THE AUSLANDER CONJECTURE FOR NIL-AFFINE CRYSTALLOGRAPHIC GROUPS

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Abstract. We study subgroups Γ in $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ acting properly discontinuously and cocompactly on N. Here N is a simply connected, connected real nilpotent Lie group of finite dimension n. This situation is a natural generalization of the so-called affine crystallographic groups. We prove that for all dimensions $1 \leq n \leq 5$ the generalized Auslander conjecture holds, i.e., that such subgroups are virtually polycyclic.

1. Introduction

A classical crystallographic group is a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. Such groups act properly discontinuously and cocompactly on $\mathbb{R}^n$. The structure of such groups is well known by the three Bieberbach theorems ([8], [16]). In fact, all these groups are finitely generated virtually abelian.

As a generalization of this concept, one also studies affine crystallographic groups. These are subgroups of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ acting crystallographically (by which we will always mean properly discontinuously and cocompactly) on $\mathbb{R}^n$. The structure of these affine crystallographic groups is not at all as well known as in the case of the classical crystallographic groups and two main questions have played a major role in the study of these groups:

- In 1977 Milnor ([15]) asked whether any torsionfree polycyclic-by-finite group could be realized as an affine crystallographic group. And conversely
- In 1964 Auslander conjectured ([4]) that any affine crystallographic group is virtually solvable (and hence polycyclic-by-finite).

It follows that a positive answer to both questions implies a complete understanding of the affine crystallographic groups. However, it has been shown that not all torsionfree polycyclic-by-finite groups do occur as an affine crystallographic group, thus answering negatively Milnor's question ([5], [6], [7]).

On the other hand, Auslander's conjecture is still open and is only known to be true in dimensions $n \leq 6$ ([2], see also [1] for a survey on the Auslander conjecture).

The negative answer to Milnor's question, suggests to consider more general classes of crystallographic groups (e.g. polynomial crystallographic groups as in [12]). In this paper we are considering crystallographic subgroups in $\mathrm{Aff}(N)$ for a simply connected, connected nilpotent Lie group N. Here N is diffeomorphic to some $\mathbb{R}^n$ and $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$

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acts on N via

$$(n,\alpha)\cdot m = n\alpha(m) \quad \forall\,(n,\alpha)\in \mathrm{Aff}(N),\ m\in N. \tag{1}$$

A crystallographic subgroup of $\mathrm{Aff}(N)$ is a subgroup acting crystallographically on N. We will call such a group a NIL-affine crystallographic group.
The notation $\mathrm{Aff}(N)$ makes sense, since this group is really the group of connection preserving diffeomorphisms of N for any left invariant affine connection on N ([13]). In this sense, studying the NIL-affine crystallographic groups is really a very natural generalization of the affine crystallographic groups. Moreover, the analogue of Milnor's question does hold in this case:

Theorem 1.1. ([10]) *Let Γ be a torsionfree polycyclic-by-finite group. Then there exists a connected and simply connected nilpotent Lie group N and an embedding $\rho:\Gamma\to\mathrm{Aff}(N)$, such that $\rho(\Gamma)$ is a crystallographic subgroup of $\mathrm{Aff}(N)$.*

Conversely, just as in the case of affine crystallographic groups, it is now very natural to ask whether every NIL-affine crystallographic group is virtually solvable:

The generalized Auslander conjecture 1.2. *Let N be a connected and simply connected nilpotent Lie group and let $\Gamma\subseteq\mathrm{Aff}(N)$ be a group acting crystallographically on N. Then Γ is virtually polycyclic.*

We expect the answer to be positive since the question is closely related to the original Auslander problem. Note that a positive answer to this generalized Auslander problem would imply a complete algebraic description of the class of NIL-affine crystallographic groups, or stated otherwise, would provide a complete geometric description of the class of polycyclic-by-finite groups.
We will prove this conjecture for all N with $\dim N\le 5$. Moreover, in the next section we will show how several concepts of classical affine geometry on $\mathbb{R}^n$ can be translated to the nilpotent case, giving more indications of a close relation between the generalized Auslander conjecture and the original one and thus providing even more evidence for a positive answer.

2. Subgroups of Aff(N) not acting properly discontinuously

In this section we generalize the criterium of [3] for the failure of proper discontinuity for certain subgroups in $\mathrm{Aff}(\mathbb{R}^n)$ to the case of subgroups in $\mathrm{Aff}(N)$. This means that we follow on one hand very closely the construction given in [3], but on the other hand really have to develop some basics for affine geometry in a nilpotent Lie group N. Here we encounter new problems, because N will not be commutative in general.

In establishing this criterium, we will exploit the structure of the *linear* parts of the affine motions involved. Let us make this more precise in what follows.
Given $\alpha\in\mathrm{Aut}(N)$, let α_* denote the induced automorphism of the Lie algebra $\mathfrak{n}$ of N. We have the identification $\mathbb{R}^n\equiv\mathfrak{n}\xrightarrow{\exp}N$, $\exp\circ\alpha_*=\alpha\circ\exp$ and $\mathrm{Aut}(N)\cong\mathrm{Aut}(\mathfrak{n})$. Denote by

$$\ell\colon N\rtimes\mathrm{Aut}(N)\to\mathrm{Aut}(\mathfrak{n}):\quad (n,\alpha)\mapsto\alpha_*$$

the projection of $g = (n, \alpha)$ to its linear part $\ell(g) = \alpha_*$. Then we have for any $g \in \mathrm{Aff}(N)$ the vector space decomposition of $\mathfrak{n}$ into a direct sum of $\ell(g)$-invariant subspaces

$$\mathfrak{n} = \mathfrak{n}^-(g) \oplus \mathfrak{n}^0(g) \oplus \mathfrak{n}^+(g) \tag{2}$$

where the spaces $\mathfrak{n}^-(g)$, $\mathfrak{n}^0(g)$, $\mathfrak{n}^+(g)$ are determined by the following conditions. Their sum is $\mathfrak{n}$ and all eigenvalues λ of the restriction $\ell(g)_{|\mathfrak{n}^-(g)}$ satisfy $|\lambda| < 1$, all eigenvalues of $\ell(g)_{|\mathfrak{n}^0(g)}$ satisfy $|\lambda| = 1$ and all eigenvalues of $\ell(g)_{|\mathfrak{n}^+(g)}$ satisfy $|\lambda| > 1$. The decomposition is not only a vector space decomposition but also a decomposition as Lie algebras. In fact, we will need that the subspaces $\mathfrak{n}^-(g)$, $\mathfrak{n}^0(g)$, $\mathfrak{n}^+(g)$ and the two direct sums

$$\mathfrak{d}^-(g) = \mathfrak{n}^-(g) \oplus \mathfrak{n}^0(g) \tag{3}$$

$$\mathfrak{d}^+(g) = \mathfrak{n}^+(g) \oplus \mathfrak{n}^0(g) \tag{4}$$

are Lie subalgebras of $\mathfrak{n}$. However, we will prove this in Lemma 2.1. So we can also fix notations for the corresponding Lie subgroups of N:

$$\begin{aligned} N^-(g) &= \exp(\mathfrak{n}^-(g)) \\ N^0(g) &= \exp(\mathfrak{n}^0(g)) \\ N^+(g) &= \exp(\mathfrak{n}^+(g)) \\ D^-(g) &= \exp(\mathfrak{d}^-(g)) \\ D^+(g) &= \exp(\mathfrak{d}^+(g)). \end{aligned}$$

Lemma 2.1. *Let $\mathfrak{g}$ be a n-dimensional real Lie algebra and $\alpha \in \mathrm{Aut}(\mathfrak{g})$. Then there exists a basis $(v_1, \ldots, v_n)$ of $\mathfrak{g}$ such that α has the following block form with respect to this basis*

$$\alpha = \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & B & 0 \\ \hline 0 & 0 & C \end{array}\right)$$

where $A \in M_k(\mathbb{R})$ such that all eigenvalues λ satisfy $|\lambda| < 1$, $B \in M_l(\mathbb{R})$ such that all eigenvalues λ satisfy $|\lambda| = 1$, and $C \in M_m(\mathbb{R})$ such that all eigenvalues λ satisfy $|\lambda| > 1$. If

$$\begin{aligned} \mathfrak{g}^-(\alpha) &= \mathrm{span}\{v_1, \ldots, v_k\} \\ \mathfrak{g}^0(\alpha) &= \mathrm{span}\{v_{k+1}, \ldots, v_{k+l}\} \\ \mathfrak{g}^+(\alpha) &= \mathrm{span}\{v_{k+l+1}, \ldots, v_{k+l+m}\} \end{aligned}$$

with $k+l+m = n$ then the subspaces $\mathfrak{g}^-(\alpha)$, $\mathfrak{g}^0(\alpha)$, $\mathfrak{g}^+(\alpha)$ and $\mathfrak{g}^-(\alpha) \oplus \mathfrak{g}^0(\alpha)$, $\mathfrak{g}^+(\alpha) \oplus \mathfrak{g}^0(\alpha)$ are Lie subalgebras of $\mathfrak{g}$.
Moreover, $\mathfrak{g}^+(\alpha)$ (resp. $\mathfrak{g}^-(\alpha)$) is an ideal of $\mathfrak{g}^+(\alpha) \oplus \mathfrak{g}^0(\alpha)$ (resp. $\mathfrak{g}^-(\alpha) \oplus \mathfrak{g}^0(\alpha)$).

Proof. The first assertion follows easily by using the real canonical Jordan form for α. Now let $\alpha = \alpha_s \alpha_u$ be the multiplicative Jordan decomposition of α. Here α_s is a semisimple automorphism, α_u is a unipotent automorphism and $\alpha_s \alpha_u = \alpha_u \alpha_s$. We have $\alpha_s, \alpha_u \in$

$\mathrm{Aut}(\mathfrak{g})$ since $\mathrm{Aut}(\mathfrak{g})$ is a linear algebraic group. In fact, α_s and α_u are represented by block matrices as above,

$$\alpha_s = \begin{pmatrix} A_s & 0 & 0 \\ 0 & B_s & 0 \\ 0 & 0 & C_s \end{pmatrix}, \quad \alpha_u = \begin{pmatrix} A_u & 0 & 0 \\ 0 & B_u & 0 \\ 0 & 0 & C_u \end{pmatrix}$$

where the subscript s means that we take the semisimple part of the matrix, and the subscript u stands for the unipotent part. Note that $\mathfrak{g}^{\varepsilon}(\alpha) = \mathfrak{g}^{\varepsilon}(\alpha_s)$ for $\varepsilon = -, 0, +$. We may assume that the matrices appearing in the representation of α_s have diagonal form. Otherwise we may pass to the complexification of $\mathfrak{g}$ where we can diagonalize. Now a direct calculation finishes the proof. Let us first check that the space $\mathfrak{h} = \mathfrak{g}^-(\alpha) \oplus \mathfrak{g}^0(\alpha)$ is a Lie subalgebra of $\mathfrak{g}$. All other cases are analogous. $\mathfrak{h}$ is spanned by all eigenvectors corresponding to an eigenvalue λ with $|\lambda| \leq 1$. So, suppose that $\{v_1, \dots v_{k+l}\}$ is a basis of $\mathfrak{h}$ such that $\alpha_s(v_i) = \lambda_i v_i$ for all i ($|\lambda_i| \leq 1$). For any $v_i, v_j \in \mathfrak{h}$, we compute that

$$\alpha_s([v_i, v_j]) = [\alpha_s v_i, \alpha_s v_j] = [\lambda_i v_i, \lambda_j v_j] = \lambda_i \lambda_j [v_i, v_j].$$

Now, clearly $|\lambda_i \lambda_j| \leq 1$, from which it follows that $[v_i, v_j] \in \mathfrak{h}$, proving that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.
Checking that $\mathfrak{g}^-(\alpha)$ is an ideal of $\mathfrak{h}$ can be done similarly, which finishes the proof. □

To be able to generalize the ideas of [3], we must be able to talk about *affine subspaces* of N. Therefore, we define a line in the Lie group N as a left coset of a 1-parameter subgroup. In other words, $L = m \cdot \exp(tA)$ for some $A \in \mathfrak{n}$ with $A \neq 0$. We say that this line is parallel to the Lie subalgebra $\mathrm{span}\{A\}$. More generally, we have the following definition.

Definition 2.2. For a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{n}$ we define an affine subspace of N to be any left coset of the form $m \cdot \exp(\mathfrak{h})$ ($m \in N$). We say that this affine subspace is *parallel* to $\mathfrak{h}$.

Define the following two subsets of $\mathrm{Aff}(N)$:

$$\Omega = \{g \in \mathrm{Aff}(N) \mid \dim \mathfrak{n}^0(g) = 1,\ \ell(g)_{|\mathfrak{n}^0(g)} = \mathrm{id}\}$$
$$\Omega_0 = \{g \in \Omega \mid g \cdot n \neq n \quad \forall n \in N\}.$$

The elements of Ω are called pseudohyperbolic, and Ω_0 consists of fixed-point-free elements of $\mathrm{Aff}(N)$ inside Ω. We will study the action of a group generated by pseudohyperbolic elements. The following lemma is needed to be able to describe the behaviour of the action of a pseudohyperbolic element g and its iterates g^n.

Lemma 2.3. *Suppose that $\alpha \in \mathrm{Aut}(N)$ satisfies the following condition: if 1 is an eigenvalue of $\alpha_* \in \mathrm{Aut}(\mathfrak{n})$ then its geometric and algebraic multiplicity coincide. Let $\mathfrak{e} = Eig(\alpha_*, 1)$ be the eigenspace to the eigenvalue 1, and $E = \exp(\mathfrak{e})$. For fixed $n \in N$ define a map $\varphi \colon N \to N$ by*

$$m \mapsto m^{-1} n \alpha(m).$$

Then there exists an $m \in N$ such that $\varphi(m) \in E$. This m is uniquely determined modulo E.

Proof. We prove the result by induction on the nilpotency class c of N. If N is abelian, i.e., for c=1 and $N=\mathbb{R}^n$, we have

$$\varphi(m) = -m + n + \alpha(m) = (\alpha - \mathrm{id}) \cdot m + n. \tag{5}$$

Let $(v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ such that the first k vectors form a basis of E. Then α is represented by the matrix

$$\alpha = \left(\begin{array}{c|c} I & * \\ \hline 0 & A \end{array}\right)$$

where $A \in M_{n-k}(\mathbb{R})$ has no eigenvalues equal to 1. Hence the block matrices on the diagonal of $\alpha - \mathrm{id}$ are 0 and $A - I$, the latter being invertible. Hence the matrix equation $(A-I)x + b = 0$ for any $b \in \mathrm{span}\{v_{k+1}, \ldots, v_n\}$ has a unique solution. This means that there is an $m \in N$ such that the last $n-k$ components of the vector $\varphi(m) = (\alpha - \mathrm{id}) \cdot m + n$ are zero, i.e., such that $\varphi(m) \in E$. Moreover the last $n-k$ components of m are uniquely determined. Hence if $m' \in N$ is another element satisfying $\varphi(m') \in E$, then $m = m' + e$ with some $e \in E$.

Now suppose that $c > 1$ and that the lemma is true for lower nilpotency classes. Let Z be the center of N and define $\overline{\varphi}\colon N/Z \to N/Z$ by

$$\overline{m} \mapsto \overline{m^{-1} n \alpha(m)} = (\overline{m})^{-1} \overline{n} \overline{\alpha(m)}$$

where $\overline{\alpha}\colon N/Z \to N/Z$ given by $\overline{m} \mapsto \overline{\alpha(m)}$ is an automorphism of N/Z. (Here we use the bar to denote the natural projection $N \to N/Z$). Note that $\overline{\varphi}$ is well-defined and that $\overline{\alpha}$ satisfies the assumption of the lemma on the eigenvalue 1. Hence we can apply the induction hypothesis, and there is an $\overline{m} \in N/Z$ such that $(\overline{m})^{-1} \overline{n} \overline{\alpha(m)} \in EZ/Z \subseteq N/Z$. Thus (for any given lift m of $\overline{m}$) we may write

$$m^{-1} n \alpha(m) = e_1 z_1$$

with some $e_1 \in E$ and $z_1 \in Z$. It follows that for any $z \in Z$

$$\begin{aligned} \varphi(mz) &= z^{-1} m^{-1} \cdot n \cdot \alpha(m) \alpha(z) \\ &= m^{-1} n \alpha(m) \cdot z^{-1} \alpha(z) \\ &= e_1 z_1 \cdot z^{-1} \alpha(z) \\ &= e_1 \cdot z^{-1} z_1 \alpha(z). \end{aligned}$$

Since $\alpha_{|Z} \in \mathrm{Aut}(Z)$ satisfies the eigenvalue 1 condition of the lemma we can again apply the induction hypothesis, with $N = Z$ and $n = z_1$. Hence we find a $z_0 \in Z$ such that $z_0^{-1} z_1 \alpha(z_0) \in E \cap Z$. It follows that

$$\varphi(mz_0) = e_1 \cdot z_0^{-1} z_1 \alpha(z_0) \in E.$$

The uniqueness up to E also follows by induction on the nilpotency class of N. □

Now, we apply this lemma to obtain some information about the action of a pseudohyperbolic element.

Proposition 2.4. *For any $g \in \Omega$ there is exactly one g-invariant line C_g parallel to $\mathfrak{n}^0(g)$.*

Proof. A line being parallel to $\mathfrak{n}^0(g)$ is of the form $m \cdot \exp(tA)$ with some $m \in N$ and an A satisfying $\mathfrak{n}^0(g) = \operatorname{span}\{A\}$. It is g-invariant if and only if there is a function $s\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(m \cdot \exp(tA)) = m \cdot \exp(s(t) \cdot A). \tag{6}$$

Writing $g = (n, \alpha)$ and applying (1) we obtain

$$\begin{aligned} g(m \cdot \exp(tA)) &= n \cdot \alpha(m) \cdot \alpha(\exp(tA)) \\ &= n \cdot \alpha(m) \cdot \exp(tA). \end{aligned} \tag{7}$$

Now, equating (6) to (7) leads to $m^{-1}n \cdot \alpha(m) = \exp((s(t) - t)A)$, which is an element of the 1-dimensional Lie group $N^0(g) = \exp(\mathfrak{n}^0(g))$. We conclude that there exists a g-invariant line parallel to $\mathfrak{n}^0(g)$ if and only if there is an $m \in N$ such that $m^{-1}n \cdot \alpha(m) \in N^0(g)$ (and then $s(t) - t$ has to be constant). But this follows by Lemma 2.3. We have $\mathfrak{e} = \mathfrak{n}^0(g)$ and $E = N^0(g)$. Moreover α satisfies the eigenvalue 1 condition. Hence there is an $m \in N$ and some $c \in \mathbb{R}$ with $m^{-1}n \cdot \alpha(m) = \exp(cA) \in N^0(g)$. The line $m \cdot \exp(tA)$ is g-invariant and we have

$$g(m \cdot \exp(tA)) = m \cdot \exp((c + t)A) = m \cdot \exp(tA) \exp(cA). \tag{8}$$

Since m was unique up to $N^0(g)$, another choice of m yields the same line. Indeed, if $m' = mn_1$ with some $n_1 = \exp(c_1 A) \in N^0(g)$ then

$$m' \exp(tA) = mn_1 \exp(tA) = m \exp((c_1 + t)A).$$

Hence this line is unique. □

The proposition above does not only give us a g-invariant line C_g, for any $g \in \Omega$, but equation (8) shows that the action on any point x of this line is by means of a constant translation. We define the *translational part* $\tau(g)$ of g by $gx = x\tau(g)$ (where x is any point $x \in C_g$). It holds that $\tau(g) \neq 1$ if and only if $g \in \Omega_0$. For $m \in \mathbb{Z}, m \neq 0$ we have

$$\begin{aligned} C_{g^m} &= C_g \\ \tau(g^m) &= \tau(g)^m. \end{aligned}$$

Let $T(g) = \log(\tau(g)) \in \mathfrak{n}^0(g)$. If $g \in \Omega_0$, then $T(g) \neq 0$, and hence every $x \in \mathfrak{d}^+(g)$ has a unique decomposition

$$x = \lambda(x)T(g) + a(x)$$

where $\lambda(x) \in \mathbb{R}$ and $a(x) \in \mathfrak{n}^+(g)$. We call $x \in \mathfrak{d}^+(g)$ *positive with respect to* $g \in \Omega_0$, if $\lambda(x) > 0$. We will write $x \succ_g 0$.

Definition 2.5. Two elements $g_1, g_2 \in \Omega_0$ will be called *transversal*, if

$$\mathfrak{n} = \mathfrak{n}^+(g_1) \oplus \mathfrak{d}^+(g_2) = \mathfrak{d}^+(g_1) \oplus \mathfrak{n}^+(g_2).$$

It is easy to see that $g_1, g_2 \in \Omega_0$ are transversal if and only if

$$\begin{aligned} \mathfrak{n}^+(g_1) \oplus \mathfrak{n}^+(g_2) \oplus (\mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2)) &= \mathfrak{n} \\ \text{and } \dim(\mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2)) &= 1. \end{aligned}$$

Let

$$S_{g_1} = \{x \in \mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2) \mid x \succ_{g_1} 0\}$$
$$S_{g_2} = \{x \in \mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2) \mid x \succ_{g_2} 0\}.$$

Definition 2.6. For two transversal elements $g_1, g_2 \in \Omega_0$ we say that they form a *positive pair* if $S_{g_1} = S_{g_2}$.

For $g \in \Omega$ note that $C_g = m \cdot N^0(g)$.
For any $x \in N$ let

$$B_g^+(x) = x \cdot N^+(g).$$

We will also use

$$E_g^+ = m \cdot D^+(g) = C_g \cdot D^+(g)$$
$$E_g^- = m \cdot D^-(g) = C_g \cdot D^-(g).$$

Note that $\mathfrak{n}^+(g)$ is a Lie ideal in $\mathfrak{d}^+(g)$ by Lemma 2.1. Hence $N^+(g)$ is a normal subgroup in $D^+(g)$. Since the intersection of $N^+(g)$ and $N^0(g)$ is trivial, every element $x \in D^+(g)$ can be written as

$$x = n^0 n^+$$

with unique elements $n^0 \in N^0(g)$ and $n^+ \in N^+(g)$. Let $x \in E_g^+$. Then it is easy to see that $B_g^+(x) \cap C_g$ consists of exactly 1 point: if $x = m \cdot n^0 n^+$ and $C_g = m \cdot N^0(g)$ then

$$m \cdot n^0 = x \cdot (n^+)^{-1}$$

is this point. Thus we can define a projection

$$P_g \colon E_g^+ \to C_g$$

by the equality $B_g^+(x) \cap C_g = \{P_g(x)\}$ for $x \in E_g^+$. The subgroup $N^+(g)$ is α-invariant, where $g = (n, \alpha)$. Therefore, if $x = m \cdot n^0 n^+ \in E_g^+$ then

$$P_g(gx) = m \cdot n^0 \tau(g) = P_g(x)\tau(g). \tag{9}$$

We are now ready to prove the obstruction criterium to proper discontinuity.

Proposition 2.7. *Assume that $g_1, g_2 \in \Omega_0$ form a positive pair. Then there exists a compact set $K \subset N$ and two sequences $\{s_i\}, \{t_i\}$ of positive integers such that*

$$\lim_{i\to\infty} s_i = \lim_{i\to\infty} t_i = \infty \quad \text{and} \quad (g_1^{-s_i} g_2^{t_i} K) \cap K \neq \emptyset.$$

In particular, the subgroup of $\mathrm{Aff}(N)$ *generated by g_1 and g_2 does not act properly discontinuously on N.*

Proof. The last part follows from the group theoretical argument given in [3], Corollary 2.3. It is independent of our generalization.

Choose a norm $\|\cdot\|$ on $\mathfrak{n}$. It defines a left-invariant metric d on N.
Take a pseudohyperbolic element $g \in \Omega_0$ and use the notations introduced above. If $x = m \cdot n^0 \cdot n^+ = P_g(x) \cdot n^+ \in E_g^+$ and $k \in \mathbb{N}$, then

$$\begin{aligned} d(g^{-k}x, P_g(g^{-k}x)) &= d(g^{-k}x, P_g(x)\tau(g)^{-k}) \\ &= d(g^{-k}(P_g(x) \cdot n^+), P_g(x)\tau(g)^{-k}) \\ &= d(g^{-k}(P_g(x))\alpha^{-k}(n^+), P_g(x)\tau(g)^{-k}) \\ &= d(P_g(x)\tau(g)^{-k}\alpha^{-k}(n^+), P_g(x)\tau(g)^{-k}) \\ &= d(\alpha^{-k}(n^+), 1) \end{aligned}$$

since the metric is left-invariant and $g^{-k}(ab) = g^{-k}(a)\alpha^{-k}(b)$ for all $a, b \in N$. Using the fact that $d(\exp(A), 1) \leq \|A\|$ we have

$$\begin{aligned} d(\alpha^{-k}(n^+), 1) &\leq \|\log(\alpha^{-k}(n^+))\| \\ &= \|\alpha_*^{-k}(\log(n^+))\| \\ &\leq ce^{-bk}\|\log(n^+)\| \end{aligned}$$

for some constants $b, c > 0$ only depending on α_*. (Use that $\alpha^{-1}_{*|\mathfrak{n}^+(g)}$ has only eigenvalues λ of modulus $|\lambda| < 1$). Thus we have

$$d(g^{-k}x, P_g(g^{-k}x)) \leq ce^{-bk}\|\log(n^+)\| \tag{10}$$

for all $k \in \mathbb{N}$ and $x \in E_g^+$.

Fix a point $m(g)$ on C_g and write $C_g = m(g)N^0(g)$. Let

$$R(g) = \{m(g)\tau(g)^t \mid 0 \leq t < 1\}.$$

For every $x \in E_g^+$ there exists a unique integer $k(x, g)$ such that $P_g(g^{k(x,g)}x) \in R(g)$.

Write $E_{g_1}^+ = m(g_1)D^+(g_1)$ and $E_{g_2}^+ = m(g_2)D^+(g_2)$. We claim that $E_{g_1}^+ \cap E_{g_2}^+$ is not empty. To show this, we use the following lemma which can be easily proved by induction on the nilpotency class of $\mathfrak{n}$.

Lemma 2.8. *Suppose that $\mathfrak{n}$ is a nilpotent Lie algebra which is the sum of two subalgebras: $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$. Let $N = \exp(\mathfrak{n})$, $A = \exp(\mathfrak{a})$ and $B = \exp(\mathfrak{b})$. Then the map $\varphi\colon A \times B \to N$, $(a, b) \mapsto ab$ is surjective.*

Since g_1, g_2 form a positive pair we have $\mathfrak{n} = \mathfrak{d}^+(g_1) + \mathfrak{d}^+(g_2)$. Hence by Lemma 2.8 we may write $m(g_1)^{-1}m(g_2) = m_1m_2$ with $m_1 \in D^+(g_1)$ and $m_2 \in D^+(g_2)$. Then

$$\begin{aligned} E_{g_2}^+ &= m(g_2)D^+(g_2) = m(g_1)m(g_1)^{-1}m(g_2)D^+(g_2) \\ &= m(g_1)m_1m_2D^+(g_2) = m(g_1)m_1D^+(g_2) \end{aligned}$$

so that $m(g_1)m_1 \in E_{g_2}^+$. On the other hand, $m(g_1)m_1 \in E_{g_1}^+$. Hence we have found an element

$$x_0 = m(g_1)m_1 \in E_{g_1}^+ \cap E_{g_2}^+.$$

Now choose a $V \in \mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2)$ such that $V \succ_{g_1} 0$ and $V \succ_{g_2} 0$. Let $v = \exp(V)$. Fix an element $x_0 \in E^+_{g_1} \cap E^+_{g_2}$ as above. Then we have for all $i \geq 0$

$$x_i := x_0 \cdot v^i \in E^+_{g_1} \cap E^+_{g_2}.$$

Let $s_i := -k(x_i, g_1)$ and $t_i := -k(x_i, g_2)$. We will make the computations for g_1 and the numbers s_i. The argument for g_2 and the numbers t_i is the same.

Let $V_1 \in \mathfrak{n}^0(g_1)$ be so that $V = V_1 + W$, where $W \in \mathfrak{n}^+(g_1)$. Since V is positive with respect to g_1 we have $V_1 = \lambda_1 T(g_1)$ where $\lambda_1 > 0$. Let $v_1 = \exp(V_1)$. Note that $v = v_1 \cdot w$ for some $w \in N^+(g_1)$: we can write

$$\log(v_1 \cdot w) = \log(v_1) + \log(w) + W_1$$

with $W_1 \in \mathfrak{n}^+(g_1)$, since $\mathfrak{n}^+(g_1)$ is a Lie ideal in $\mathfrak{n}^0(g_1) \oplus \mathfrak{n}^+(g_1)$ and $V = V_1 + W$. It follows that $v^i = v_1^i \cdot w_2$ for some $w_2 \in N^+(g_1)$.

We want to compute $P_{g_1}(g_1^k x_i)$ for any $k \in \mathbb{Z}$. We have

$$P_{g_1}(g_1^k x_i) = P_{g_1}(g_1^k(x_0 \cdot v^i)) = P_{g_1}(g_1^k(x_0) \cdot \alpha_1^k(v^i))$$

(Of course α_1 denotes the $\mathrm{Aut}(N)$-part of g_1). There exists a $w_3 \in N^+(g_1)$ such that

$$\alpha_1^k(v^i) = \alpha_1^k((v_1)^i)\alpha_1^k(w_2) = v_1^i \cdot w_3 = \tau(g_1)^{\lambda_1 i} w_3$$

since $V_1 = \lambda_1 T(g_1)$. Writing $x_0 = m(g_1) \cdot n^0 \cdot n^+$ there exists a $w_4 \in N^+(g_1)$ such that

$$g_1^k(x_0) = g_1^k(m(g_1) \cdot n^0)\alpha_1^k(n^+) = m(g_1)n^0 \cdot \tau(g_1)^k \cdot w_4$$

where $n^0 = \tau(g_1)^{r_0} \in N^0(g_1)$. So we obtain that

$$\begin{aligned} P_{g_1}(g_1^k x_i) &= P_{g_1}(g_1^k(x_0) \cdot \alpha_1^k(v^i)) \\ &= P_{g_1}(m(g_1)n^0\tau(g_1)^k\tau(g_1)^{\lambda_1 i} \cdot \tau(g_1)^{-\lambda_1 i} w_4 \tau(g_1)^{\lambda_1 i} w_3) \\ &= m(g_1)\tau(g_1)^{r_0 + k + \lambda_1 i}. \end{aligned}$$

This lies in $R(g_1) = \{m(g_1)\tau(g_1)^t \mid 0 \leq t < 1\}$ if $0 \leq r_0 + \lambda_1 i + k < 1$. In this case k is the unique integer $k(x_i, g_1) = -s_i$ with this property. Hence we have $0 \leq r_0 + \lambda_1 i - s_i < 1$ for all $i \geq 1$ and

$$\lim_{i\to\infty} \frac{i}{s_i} = \lim_{i\to\infty} \frac{i}{r_0 + \lambda_1 i} \leq \frac{1}{\lambda_1} > 0. \tag{11}$$

Write $x_i = P_{g_1}(x_i) \cdot n_i^+ \in E^+_{g_1}$ with $n_i^+ \in N^+(g_1)$. We have $x_i = x_0 \cdot v^i = m(g_1) \cdot n^0 \cdot n^+ \cdot v^i$. Let $(W_1, \ldots, W_n)$ be a basis of the nilpotent Lie algebra $\mathfrak{n}^+(g_1)$. Using Mal'cev's theorem we can find polynomials $p_1(i), \ldots, p_n(i)$ such that

$$n^+ \cdot v^i = \tau_{g_1}^{\lambda_1 i} \exp(p_1(i)W_1 + \cdots + p_n(i)W_n).$$

Hence $n_i^+ = \exp(p_1(i)W_1 + \cdots + p_n(i)W_n)$. So there exists a polynomial $P(i)$ such that

$$\|\log(n_i^+)\| \leq P(i).$$

Using (10), (11) and $b, \lambda_1 > 0$ we obtain

$$0 \le \lim_{i\to\infty} d(g_1^{-s_i}x_i, P_{g_1}(g_1^{-s_i}x_i)) \le \lim_{i\to\infty} ce^{-bs_i}\|\log(n_i^+)\|$$
$$\le \lim_{i\to\infty} ce^{-b\lambda_1 i}P(i) = 0.$$

It follows that

$$\lim_{i\to\infty} d(g_1^{-s_i}x_i, R(g_1)) = 0.$$

Hence there exists an upper bound M_1 for all these distances. Thus, the compact set

$$K_1 = \{x \in N \mid d(x, R(g_1)) \le M_1\}$$

contains all $g_1^{-s_i}x_i$.

We can obtain in the same way a bound M_2 for the distances to $R(g_2)$, and define the compact set

$$K = \{x \in N \mid d(x, R(g_1)) \le M_1\} \cup \{x \in N \mid d(x, R(g_2)) \le M_2\}.$$

Clearly $g_1^{-s_i}x_i \in K$, $g_2^{-t_i}x_i \in K$ and $g_1^{-s_i}x_i = (g_1^{-s_i}g_2^{t_i})g_2^{-t_i}x_i$, so that

$$g_1^{-s_i}g_2^{t_i}K \cap K \ne \emptyset$$

for all s_i and t_i.

□

3. Subgroups of Aff(N) for N two-step nilpotent

In this short section we show that the generalized Auslander conjecture reduces to the ordinary one if N is two-step nilpotent. Indeed, if N is two-step nilpotent, a faithful affine representation

$$\lambda\colon \operatorname{Aff}(N) = N \rtimes \operatorname{Aut}(N) \to \operatorname{Aff}(\mathbb{R}^n)$$

was constructed in Theorem 4.1 in [11]. This representation satisfies the following:

- Let $i\colon N \hookrightarrow \operatorname{Aff}(N)$ be the embedding given by $n \mapsto (n, id)$. Then, the composition $\lambda \circ i\colon N \to \operatorname{Aff}(\mathbb{R}^n)$ defines a simply transitive action of N on $\mathbb{R}^n$. For $n \in N$, $x \in \mathbb{R}^n$ it is given by
$$n \cdot x = \lambda(n, id)(x) \in \mathbb{R}^n.$$
- λ maps the subgroup $\operatorname{Aut}(N)$ of $\operatorname{Aff}(N)$ into the subgroup $\operatorname{GL}_n(\mathbb{R})$ of $\operatorname{Aff}(\mathbb{R}^n)$. It follows that for every $\alpha \in \operatorname{Aut}(N)$ and for the zero vector $0 \in \mathbb{R}^n$, we have that
$$\lambda(1, \alpha)(0) = 0.$$

The following proposition yields the desired reduction of the generalized Auslander conjecture to the ordinary one:

Proposition 3.1. *Let N be a simply connected, connected 2-step nilpotent Lie group. Assume that $\Gamma \le \operatorname{Aff}(N)$ acts crystallographically on N. Then Γ also admits an affine crystallographic action on $\mathbb{R}^n$.*

Proof. Let $\lambda : \text{Aff}(N) \to \text{Aff}(\mathbb{R}^n)$ be the faithful representation mentioned above. As λ lets N act simply transitively on $\mathbb{R}^n$, the evaluation map

$$e_v : N \to \mathbb{R}^n : n \mapsto n \cdot 0$$

is a diffeomorphism.

Now, $\text{Aff}(N)$ acts on N (via $(n,\alpha)\cdot m = n\alpha(m)$ as before) and on $\mathbb{R}^n$ (using $\lambda(n,\alpha)$). We can check that e_v is an $\text{Aff}(N)$–equivariant map, i.e. the following diagram is commutative for any $(n,\alpha) \in \text{Aff}(N)$

$$\begin{array}{ccc} N & \xrightarrow{e_v} & \mathbb{R}^n \\ {\scriptstyle (n,\alpha)\cdot}\downarrow & & \downarrow{\scriptstyle (n,\alpha)\cdot} \\ N & \xrightarrow[e_v]{} & \mathbb{R}^n \end{array}$$

Indeed, let $m \in N$, then

$$\begin{aligned} (n,\alpha)\cdot e_v(m) &= \lambda(n,\alpha)(m\cdot 0) \\ &= \lambda(n,\alpha)(\lambda(m,id)(0)) \\ &= \lambda(n\alpha(m),\alpha)(0) \\ &= \lambda(n\alpha(m),id)(\lambda(1,\alpha)(0)) \\ &= \lambda(n\alpha(m),id)(0) \\ &= e_v(n\alpha(m)) \\ &= e_v((n,\alpha)\cdot m) \end{aligned}$$

Using this commutative diagram it is now easy to see that a subgroup Γ of $\text{Aff}(N)$ acts crystallographically on N, if and only if it also acts crystallographically (and affinely) on $\mathbb{R}^n$. □

4. The conjecture in low dimensions

In this section we prove that the generalized Auslander conjecture is true in dimensions $n \leq 5$. We have to deal with three cases. In the first case, the automorphism group of N is already virtually solvable. Then $\text{Aff}(N)$ is clearly virtually solvable, and hence all its subgroups Γ are virtually solvable. In the second case, N is 2-step nilpotent. Then the claim follows from Proposition 3.1. If neither the first case nor the second case applies to N, the claim is more difficult to prove. We have to make an appeal to the results of section 2. However, in dimension ≤ 5 there is only one Lie group N for which this problem arises.

Theorem 4.1. *Let N be a simply connected and connected nilpotent Lie group of dimension n with $1 \leq n \leq 5$. Let $\Gamma \leq \text{Aff}(N)$ act crystallographically on N. Then Γ is virtually polycyclic.*

Proof. It is enough to show that any such Γ is virtually solvable. Let $\mathfrak{n}$ be the Lie algebra of N. All nilpotent Lie algebras of dimension $n \leq 3$ are nilpotent of class ≤ 2. If $\dim \mathfrak{n} = 4$ then $\mathfrak{n}$ is either of class ≤ 2 or isomorphic to the generic filiform Lie algebra $\mathfrak{n}_4 \colon [x_1,x_2] = x_3; [x_1,x_3] = x_4$. As the derivation algebra of $\mathfrak{n}_4$ consists of lower-triangular matrices with respect to this basis, it is solvable. We can conclude that $\mathrm{Aut}(N_4)$, where $N_4 = \exp(\mathfrak{n}_4)$, is virtually solvable, which finishes the argument in dimension 4.

In dimension 5 we use the list of all nilpotent Lie algebras as given in [14]. It consists of 6 indecomposable Lie algebras $\mathfrak{g}_{5,1}, \dots, \mathfrak{g}_{5,6}$ and 3 decomposables. The algebras $\mathfrak{g}_{5,1}, \mathfrak{g}_{5,2}$ and two of the decomposable ones are nilpotent of class ≤ 2. The derivation algebra of the other decomposable one, namely $\mathrm{Der}(\mathfrak{n}_4 \oplus \mathbb{R})$, and the derivation algebras $\mathrm{Der}(\mathfrak{g}_{5,3}), \mathrm{Der}(\mathfrak{g}_{5,5})$ and $\mathrm{Der}(\mathfrak{g}_{5,6})$ are clearly solvable. Hence it only remains to consider the Lie algebra

$$\mathfrak{n} = \mathfrak{g}_{5,4} : \quad [x_1, x_2] = x_3;\ [x_1, x_3] = x_4;\ [x_2, x_3] = x_5.$$

This is the free 3-step nilpotent 2-generated Lie algebra of dimension 5. Let $N = \exp(\mathfrak{n})$ and assume that $\Gamma \leq \mathrm{Aff}(N)$ acts crystallographically. Assume furthermore that Γ is not virtually solvable. Then also the image of Γ inside $\mathrm{Aut}(\mathfrak{n})$ under the map

$$\ell \colon N \rtimes \mathrm{Aut}(N) \to \mathrm{Aut}(\mathfrak{n})$$

is not virtually solvable. We will show that this leads to a contradiction.
A simple calculation shows that, with respect to the basis $x_1, \dots, x_5$, $\mathrm{Aut}(\mathfrak{n})$ consists of matrices of the form

$$A_{\alpha_i,\beta_i} = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & 0 & 0 \\ \alpha_3 & \beta_3 & \gamma & 0 & 0 \\ \alpha_4 & \beta_4 & * & \alpha_1\gamma & \beta_1\gamma \\ \alpha_5 & \beta_5 & * & \alpha_2\gamma & \beta_2\gamma \end{pmatrix}$$

where $\gamma = \alpha_1\beta_2 - \beta_1\alpha_2$ is the determinant of the 2×2-matrix in the left upper corner (the entries denoted by $*$'s are also determined by the first two columns, but they do not play a role in what follows). Consider the homomorphism $\rho \colon \mathrm{Aut}(\mathfrak{n}) \to \mathrm{GL}_2(\mathbb{R})$ given by

$$A_{\alpha_i,\beta_i} \mapsto \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}.$$

It follows that $\rho(\ell(\Gamma))$ is not virtually solvable, because $\ell(\Gamma)$ is not. Hence, by Tits' alternative $\rho(\ell(\Gamma))$ contains a non-abelian free subgroup F_1. But then, its derived subgroup $F_2 = [F_1, F_1]$ is also a non-abelian free subgroup of $\rho(\ell(\Gamma))$, satisfying $F_2 \subseteq [\mathrm{GL}_2(\mathbb{R}), \mathrm{GL}_2(\mathbb{R})] = \mathrm{SL}_2(\mathbb{R})$. It is well known that for any free non-abelian subgroup $F_2 \subseteq SL_2(\mathbb{R})$ there exists an element $g \in F_2$ such that g has no eigenvalues of modulus 1 (see [9]). Thus there exists a $g_1 \in \Gamma$ such that $\rho(\ell(g_1)) \in F_2$ and $\rho(\ell(g_1))$ has no eigenvalue of modulus 1. We denote the eigenvalues of $\rho(\ell(g_1))$ by λ and $1/\lambda$, with $|\lambda| > 1$. It is easy to see that there is a basis (A, B, C, D, E) of $\mathfrak{n}$ with brackets $[A, B] = C$, $[A, C] = D$, $[B, C] = E$ such that

$$\ell(g_1) = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & 1/\lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ * & 0 & 0 & \lambda & 0 \\ 0 & * & 0 & 0 & 1/\lambda \end{pmatrix}.$$

Using the notation of the decompositions (2) and (3) we find that

$$\begin{aligned}\mathfrak{n}^0(g_1) &= \langle C\rangle, \quad \mathfrak{n}^+(g_1) = \langle A, D\rangle \\ \mathfrak{d}^+(g_1) &= \langle A, C, D\rangle\end{aligned}$$

This shows that $T(g_1) = \alpha C$ for some non-zero α. By rescaling our basis vectors, we may assume that $T(g_1) = C$.

Besides our fixed element $g_1 \in \Gamma$, we now choose an element $h \in \Gamma$ such that $\rho(\ell(h)) \in F_2$ and $\rho(\ell(h))$ does not commute with $\rho(\ell(g_1))$ (this is possible since F_2 is free and non-abelian). We then consider $g_2 = hg_1h^{-1}$. It follows that $\langle \rho(\ell(g_1)), \rho(\ell(g_2))\rangle$ and hence $\langle \ell(g_1), \ell(g_2)\rangle$ are free groups. Note that the automorphism $\ell(g_2)$ has exactly the same eigenvalues as $\ell(g_1)$. Then there exists a nonzero element of the form $\alpha A + \beta B$ such that

$$\ell(g_2)(\alpha A + \beta B) = \lambda(\alpha A + \beta B) \bmod \langle C, D, E\rangle \quad \text{and} \quad \ell(g_2)(C) = C \bmod \langle D, E\rangle.$$

Here we have $\beta \neq 0$, otherwise $\langle \ell(g_1), \ell(g_2)\rangle$ would be a solvable group. Note that

$$\begin{aligned}\ell(g_2)(\alpha D + \beta E) &= \ell(g_2)([\alpha A + \beta B, C]) \\ &= [\ell(g_2)(\alpha A + \beta B), \ell(g_2)(C)] \\ &= [\lambda(\alpha A + \beta B), C] \\ &= \lambda([\alpha A + \beta B, C]) \\ &= \lambda(\alpha D + \beta E).\end{aligned}$$

It follows that $\dim \mathfrak{n}^0(g_2) = 1$, $\dim \mathfrak{n}^+(g_2) = 2$, so there exist scalars $\alpha, \beta \neq 0, \gamma, \delta, \varepsilon, \mu, \nu$ with

$$\begin{aligned}\mathfrak{n}^0(g_2) &= \langle C + \mu D + \nu E\rangle \\ \mathfrak{n}^+(g_2) &= \langle \alpha D + \beta E, \alpha A + \beta B + \gamma C + \delta D + \varepsilon E\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\mathfrak{n} &= \mathfrak{n}^+(g_1) \oplus \mathfrak{n}^0(g_2) \oplus \mathfrak{n}^+(g_2) \\ &= \mathfrak{n}^+(g_1) \oplus \mathfrak{n}^0(g_1) \oplus \mathfrak{n}^+(g_2).\end{aligned}$$

It follows that g_1 and g_2 are transversal elements. (Note that g_1 and g_2 act fixed-point-free because $\langle g_1, g_2\rangle$ is a free group and hence torsionfree.) Hence $\mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2)$ is a 1-dimensional vector space. We want to show that we can find a positive pair, so that our desired contradiction follows prom proposition 2.7. Let $V \in \mathfrak{d}^+(g_1) \cap \mathfrak{d}^+(g_2)$ be a non-zero vector. This implies that there are scalars k, l, m and r, s, t such that

$$\begin{aligned}V &= kA + lC + mD \\ &= r(\alpha A + \beta B + \gamma C + \delta D + \varepsilon E) + s(C + \mu D + \gamma E) + t(\alpha D + \beta E).\end{aligned}$$

Since $\beta \neq 0$ we have $r = k = 0$ and $s = l$. For $l = 0$ we would obtain $V = 0$, hence we have $l \neq 0$.

As $T(g_1) = C$, the semiline S_{g_1} consists of those $V = lC + mD$ with $l > 0$. On the other hand, $T(g_2) = \xi(C + \mu D + \gamma E)$ for some non-zero ξ. We distinguish two possibilities:

- If $\xi > 0$, then it is obvious that $S_{g_2} = S_{g_1}$ and hence g_1 and g_2 form a positive pair.
- However, if ξ is negative, we can start all over again and consider the pair g_1 and g_2^{-1}. As $T(g_2^{-1}) = -T(g_2)$, we obtain that in this case $S_{g_2^{-1}} = S_{g_1}$ and hence g_1 and g_2^{-1} form a postive pair.

□

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