a} \hookrightarrow \mathfrak{gl}_{n-\mu(\mathfrak{g})} \oplus \mathbb{C}^\ell$. Denote by $\alpha(\mathfrak{g})$ the maximal dimension of a commutative subalgebra of $\mathfrak{g}$. We have $\alpha(\mathfrak{a}) = k$ and

$$\alpha(\mathfrak{gl}_m(\mathbb{C}) \oplus \mathbb{C}^\ell) = \alpha(\mathfrak{gl}_m(\mathbb{C})) + \alpha(\mathbb{C}^\ell) = \lfloor m^2/4 \rfloor + 1 + \ell,$$

since α is additive, see [9]. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{b}$ then $\alpha(\mathfrak{a}) \leq \alpha(\mathfrak{b})$. It follows that

$$k \leq \lfloor (n-\mu(\mathfrak{s}))^2/4 \rfloor + 1 + \ell.$$

This implies $n \geq \lceil 2\sqrt{k-\ell-1} \rceil + \mu(\mathfrak{s}) = \mu(\mathbb{C}^{k-\ell}) + \mu(\mathfrak{s})$. Together with lemma 2.9 the formula of theorem 1.3 follows. □

Finally, the following result can be derived from the above corollary in a similar way.

Proposition 2.20. *Let $\mathfrak{s}$ be a semisimple Lie algebra and $\mathfrak{g}$ be a perfect Lie algebra, i.e., satisfying $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Then we have $\mu(\mathfrak{s} \oplus \mathfrak{g}) = \mu(\mathfrak{s}) + \mu(\mathfrak{g})$.*

Remark 2.21. Theorem 1.3 can be used to classify all reductive subalgebras, up to isomorphism, of $\mathfrak{gl}_n(\mathbb{C})$. As an example, for $n = 4$ we obtain (note that $\mu(\mathbb{C}^5) = 4$)

$$\begin{aligned} &\mathbb{C}^i, i = 1, \ldots, 5 \\ &A_k \oplus \mathbb{C}^i, k = 1,2,3,\ 0 \leq i \leq 4-k \\ &A_1 \oplus A_1 \oplus \mathbb{C}^i, i = 0,1,2 \\ &C_2 \oplus \mathbb{C}^i, i = 0,1. \end{aligned}$$

2.5. **Maximal reductive subalgebras.** If we have a faithful representation $\varphi\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ then $\varphi(\mathfrak{g})$ lies in a maximal reductive subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, or $\mathfrak{g} \cong \mathfrak{gl}_n(\mathbb{C})$. There is a complete classification of all maximal reductive Lie subalgebras in semisimple Lie algebras, due to Malcev [9], Dynkin [5], [6] and Borel [3]. Hence one might wonder if one can use this classification to give another proof of theorem 1.3. However it turns out that this may be quite complicated in general. In some cases, we can give a nice, short proof. Consider the following easy example.

Example 2.22. *We have $\mu(A_1 \oplus \mathbb{C}^4) = 5$.*

In fact, it is obvious that $\mathfrak{g} = A_1 \oplus \mathbb{C}^4$ has a faithful representation of dimension 5: the direct sum of the natural representations of $A_1 \oplus \mathbb{C} = \mathfrak{gl}_2(\mathbb{C})$ and $\mathbb{C}^3$. It remains to show that $\mathfrak{g}$ cannot be faithfully embedded into $\mathfrak{gl}_4(\mathbb{C})$. Suppose it can, i.e., $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_4(\mathbb{C})$. Denote by $\pi\colon \mathfrak{gl}_4(\mathbb{C}) \to \mathfrak{gl}_4(\mathbb{C})/Z$ the natural projection, where Z is the center of $\mathfrak{gl}_4(\mathbb{C})$ with $\dim Z = 1$. Then we claim that $\pi(\mathfrak{g})$ is a reductive Lie subalgebra of A_3, which is either isomorphic to $\mathfrak{g}$, or to $A_1 \oplus \mathbb{C}^3$: let $z \in Z \cap \mathfrak{g}$. Then $[x, z] = 0$ for all $x \in \mathfrak{g}$, i.e., $Z \cap \mathfrak{g} \subseteq Z(\mathfrak{g})$ and $\dim(Z \cap \mathfrak{g}) \leq 1$. It follows that $\pi(\mathfrak{g}) \cong A_1 \oplus Z(\mathfrak{g})/(Z \cap \mathfrak{g})$. If $Z \cap \mathfrak{g} = 0$, then $\pi(\mathfrak{g}) \cong \mathfrak{g}$, otherwise $\dim(Z \cap \mathfrak{g}) = 1$, so that $\pi(\mathfrak{g}) \cong A_1 \oplus \mathbb{C}^3$. So $A_1 \oplus \mathbb{C}^4$ or $A_1 \oplus \mathbb{C}^3$ is a reductive subalgebra of A_3, hence lies in a maximal one. But these are exactly the following ones:

$$C_2, A_2 \oplus \mathbb{C}, A_1 \oplus A_1, A_1 \oplus A_1 \oplus \mathbb{C}.$$

Here $A_1 \oplus A_1$ is not contained in the last one. Consider again $\alpha(\mathfrak{g})$, the maximal dimension of a commutative subalgebra of $\mathfrak{g}$. Malcev computed $\alpha(\mathfrak{g})$ for reductive Lie algebras. For a new proof of this result see the nice article of Suter [10]. We have $\alpha(A_1 \oplus \mathbb{C}^3) = 4$ but $\alpha(C_2) = \alpha(A_2 \oplus \mathbb{C}) = \alpha(A_1 \oplus A_1 \oplus \mathbb{C}) = 3$ and $\alpha(A_1 \oplus A_1) = 2$. If $\mathfrak{h}_1 \subset \mathfrak{h}_2$ for reductive Lie algebras then $\alpha(\mathfrak{h}_1) \leq \alpha(\mathfrak{h}_2)$. It follows that $A_1 \oplus \mathbb{C}^3$ and hence also $\mathfrak{g}$ cannot be a subalgebra of one of the maximal reductive subalgebras of A_3. This is a contradiction. □

In this way one can also prove more generally that

$$\mu(A_1 \oplus \mathbb{C}^k) = 2 + \lceil 2\sqrt{k-2} \rceil, \quad k \geq 3.$$

The following example, however, shows that this method of using Dynkin's results will become very complicated in general.

Example 2.23. *Show that $\mu(A_1 \oplus C_3 \oplus \mathbb{C}^6) = 12$.*

Assume that $\mathfrak{g} = A_1 \oplus C_3 \oplus \mathbb{C}^6$ could be embedded into $\mathfrak{gl}_{11}(\mathbb{C}) = A_{10} \oplus \mathbb{C}$. Then we may assume that $A_1 \oplus C_3 \oplus \mathbb{C}^5$ is a reductive subalgebra of A_{10}. Passing to maximal reductive subalgebras we may assume that $A_1 \oplus C_3 \oplus \mathbb{C}^5$ is a reductive subalgebra of one of the following algebras:

$$A_9 \oplus \mathbb{C}, A_1 \oplus A_8 \oplus \mathbb{C}, A_2 \oplus A_7 \oplus \mathbb{C}, A_3 \oplus A_6 \oplus \mathbb{C}, A_4 \oplus A_5 \oplus \mathbb{C}, B_5.$$

The invariant α of these Lie algebras is given by $26, 22, 19, 17, 16, 11$ respectively, whereas $\alpha(A_1 \oplus C_3 \oplus \mathbb{C}^5) = 12$. Unfortunately, the only possibility which can be excluded immediately then is B_5. Then we have to treat all the other cases, which ramify to even more cases in the next step, repeating this kind of argument. Moreover, the maximal reductive subalgebras of other types, different from A_n, play a role.

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