Degenerations and Contractions of Lie algebras and Algebraic groups

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1. DEGENERATIONS, CONTRACTIONS AND DEFORMATIONS OF LIE ALGEBRAS

Degenerations, contractions and deformations play an important role in mathematics and physics. Unfortunately there are many different definitions and special cases of these notions. We try to give a general definition which unifies these notions and shows the connections among them.

1.1. **Degenerations via orbit closures.** Let $\mathfrak{g}$ be an n-dimensional vector space over a field k. Denote by $\mathcal{L}_n(k)$ the *variety of Lie algebra laws*. The general linear group $GL_n(k)$ acts by base changes on $\mathfrak{g}$, and hence on $\mathcal{L}_n(k)$. One denotes by $O(\mu)$ the orbit of μ under the action of $GL_n(k)$, and by $\overline{O(\mu)}$ the closure of the orbit with respect to the Zariski topology.

Definition 1.1. *Let $\lambda, \mu \in \mathcal{L}_n(k)$ be two Lie algebra laws. We say that λ degenerates to μ, if $\mu \in \overline{O(\lambda)}$. This is denoted by $\lambda \to_{\deg} \mu$.*

Degeneration defines an order relation on the orbit space of n-dimensional Lie algebra laws by $O(\mu) \le O(\lambda) \iff \mu \in \overline{O(\lambda)}$. For example, every law $\lambda \in \mathcal{L}_n(k)$ degenerates to the abelian law $\lambda_0 \in \mathcal{L}_n(k)$. In general, it is quite difficult to see whether there exists a degeneration $\lambda \to_{\deg} \mu$ between two Lie algebra laws $\lambda, \mu \in \mathcal{L}_n(k)$. It is also interesting to investigate the varieties $\mathcal{L}_n(k)$ and the orbit closures over $\mathbb{R}$ or $\mathbb{C}$ in low dimensions, see [1], [2], [3], [7],[8].

1.2. **Degenerations, contractions and deformations.** Let k be a field and A be a discrete valuation ring (DVR). Grunewald and O'Halloran [5] proved a result which shows that the definition of degeneration generalizes as follows:

Definition 1.2. *Let $\mathfrak{g}$ be a Lie algebra over k and A a discrete valuation k-algebra with residue field k. Then a Lie algebra $\mathfrak{a}$ over A is a degeneration of $\mathfrak{g}$ over A, if there exists a finite extension L/K of the quotient field K of A, such that $\mathfrak{a} \otimes_A L \cong \mathfrak{g} \otimes_k L$. The Lie algebra $\mathfrak{g}_0 := \mathfrak{a} \otimes_A k$ is called the limit algebra of the degeneration.*

Remark 1.3. The limit algebra here is also a degeneration in the sense of orbit closure. In this formulation we see that there is a close relationship between deformations and degenerations: If $\mathfrak{a}$ is a degeneration of $\mathfrak{g}$, and $\mathfrak{g}_0$ is isomorphic to the limit algebra of $\mathfrak{a}$ via $\varphi\colon \mathfrak{g}_0 \to \mathfrak{a} \otimes_A k$, then $(\mathfrak{a}, \varphi)$ is a deformation of $\mathfrak{g}_0$.

Definition 1.4. *Let $\mu_1 \in \mathcal{L}_n(k)$ and $\mathfrak{g} = (V, \mu_1)$. Let A be a discrete valuation k-algebra with residue field k and quotient field K. Let $\varphi \in \mathrm{End}(V_A) \cap GL(V_K)$. If $\mu = \varphi \cdot \mu_1$ is in $\mathcal{L}_n(A)$, and hence μ defines a Lie algebra $\mathfrak{a} = (V_A, \mu)$ over A, then $\mathfrak{a}$ is called a contraction of $\mathfrak{g}$ via φ. The Lie algebra $\mathfrak{g}_0 := \mathfrak{a} \otimes_A k$ is called the limit algebra of the contraction.*

Lemma 1.5. *Every contraction of $\mathfrak{g}$ is also a degeneration of $\mathfrak{g}$.*

Remark 1.6. The converse of this lemma in general is not clear. For $\mathbb{R}$ and $\mathbb{C}$ however every degeneration is isomorphic to a contraction (see [9]).

Definition 1.7. *Let $\mathfrak{a}$ be a degeneration of $\mathfrak{g}$ and $\varphi\colon \mathfrak{a}\otimes_A K \to \mathfrak{g}\otimes_k K$ be an isomorphism. Then the pair $(\mathfrak{a},\varphi)$ is called a generalized contraction of $\mathfrak{g}$ with φ.*

2. Degenerations and Contractions of algebraic groups

We want to transfer the notions to algebraic groups. Note that in the case of Lie algebras the underlying space does not change (under deformation, degeneration or contraction). This will be different for algebraic groups, where the underlying variety will also change. The main results here are due to C. Daboul, see [4].

2.1. **Affine group schemes.** Let A be a ring. The *spectrum* of A is the pair $(\mathrm{Spec}(A),\mathcal{O})$ consisting of the topological space $\mathrm{Spec}(A)$ together with its structure sheaf $\mathcal{O}$. If $\mathfrak{p}$ is a point in $\mathrm{Spec}(A)$, then the stalk $\mathcal{O}_\mathfrak{p}$ at $\mathfrak{p}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_\mathfrak{p}$. Consequently, $\mathrm{Spec}(A)$ is a locally ringed space. Every sheaf of rings of this form is called an affine scheme:

Definition 2.1. *A locally ringed space $(X,\mathcal{O}_X)$ is called an affine scheme, if it is isomorphic to $\mathrm{Spec}(R)$ of some ring R, i.e., if $(X,\mathcal{O}_X)\cong(\mathrm{Spec}(R),\mathcal{O}_{\mathrm{Spec}(R)})$.*

An affine scheme X is called an *A-scheme*, if its coordinate ring is an A-algebra. For $\mathfrak{p}=(0)$ we call the fibre $X_\mathfrak{p}$ the *generic fibre of* X and denote it by X_K. The fibre $X_\mathfrak{m}$ is called the *special fibre* and is denoted by X_k. There is the notion of a *smooth affine A-scheme*, see [6]. Note that algebraic groups over a field k of characteristic zero are smooth affine k-schemes. If we have a smooth affine group scheme $\mathcal{G}$ over A then we can define its Lie algebra $\mathrm{Lie}(\mathcal{G})$ via $\mathcal{G}$-invariant derivations.

2.2. **Degenerations, contractions and liftings.** An affine group scheme over A can be considered as a family of affine group schemes over the residue fields k_t, where $t\in\mathrm{Spec}(A)$. Its fibres $\mathcal{G}_t$ are in fact affine group schemes with coordinate rings $k_t[\mathcal{G}_t]$. Hence we have $\mathcal{G}_t=\mathcal{G}_{k_t}$, and we use both notations. In particular we write $\mathcal{G}_K$ for the generic fibre of $\mathcal{G}$, where K is the quotient field of A.

Definition 2.2. *Let A be a discrete valuation k-algebra with residue field k and quotient field K. A degeneration of an affine algebraic group G over k is a smooth affine group scheme $\mathcal{G}$ over A, such that there is a field extension L/K of finite degree, such that G_L is isomorphic to $\mathcal{G}_L$.*

The special fiber $\mathcal{G}_k$ then is called the *limit group* of the degeneration.

Definition 2.3. *Let A be a discrete valuation k-algebra with residue field k and quotient field K. A generalized contraction of an affine algebraic group G over k is a pair $(\mathcal{G},\Phi)$ consisting of a degeneration $\mathcal{G}$ of G and an isomorphism of K-group schemes $\Phi\colon\mathcal{G}_K\to G_K$. The pair $(\mathcal{G},\Phi)$ is called a contraction, if in addition $\Phi^{\#}(A[G])\subseteq A[\mathcal{G}]$, where $\Phi^{\#}$ denotes the dual map.*

Let G be an affine algebraic group. If $(\mathcal{G}, \Phi)$ is a contraction of G then $(\mathfrak{a}, \varphi) = (\mathrm{Lie}(\mathcal{G}), d\Phi)$ is a contraction of $\mathfrak{g} = \mathrm{Lie}(G)$. The same is true for a generalized contraction. One can show that every generalized contraction of an affine algebraic group is isomorphic to a contraction. Hence each degeneration of a Lie algebra which corresponds to a degeneration of an affine algebraic group is isomorphic to a contraction.

Definition 2.4. *Let $\mathfrak{a}$ be a deformation or a degeneration of $\mathfrak{g}$ over A. Then a smooth A-group scheme $\mathcal{G}$ with $\mathrm{Lie}(\mathcal{G}) \cong \mathfrak{a}$ is called a lifting of $\mathfrak{a}$.*

If G is an affine algebraic group over k with Lie algebra $\mathfrak{g}$, and if $\mathfrak{a}$ is a degeneration of $\mathfrak{g}$ over a discrete valuation k-algebra A, then we would like to find a lifting of $\mathfrak{a}$ with generic fibre G.

Proposition 2.5. *Let $\mathfrak{a}$ be a degeneration of $\mathfrak{g}$. Suppose that there exists a conserved representation of $\mathfrak{a}$, which is the derivative of a faithful representation of G. Then we can construct a lifting of the degeneration.*

A main ingredient in the proof is the closure of representations in the sense of schemes. This result applies to many degenerations: if, for example, the center of the limit algebra is trivial, then the adjoint representation is conserved and the condition is satisfied. On the other hand one can use the Neron-Blowup for schemes to obtain the following result:

Proposition 2.6. *All Inönü-Wigner contractions of Lie algebras can be lifted to the group level.*

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