LIE ALGEBRA PREDERIVATIONS AND STRONGLY NILPOTENT LIE ALGEBRAS

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Abstract. We study Lie algebra prederivations. A Lie algebra admitting a non-singular prederivation is nilpotent. We classify filiform Lie algebras admitting a non-singular prederivation but no non-singular derivation. We prove that any 4-step nilpotent Lie algebra admits a non-singular prederivation.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over a field k and $\mathrm{Der}(\mathfrak{g})$ its derivation algebra. It is a natural question whether $\mathfrak{g}$ admits a non-singular derivation or not. Jacobson has proved [6] that any Lie algebra over a field of characteristic zero admitting a non-singular derivation must be nilpotent. He also asked for the converse, whether any nilpotent Lie algebra admits a non-singular derivation. As it turned out, this was not the case. Dixmier and Lister [3] constructed nilpotent Lie algebras possessing only nilpotent derivations. They called this class of Lie algebras *characteristically nilpotent* Lie algebras. This class has been studied extensively later on [8].

On the other hand there exist various generalizations of Lie algebra derivations, see for example [7], [10]. For so called *prederivations* Jacobson's theorem is also true: any Lie algebra over a field of characteristic zero admitting a non-singular prederivation is nilpotent [1]. Lie algebra prederivations have been studied in connection with bi-invariant semi-Riemannian metrics on Lie groups [10]. The Lie algebra of prederivations $\mathrm{Pder}(\mathfrak{g})$ forms a subalgebra of $\mathfrak{gl}(\mathfrak{g})$ containing the algebra $\mathrm{Der}(\mathfrak{g})$. Note that a prederivation is just a derivation of the *Lie triple system* induced by $\mathfrak{g}$. A Lie triple system over k is a vector space $\mathcal{T}$ with a trilinear mapping $(x, y, z) \mapsto [x, y, z]$ satisfying the following axioms [5]:

$$[x, y, z] = -[x, z, y]$$
$$[x, y, z] + [y, z, x] + [z, x, y] = 0$$
$$[[x, y, z], a, b] - [[x, a, b], y, z] = [x, [y, a, b]] + [x, y, [z, a, b]]$$

Any Lie algebra is at the same time a Lie triple system via $[x, y, z] := [x, [y, z]]$. As before in the derivation case there exist nilpotent Lie algebras possessing only nilpotent prederivations. In analogy to characteristically nilpotent Lie algebras we call a nilpotent Lie algebra *strongly nilpotent* if all its prederivations are nilpotent.

We classify strongly nilpotent Lie algebras in dimension 7 and filiform Lie algebras of dimension $n \leq 11$ admitting a non-singular prederivation but no non-singular derivation. The existence of a non-singular prederivation is useful for the construction of affine structures [2] on the Lie algebra.

1991 *Mathematics Subject Classification.* Primary 17B30.

2. Prederivations

Let $\mathfrak{g}$ be a Lie algebra over a field k and $\mathrm{Aut}(\mathfrak{g})$ its automorphism group. Define a pre-automorphism of $\mathfrak{g}$ to be a bijective linear map $A : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$A([x, [y, z]]) = [A(x), [A(y), [A(z)]]]$$

for all $x, y, z \in \mathfrak{g}$. The set of all pre-automorphisms forms a subgroup $\mathcal{M}(\mathfrak{g})$ of $GL(\mathfrak{g})$. This group plays an important role in the study of Lie groups which are endowed with a bi-invariant pseudo-Riemannian metric [10]. Over the real numbers $\mathcal{M}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$. Its Lie algebra is denoted by $\mathrm{Pder}(\mathfrak{g})$ and consists of so called prederivations. One may extend this definition for any field k.

2.1. **Definition.** A linear map $P : \mathfrak{g} \to \mathfrak{g}$ is called a prederivation of $\mathfrak{g}$ if

$$P([x, [y, z]]) = [P(x), [y, z]] + [x, [P(y), z]] + [x, [y, P(z)]]$$

for every $x, y, z \in \mathfrak{g}$.

The set of all prederivations of $\mathfrak{g}$ forms a subalgebra $\mathrm{Pder}(\mathfrak{g})$ of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$ containing the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g})$:

2.2. **Lemma.** *It holds* $\mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{Pder}(\mathfrak{g})$.

Proof. Let $D \in \mathrm{Der}(\mathfrak{g})$. Then by definition

$$D([x, [y, z]]) = [x, D([y, z])] + [D(x), [y, z]]$$

Substituting $D([y, z]) = [D(y), z] + [y, D(z)]$ we obtain

$$D([x, [y, z]]) = [x, [D(y), z]] + [x, [y, D(z)]] + [D(x), [y, z]].$$

□

Clearly we have equality for abelian Lie algebras. This is also known to be true for semisimple Lie algebras over a field k of characteristic zero [10].

2.3. **Proposition.** *Every prederivation of a finite-dimensional semisimple Lie algebra over k is a derivation and hence an inner derivation:* $\mathrm{Der}(\mathfrak{g}) = \mathrm{Pder}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

In the case of solvable Lie algebras equality does not hold in general. In this paper we are mostly interested in nilpotent Lie algebras.

2.4. **Definition.** Let $\mathfrak{g}$ be a Lie algebra and $\{\mathfrak{g}^k\}$ its lower central series defined by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ for $k \geq 1$. Recall that $\mathfrak{g}$ is said to be nilpotent of degree p, or nilindex p, if there exists an integer p such that $\mathfrak{g}^p = 0$ and $\mathfrak{g}^{p-1} \neq 0$. A nilpotent Lie algebra of dimension n and nilindex $p = n - 1$ is called *filiform.*

Let $\mathfrak{g}$ be a n-dimensional nilpotent Lie algebra. The descending central series

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \ldots \supseteq \mathfrak{g}^p = (0)$$

defines a positive filtration $\mathcal{F}$ of $\mathfrak{g}$. For each $k = 0, 1, \ldots, p-1$ choose a linear subspace V_k so that

$$\mathfrak{g}^k = V_{k+1} \oplus \mathfrak{g}^{k+1}$$

Then as a vector space

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_p \tag{1}$$

2.5. Proposition. *Let $\mathfrak{g}$ nilpotent of degree $p > 1$ and of dimension $n \geq 3$. Then* $\dim \operatorname{Pder}(\mathfrak{g}) > \dim \operatorname{Der}(\mathfrak{g})$.

Proof. It is well known that $\dim V_1 = \dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) \geq 2$ for a nilpotent Lie algebra. Here V_1 denotes the subspace defined by the above filtration $\mathcal{F}$, i.e., with $\mathfrak{g} = V_1 \oplus [\mathfrak{g},\mathfrak{g}]$. Chose a basis $(e_1,\dots,e_r)$ of V_1, $r \geq 2$. Since $\mathfrak{g}^2$ is strictly contained in $\mathfrak{g}^1$, there exists a commutator of elements in V_1 which is not contained in $\mathfrak{g}^2$. Hence we may assume

$$[e_1, e_2] = e_j$$

such that $e_j \in V_2$ and $(e_1,\dots,e_r,e_{r+1},\dots,e_j,e_{j+1},\dots,e_n)$ is a basis of $\mathfrak{g}$ with $\mathfrak{g}^2 = \operatorname{span}\{e_{j+1},\dots,e_n\}$. Let $Z(\mathfrak{g})$ denote the center of $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent we may choose a non-zero $z \in Z(\mathfrak{g})$. Define a linear map $P : \mathfrak{g} \to \mathfrak{g}$ by

$$P(e_i) = \begin{cases} z & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then P is a prederivation of $\mathfrak{g}$. Indeed, $P([a,[b,c]]) = 0$ for all $a,b,c \in \mathfrak{g}$ by construction, and $[P(a),[b,c]] = [a,[P(b),c]] = [a,[b,P(c)]] = 0$, since $P(\mathfrak{g}) \subset Z(\mathfrak{g})$. On the other hand, assume that this P would be a derivation. Then

$$z = P(e_j) = P([e_1,e_2]) = [P(e_1),e_2] + [e_1,P(e_2)] = 0$$

which is a contradiction. □

2.6. Lemma. *For $P \in \operatorname{Pder}(\mathfrak{g})$ let $\omega(x,y) = P([x,y]) - [x,P(y)] + [y,P(x)]$. Then*

$$[x,\omega(y,z)] + \omega(x,[y,z]) = 0.$$

Proof. Note that $\omega \in B^2(\mathfrak{g},\mathfrak{g})$ is a 2-coboundary for the Lie algebra cohomology with the adjoint module. We have

$$\begin{aligned}
[x,\omega(y,z)] &= [x,P([y,z]) - [x,[y,P(z)]] + [x,[z,P(y)]] \\
\omega(x,[y,z]) &= P([x,[y,z]]) - [x,P([y,z])] + [[y,z],P(x)]
\end{aligned}$$

and hence, since P is a prederivation

$$\begin{aligned}
[x,\omega(y,z)] + \omega(x,[y,z]) &= P([x,[y,z]]) - [P(x),[y,z]] - [x,[P(y),z]] - [x,[y,P(z)]] \\
&= 0
\end{aligned}$$

□

Let us now study Lie algebras admitting a non-singular prederivation.

2.7. *Remark.* There are many reasons why such algebras are interesting. One is the fact that a non-singular derivation D implies an *affine structure* on the Lie algebra via the representation $\theta(x) = D^{-1} \circ \operatorname{ad}(x) \circ D$. The existence of affine structures is a difficult problem with an interesting history. For details see [2]. If there is no such derivation, it is still useful to have a non-singular prederivation. The idea is to construct a bilinear product $x \cdot y := \theta(x)y$ on the Lie algebra by

$$\theta(x) = P^{-1} \circ \operatorname{ad}(x) \circ P + \frac{1}{2} P^{-1} \circ \omega(x)$$

where $\omega(x)y = \omega(x,y)$ as above. If $P \in \operatorname{Der}(\mathfrak{g})$ then $\omega(x) = 0$ and we are back to the classical construction. In general we have always $x \cdot y - y \cdot x = [x,y]$, but θ might not be a representation. However, in many cases this construction with non-singular prederivations yields affine structures. In proposition 2.15 we will study 7-dimensional Lie algebras having only singular derivations. Many of them have a non-singular prederivation. For example, consider

$$\begin{aligned}\mathfrak{g}_{7,5} =&< e_1,\dots,e_7 \mid [e_1,e_i]=e_{i+1}, i=2,3,6\;,\; [e_1,e_4]=e_6+e_7, [e_2,e_3]=e_5,\\ &[e_2,e_5]=e_6, [e_3,e_5]=e_7 >\end{aligned}$$

Then it is easy to verify that the prederivation P defined by $P(e_1)=e_1, P(e_2)=e_2, P(e_3)=2e_3+e_4, P(e_4)=3e_4, P(e_5)=3e_5, P(e_6)=4e_6, P(e_7)=5e_7$ induces an affine structure.

The following generalization of Jacobson's theorem in [6] follows easily from Theorem 1 in [1] by using the same arguments over $k\otimes_{\mathbb{R}}\mathbb{C}$:

2.8. Proposition. *Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic zero admitting a non-singular prederivation. Then $\mathfrak{g}$ is nilpotent.*

In analogy with characteristically nilpotent Lie algebras one might ask whether there exist nilpotent Lie algebras possessing only singular prederivations. This is indeed the case. We define a subclass of characteristically nilpotent Lie algebras as follows:

2.9. Definition. A Lie algebra $\mathfrak{g}$ over k is called *strongly nilpotent* if all its prederivations are nilpotent.

2.10. *Remark.* Any strongly nilpotent Lie algebra is characteristically nilpotent, but the converse is not true in general: consider the following 7-dimensional Lie algebra, defined by

$$\begin{aligned}[e_1,e_i]&=e_{i+1},\; 2\le i\le 6\\ [e_2,e_3]&=e_6+e_7\\ [e_2,e_4]&=e_7\end{aligned}$$

It is not difficult to see that all derivations are nilpotent (see also [9]). On the other hand, $P=\operatorname{diag}(1,3,3,5,5,7,7)$ is a non-singular prederivation. One has $\dim\operatorname{Pder}(\mathfrak{g})=16$ and $\dim\operatorname{Der}(\mathfrak{g})=11$.

The following example presents Lie algebras possessing only nilpotent prederivations.

2.11. Proposition. *Let $\mathfrak{g}$ be the n-dimensional Lie algebra with basis $(e_1,\dots,e_n)$, $n\ge 7$ and defining brackets*

$$\begin{aligned}[e_1,e_i]&=e_{i+1},\; 2\le i\le n-1\\ [e_2,e_3]&=e_{n-1}\\ [e_2,e_4]&=e_n\\ [e_2,e_5]&=-e_n\\ [e_3,e_4]&=e_n\end{aligned}$$

Then $\mathfrak{g}$ is strongly nilpotent.

Proof. Let $P \in \operatorname{Pder}(\mathfrak{g})$ and write

$$P(e_1) = \sum_{i=1}^{n} \alpha_i e_i$$
$$P(e_2) = \sum_{i=1}^{n} \beta_i e_i$$
$$P(e_3) = \sum_{i=1}^{n} \gamma_i e_i$$

Note that $[e_2, e_{n-1}] = 0$ since $n \geq 7$. Using the identity

$$P([e_i,[e_j,e_k]]) = [P(e_i),[e_j,e_k]] + [e_i,[P(e_j),e_k]] + [e_i,[e_j,P(e_k)]]$$

for various (i,j,k) we will obtain that the associated matrix of P is strictly lower-triangular. For $(i,j,k) = (1,1,2), (1,2,4), (2,1,4), (1,2,3), (2,1,3), (3,1,3)$ and $(2,1,2)$ we obtain

$$\begin{aligned}
P(e_4) &= (2\alpha_1 + \beta_2)e_4 + \beta_3 e_5 + \ldots + \beta_{n-4}e_{n-2} + \delta_1 e_{n-1} + \delta_2 e_n \\
0 &= \beta_1 e_6 \\
P(e_n) &= (3\alpha_1 + 2\beta_2)e_n \\
P(e_n) &= -\gamma_1 e_4 + (\alpha_1 + \beta_2 + \gamma_3)e_n \\
P(e_n) &= \gamma_2 e_{n-1} + (3\alpha_1 + 2\beta_2 + \beta_3 - \gamma_4)e_n \\
P(e_n) &= (5\alpha_1 + 2\beta_2)e_n \\
P(e_{n-1}) &= (\alpha_1 + 2\beta_2)e_{n-1} + (\beta_3 - 2\beta_4)e_n
\end{aligned}$$

It follows $\beta_1 = \gamma_1 = \gamma_2 = \alpha_1 = 0$ and $\gamma_3 = \beta_2$. Now $(i,j,k) = (1,1,k)$ for $k = 3, \ldots, n-2$ yields successively

$$\begin{aligned}
P(e_5) &= \beta_2 e_5 + \gamma_4 e_6 + \ldots + (2\alpha_2 + \alpha_3 + \gamma_{n-2})e_n \\
P(e_6) &= \beta_2 e_6 + \beta_3 e_7 + \ldots + (\beta_{n-4} - \alpha_2)e_n \\
P(e_7) &= \beta_2 e_7 + \gamma_4 e_8 + \ldots + \gamma_{n-4}e_n \\
\vdots &= \vdots \\
P(e_n) &= \beta_2 e_n
\end{aligned}$$

Comparing with $P(e_n) = 2\beta_2 e_n$ we obtain $\beta_2 = 0$ and P is nilpotent. $\square$

2.12. *Example.* Let $n = 7$ and $\mathfrak{g}$ as above. Then we see that the algebra $\operatorname{Pder}(\mathfrak{g})$ is given by the set of the following matrices:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_3 & \beta_3 & 0 & 0 & 0 & 0 & 0 \\
\alpha_4 & \frac{\alpha_2}{2} & \beta_3 & 0 & 0 & 0 & 0 \\
\alpha_5 & \beta_5 & \gamma_5 & \beta_3 & 0 & 0 & 0 \\
\alpha_6 & \beta_6 & \gamma_6 & \frac{3\alpha_2}{2} & \beta_3 & 0 & 0 \\
\alpha_7 & \beta_7 & \gamma_7 & \beta_5 - \alpha_3 - \alpha_4 & \gamma_5 + 2\alpha_2 + \alpha_3 & \beta_3 - \alpha_2 & 0
\end{pmatrix}$$

One has $\dim \operatorname{Pder}(\mathfrak{g}) = 13$ and $\dim \operatorname{Der}(\mathfrak{g}) = 10$.

We can also construct a series of characteristically nilpotent Lie algebras which are not strongly nilpotent. Let $\mathfrak{g}_n(\alpha)$ for $\alpha \neq 0$ be the Lie algebras with basis $(e_1, \dots, e_n)$, $n \geq 7$ and defining brackets

$$\begin{aligned} [e_1, e_i] &= e_{i+1},\ 2 \leq i \leq n-1 \\ [e_2, e_3] &= e_5 + \alpha e_n \\ [e_2, e_j] &= e_{j+2},\ 4 \leq j \leq n-2 \end{aligned}$$

Over the complex numbers, $\mathfrak{g}_n(\alpha) \cong \mathfrak{g}_n(1) = \mathfrak{g}_n$. An isomorphism is given by $\varphi(e_i) = \lambda^i e_i$ with a λ satisfying $\lambda^{n-5} = \alpha^{-1}$.

2.13. **Proposition.** *For every $n \geq 7$ the Lie algebra $\mathfrak{g}_n$ is characteristically nilpotent, but not strongly nilpotent.*

Proof. We construct a non-singular prederivation of $\mathfrak{g}_n$ by

$$P(e_i) = \begin{cases} 3e_i + (5-n)e_{n-2} & \text{if } i = 3, \\ 5e_i + (5-n)e_n & \text{if } i = 5, \\ ie_i & \text{otherwise.} \end{cases}$$

Then $\det P = n! \neq 0$. We have to ckeck the identity

$$P([e_i, [e_j, e_k]]) = [P(e_i), [e_j, e_k]] + [e_i, [P(e_j), e_k]] + [e_i, [e_j, P(e_k)]]$$

for $1 \leq i, j, k \leq n$, where we may assume that $j < k$ and $i \leq k$. The identity clearly holds in all cases where $P(e_3)$ or $P(e_5)$ is not involved since then $P(e_i) = ie_i$ and hence $P([e_i, e_j]) = [P(e_i), e_j] + [e_i, P(e_j)]$ for $i, j, i+j \neq 3, 5$. The term $[e_i, [e_j, e_k]]$ belongs to $\operatorname{span}\{e_4, \dots e_n\}$. It equals e_5 for $(i, j, k) = (1, 1, 3)$ or $(i, j, k) = (2, 1, 2)$ in which case the above identity reads as $P(e_5) = 2e_5 + 3e_5 + (5-n)e_n$, respectively $P(e_5 + e_n) = (2+1+2)(e_5 + e_n)$. It remains to ckeck the cases where i, j or k equals 3 or 5. But this is easily done. We have $[e_i, P(e_5)] = 5[e_i, e_5]$ since e_n belongs to the center of $\mathfrak{g}_n$. It holds $[e_i, P(e_3)] = 0$ for $i \geq 3$ and $[e_1, P(e_3)] = 3e_4 + (5-n)e_{n-1}$, $[e_2, P(e_3)] = 3e_5 + 3e_n$.
Now assume that $D \in \operatorname{Der}(\mathfrak{g}_n)$ is a derivation. Write

$$\begin{aligned} D(e_1) &= \sum_{i=1}^{n} \zeta_i e_i \\ D(e_2) &= \sum_{i=1}^{n} \mu_i e_i \end{aligned}$$

Using $D(e_i) = [D(e_1), e_{i-1}] + [e_1, D(e_{i-1})]$ we compute $D(e_3), D(e_4), \dots D(e_n)$ successively. We have

$$\begin{aligned} D(e_3) &= (\zeta_1 + \mu_2)e_3 + \mu_3 e_4 + (\mu_4 - \zeta_3)e_5 + \ldots + (\mu_{n-1} - \zeta_{n-2} - \zeta_3)e_n \\ D(e_4) &= (2\zeta_1 + \mu_2)e_4 + (\zeta_2 + \mu_3)e_5 + \ldots + (\zeta_2 + \mu_{n-2} - \zeta_{n-3})e_n \\ D(e_5) &= (3\zeta_1 + \mu_2)e_5 + (\mu_3 + 2\zeta_2)e_6 + \ldots + (\mu_{n-3} - \zeta_{n-4})e_n \\ \vdots\ &=\ \vdots \\ D(e_n) &= ((n-2)\zeta_1 + \mu_2)e_n \end{aligned}$$

Then $D([e_2,e_3]) = [D(e_2),e_3] + [e_2,D(e_3)]$ is, for $n \ge 7$, equivalent to

$$-\mu_1 e_4 + (2\zeta_1 - \mu_2)e_5 + 2\zeta_2 e_6 + ((n-3)\zeta_1 - \mu_2)e_n = 0,$$

which implies $\mu_1 = 0$, $\mu_2 = 2\zeta_1$, $(n-5)\zeta_1 = 0$ and hence $\mu_1 = \mu_2 = \zeta_1 = 0$. It follows that the matrix for D is strictly lower triangular, hence D is nilpotent. Note that we have used $n \ge 7$ in the computation. For $n \le 6$ there exists always a non-singular derivation of $\mathfrak{g}_n$. □

Let $\mathfrak{g}$ a p-step nilpotent Lie algebra. If $p < 3$ then $\mathfrak{g}$ always admits a non-singular derivation. However there are examples of 3-step nilpotent Lie algebras possessing only nilpotent derivations [3]. The result for prederivations is as follows.

2.14. **Proposition.** *For $p < 5$ any p-step nilpotent Lie algebra admits a non-singular prederivation.*

Proof. For a 2-step nilpotent or abelian Lie algebra, every linear map $P : \mathfrak{g} \to \mathfrak{g}$ is a prederivation. Then the claim is obvious. Hence we may assume that $p = 3$ or $p = 4$, and $\mathfrak{g}^4 = 0$. Then (1) says

$$\mathfrak{g} = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

Let $(e_1, \dots, e_n)$ be a basis of $\mathfrak{g}$ adapted to this decomposition, i.e., which is the union of the bases for V_i. Define a linear map $P : \mathfrak{g} \to \mathfrak{g}$ by

$$P(e_i) = \begin{cases} e_i & \text{if } e_i \in V_1 \text{ or } V_2, \\ 3e_i & \text{if } e_i \in V_3 \text{ or } V_4. \end{cases}$$

In particular $P(e_i) = 3e_i$ for all $e_i \in \mathfrak{g}^2$. Writing $P(e_i) = \zeta_i e_i$ for $i = 1, \dots, n$, P is a prederivation if and only if

$$P([e_i,[e_j,e_k]]) = (\zeta_i + \zeta_j + \zeta_k)[e_i,[e_j,e_k]]$$

for all i,j,k. The term $[e_i,[e_j,e_k]]$ is zero for $e_j \in \mathfrak{g}^2$ or $e_k \in \mathfrak{g}^2$ because of $\mathfrak{g}^4 = 0$. If $e_i \in \mathfrak{g}^2$, then we apply the Jacobi identity to see that

$$[e_i,[e_j,e_k]] = [e_j,[e_i,e_k]] + [e_k,[e_i,e_j]]$$

is contained in $\mathfrak{g}^4$ which is zero. Hence the above identity for P is trivially satisfied if not $e_i, e_j, e_k \in V_1$ or V_2. But then we have $[e_i,[e_j,e_k]] \in \mathfrak{g}^2$ and the identity is equivalent to $P([e_i,[e_j,e_k]]) = 3[e_i,[e_j,e_k]]$. Hence P is a non-singular prederivation. □

How big is the class of strongly nilpotent Lie algebras ? In dimension 7 there are already infinitely many complex non-isomorphic strongly nilpotent Lie algebras:

2.15. **Proposition.** *Any strongly nilpotent complex Lie algebra of dimension 7 is isomorphic to one of the following algebras:*

$$\begin{aligned}
\mathfrak{g}_{7,1} &= < e_1, \dots, e_7 \mid [e_1,e_i] = e_{i+1}, 2 \le i \le 6,\ [e_2,e_3] = e_6, [e_2,e_4] = e_7, \\
&\qquad [e_2,e_5] = e_7, [e_3,e_4] = -e_7 > \\
\mathfrak{g}_{7,4}^{\lambda} &= < e_1, \dots, e_7 \mid [e_1,e_2] = e_3, [e_1,e_3] = e_4, [e_1,e_4] = e_6 + \lambda e_7, [e_1,e_5] = e_7, \\
&\qquad [e_1,e_6] = e_7, [e_2,e_3] = e_5, [e_2,e_4] = e_7, [e_2,e_5] = e_6, [e_3,e_5] = e_7 > \\
\mathfrak{g}_{7,7} &= < e_1, \dots, e_7 \mid [e_1,e_i] = e_{i+1}, i = 2,3,6,\ [e_1,e_4] = e_7, [e_1,e_5] = e_7, \\
&\qquad [e_2,e_3] = e_5, [e_2,e_4] = e_7, [e_2,e_5] = e_6, [e_3,e_5] = e_7 >
\end{aligned}$$

These algebras are pairwise non-isomorphic except for $\mathfrak{g}_{7,4}^{\lambda} \simeq \mathfrak{g}_{7,4}^{-\lambda}$.

Proof. Since a characteristically nilpotent complex Lie algebra of dimension 7 is indecomposable, we may take the classification of such algebras from [9]: the list contains the algebras $\mathfrak{g}_{7,1}, \ldots, \mathfrak{g}_{7,8}$, where $\mathfrak{g}_{7,4}$ depends on a parameter $\lambda \in \mathbb{C}$. Now one has to compute the algebra of prederivations in each case. We did this, but it does not seem useful to write down all the computations here. The following table shows the results:

Algebra	$\dim \mathrm{Der}(\mathfrak{g})$	$\dim \mathrm{Pder}(\mathfrak{g})$	P^{-1} exists
$\mathfrak{g}_{7,1}$	10	13	–
$\mathfrak{g}_{7,2}$	10	13	✓
$\mathfrak{g}_{7,3}$	11	16	✓
$\mathfrak{g}_{7,4}$	10	12	–
$\mathfrak{g}_{7,5}$	10	13	✓
$\mathfrak{g}_{7,6}$	10	13	✓
$\mathfrak{g}_{7,7}$	10	12	–
$\mathfrak{g}_{7,8}$	10	14	✓

The algebras without a non-singular prederivation are in fact strongly nilpotent. □

3. Prederivations of filiform Lie algebras

If $\mathfrak{g}$ is a filiform Lie algebra of dimension n, then there exists an adapted basis $(e_1, \ldots, e_n)$ for $\mathfrak{g}$, see [2]. The brackets of such a filiform Lie algebra with respect to the basis $(e_1, \ldots, e_n)$ are then given by

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, n-1 \tag{2}$$

$$[e_i, e_j] = \sum_{r=1}^{n} \left(\sum_{\ell=0}^{[(j-i-1)/2]} (-1)^{\ell} \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,\, r-j+i+2\ell+1} \right) e_r, \quad 2 \le i < j \le n. \tag{3}$$

with constants $\alpha_{k,s}$ which are zero for all pairs (k,s) not in the index set $\mathcal{I}_n$. Here $\mathcal{I}_n$ is given by

$$\mathcal{I}_n^0 = \{(k,s) \in \mathbb{N} \times \mathbb{N} \mid 2 \le k \le [n/2],\ 2k+1 \le s \le n\},$$
$$\mathcal{I}_n = \begin{cases} \mathcal{I}_n^0 & \text{if } n \text{ is odd,} \\ \mathcal{I}_n^0 \cup \{(\frac{n}{2}, n)\} & \text{if } n \text{ is even.} \end{cases}$$

For filiform Lie algebras we study the conditions for the existence of a non-singular prederivation. As it turns out there are only a few algebras possessing a non-singular prederivation but no non-singular derivation:

3.1. Proposition. *Up to isomorphism there are the following filiform Lie algebras $\mathfrak{g}$ of dimension $n \le 11$ over $\mathbb{C}$ possessing a non-singular prederivation but no non-singular derivation:*

$\dim \mathfrak{g}$	*Algebra*
7	μ_7^4, μ_7^6
8	$\mu_8^{11}(0), \mu_8^{13}$
9	$\mu_9^{28}(0), \mu_9^{30}, \mu_9^{32}, \mu_9^{34}$
10	$\mu_{10}^{8}(0,0,0), \mu_{10}^{15}(0,0), \mu_{10}^{48}$
11	$\mu_{11}^{4}(0,0), \mu_{11}^{16}, \mu_{11}^{71}, \mu_{11}^{62}(0,\beta,0,0), \beta \neq 0,$
	$\mu_{11}^{89}, \mu_{11}^{81}, \mu_{11}^{96a}, \mu_{11}^{101a}$

Here we use the notation from the classification list of [4].

3.2. *Remark.* We have found 2 new filiform Lie algebras $\mu_{11}^{96a}, \mu_{11}^{101a}$ given by

$$\mu_{11}^{96a} = \mu_0 + \Psi_{1,8} + \Psi_{1,10}$$
$$\mu_{11}^{101a} = \mu_0 + \Psi_{1,9} + \Psi_{1,10}$$

It seems that they are not isomorphic to one of the algebras in the list of [4].

Proof. Since the filiform Lie algebras in question are classified, it would be possible to prove the result by calculating the derivations and prederivations separately for each algebra of the classification list. We proceed differently, however. Since we may write any filiform Lie algebra with respect to an adapted basis as in (2) and (3), we can determine the algebras possessing a non-singular prederivation, but no non-singular derivation, with respect to the structure constants $\alpha_{i,j}$. Then we obtain a small list of algebras not necessarily beeing non-isomorphic. The result follows then by determining the isomorphisms between the remaining algebras. In all but two cases (see above) we could easily find an isomorphism to an algebra of the list in [4].
We present the computations for $\dim \mathfrak{g} = 11$. The Lie brackets relative to an adapted basis $(e_1, \dots, e_{11})$ are given by (2) and (3):

$$
\begin{aligned}
[e_1, e_i] &= e_{i+1},\ 2 \le i \le 9 \\
[e_2, e_3] &= \alpha_{2,5}e_5 + \alpha_{2,6}e_6 + \alpha_{2,7}e_7 + \alpha_{2,8}e_8 + \alpha_{2,9}e_9 + \alpha_{2,10}e_{10} + \alpha_{2,11}e_{11} \\
[e_2, e_4] &= \alpha_{2,5}e_6 + \alpha_{2,6}e_7 + \alpha_{2,7}e_8 + \alpha_{2,8}e_9 + \alpha_{2,9}e_{10} + \alpha_{2,10}e_{11} \\
[e_2, e_5] &= (\alpha_{2,5} - \alpha_{3,7})e_7 + (\alpha_{2,6} - \alpha_{3,8})e_8 + (\alpha_{2,7} - \alpha_{3,9})e_9 \\
&\quad + (\alpha_{2,8} - \alpha_{3,10})e_{10} + (\alpha_{2,9} - \alpha_{3,11})e_{11} \\
[e_2, e_6] &= (\alpha_{2,5} - 2\alpha_{3,7})e_8 + (\alpha_{2,6} - 2\alpha_{3,8})e_9 + (\alpha_{2,7} - 2\alpha_{3,9})e_{10} + (\alpha_{2,8} - 2\alpha_{3,10})e_{11} \\
[e_2, e_7] &= (\alpha_{2,5} - 3\alpha_{3,7} + \alpha_{4,9})e_9 + (\alpha_{2,6} - 3\alpha_{3,8} + \alpha_{4,10})e_{10} \\
&\quad + (\alpha_{2,7} - 3\alpha_{3,9} + \alpha_{4,11})e_{11} \\
[e_2, e_8] &= (\alpha_{2,5} - 4\alpha_{3,7} + 3\alpha_{4,9})e_{10} + (\alpha_{2,6} - 4\alpha_{3,8} + 3\alpha_{4,10})e_{11} \\
[e_2, e_9] &= (\alpha_{2,5} - 5\alpha_{3,7} + 6\alpha_{4,9} - \alpha_{5,11})e_{11} \\
[e_3, e_4] &= \alpha_{3,7}e_7 + \alpha_{3,8}e_8 + \alpha_{3,9}e_9 + \alpha_{3,10}e_{10} + \alpha_{3,11}e_{11} \\
[e_3, e_5] &= \alpha_{3,7}e_8 + \alpha_{3,8}e_9 + \alpha_{3,9}e_{10} + \alpha_{3,10}e_{11} \\
[e_3, e_6] &= (\alpha_{3,7} - \alpha_{4,9})e_9 + (\alpha_{3,8} - \alpha_{4,10})e_{10} + (\alpha_{3,9} - \alpha_{4,11})e_{11} \\
[e_3, e_7] &= (\alpha_{3,7} - 2\alpha_{4,9})e_{10} + (\alpha_{3,8} - 2\alpha_{4,10})e_{11} \\
[e_3, e_8] &= (\alpha_{3,7} - 3\alpha_{4,9} + \alpha_{5,11})e_{11}
\end{aligned}
$$

$$\begin{aligned}
[e_4, e_5] &= \alpha_{4,9}e_9 + \alpha_{4,10}e_{10} + \alpha_{4,11}e_{11} \\
[e_4, e_6] &= \alpha_{4,9}e_{10} + \alpha_{4,10}e_{11} \\
[e_4, e_7] &= (\alpha_{4,9} - \alpha_{5,11})e_{11} \\
[e_5, e_6] &= \alpha_{5,11}e_{11}
\end{aligned}$$

The Jacobi identity is satisfied if and only if:

$$\begin{aligned}
0 &= \alpha_{4,9}(2\alpha_{2,5} + \alpha_{3,7}) - 3\alpha_{3,7}^2 \\
0 &= \alpha_{4,10}(2\alpha_{2,5} + \alpha_{3,7}) + 3\alpha_{4,9}(\alpha_{2,6} + \alpha_{3,8}) - 7\alpha_{3,7}\alpha_{3,8} \\
0 &= \alpha_{5,11}(2\alpha_{2,5} - \alpha_{3,7} - \alpha_{4,9}) + \alpha_{4,9}(6\alpha_{4,9} - 4\alpha_{3,7}) \\
0 &= \alpha_{4,12}(2\alpha_{2,7} + \alpha_{3,9}) + \alpha_{4,11}(2\alpha_{2,5} + \alpha_{3,7}) + 3\alpha_{4,10}(\alpha_{2,6} + \alpha_{3,8}) - 4\alpha_{3,8}^2 \\
&\quad + 2\alpha_{4,9}(2\alpha_{2,7} + 3\alpha_{3,9}) - 8\alpha_{3,7}\alpha_{3,9}
\end{aligned}$$

We will denote the fact that there exists a non-singular prederivation $P \in \mathrm{Pder}(\mathfrak{g})$ simply by "P^{-1} exists". We have to distinguish several cases. Consider first the case

$$2\alpha_{2,5} + \alpha_{3,7} = 0$$

If $\alpha_{4,9} \neq 0$ then P^{-1} exists if and only if

$$\begin{aligned}
\alpha_{2,6} &= 0 \\
\alpha_{2,8} &= 0 \\
\alpha_{3,11} &= (\alpha_{3,10}\alpha_{4,10} + 4\alpha_{4,9}\alpha_{2,9} + \alpha_{3,9}^2)/\alpha_{4,9} \\
\alpha_{4,11} &= (\alpha_{4,10}^2 + 6\alpha_{4,9}\alpha_{3,9})/\alpha_{4,9}
\end{aligned}$$

A non-singular derivation D^{-1} exists if and only if the above conditions are satisfied. Hence in this case "P^{-1} exists if and only if D^{-1} exists".
In the following let $\alpha_{4,9} = 0$. For $\alpha_{2,6}, \alpha_{3,8} \neq 0$ we have: P^{-1} exists if and only if D^{-1} exists.
For $\alpha_{2,6} \neq 0$, $\alpha_{3,8} = 0$ P^{-1} exists if and only if

$$\begin{aligned}
\alpha_{2,i} &= 0, \; i = 7, 9, 11 \\
\alpha_{2,10} &= 4\alpha_{2,8}^2/(3\alpha_{2,6}) \\
\alpha_{3,i} &= 0, \; i = 7, 8, 9, 10, 11 \\
\alpha_{4,i} &= 0, \; i = 9, 10, 11 \\
\alpha_{5,11} &= 0
\end{aligned}$$

However D^{-1} exists if and only if the above conditions hold *and* $\alpha_{2,11} = 0$. Hence we have the following algebra, given by 16 structure constants $(\alpha_{2,5}, \alpha_{2,6}, \dots, \alpha_{5,11})$ as follows:

$$\left(0, a_2, 0, a_4, 0, \frac{4a_4^2}{3a_2}, a_7, 0, 0, 0, 0, 0, 0, 0, 0, 0\right)$$

where $a_2, a_7 \neq 0$. This algebra is isomorphic over $\mathbb{C}$ to μ_{11}^{71} given by

$$(0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

In the following let $\alpha_{2,6}=0$. Consider the case $\alpha_{2,7},\alpha_{3,9}\neq 0$. If $\alpha_{4,11}\neq 2\alpha_{3,9}$ then P^{-1} exists if and only if D^{-1} exists. If $\alpha_{4,11}=2\alpha_{3,9}$ then P^{-1} exists if and only if

$$\begin{aligned}\alpha_{2,8}&=0\\ \alpha_{3,10}&=0\\ \alpha_{3,11}&=\alpha_{4,11}\alpha_{2,9}/\alpha_{3,9}\\ \alpha_{4,10}&=0\\ \alpha_{5,11}&=0\end{aligned}$$

On the other hand D^{-1} exists if and only if in addition $\alpha_{2,11}=\alpha_{2,9}^2/\alpha_{3,9}$. We obtain the algebra

$$(0,0,a_3,0,a_5,a_6,a_7,0,0,a_{10},0,2a_5,0,0,2a_{10},0)$$

where $a_3\neq 0, a_7\neq a_5^2/a_{10}$. It is isomorphic over $\mathbb{C}$ to $\mu_{11}^{62}(0,\beta,0,0),\beta=a_3/(2a_{10})\neq 0$ given by

$$(0,0,\beta,0,0,0,1,0,0,\frac{1}{2},0,0,0,0,1,0)$$

Consider the case $\alpha_{2,7}\neq 0,\alpha_{3,9}=0$. If $\alpha_{4,11}\neq 0$ then P^{-1} exists if and only if D^{-1} exists. If $\alpha_{4,11}=0$ then P^{-1} exists if and only if

$$\begin{aligned}\alpha_{2,i}&=0,\ i=8,9\\ \alpha_{3,i}&=0,\ i=7,8,10,11\\ \alpha_{4,10}&=0\\ \alpha_{5,11}&=0\end{aligned}$$

whereas D^{-1} exists if and only in addition $\alpha_{2,11}=0$. We obtain the algebra

$$(0,0,a_3,0,0,a_6,a_7,0,0,0,0,0,0,0,0,0)$$

where $a_3,a_7\neq 0$. It is isomorphic over $\mathbb{C}$ to μ_{11}^{81} given by

$$(0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0)$$

In the following let $\alpha_{2,7}=0$. If $\alpha_{3,8}\neq 0$ then P^{-1} exists if and only if D^{-1} exists. This is also true for $\alpha_{3,8}=0,\alpha_{3,9}\neq 0$. Assume in the following $\alpha_{3,8}=\alpha_{3,9}=0$. If $\alpha_{2,8}\neq 0$ then P^{-1} exists if and only if D^{-1} exists except for the case where $\alpha_{3,10}=0,\alpha_{2,11}\neq 0$. We obtain the algebra

$$(0,0,0,a_4,0,0,a_7,0,0,0,0,0,0,0,0,0)$$

where $a_4,a_7\neq 0$. It is isomorphic over $\mathbb{C}$ to μ_{11}^{89} given by

$$(0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0)$$

Assume in the following $\alpha_{2,8}=0$. If $\alpha_{3,11}\neq 0$ then P^{-1} exists if and only if D^{-1} exists. Assume $\alpha_{3,11}=0$. If $\alpha_{2,9}\neq 0$ then P^{-1} exists if and only if

$$\begin{aligned}\alpha_{2,10}&=0\\ \alpha_{3,10}&=0\\ \alpha_{4,i}&=0,\ i=10,11\\ \alpha_{5,11}&=0\end{aligned}$$

However D^{-1} exists if and only if in addition $\alpha_{2,11}=0$. We obtain the algebra

$$(0,0,0,0,a_5,0,a_7,0,0,0,0,0,0,0,0,0)$$

where $a_5, a_7 \neq 0$. It is isomorphic over $\mathbb{C}$ to

$$(0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0)$$

For $\alpha_{2,9} = 0$ we have always that P^{-1} exists if and only if D^{-1} exists except for the algebra

$$(0,0,0,0,0,a_6,a_7,0,0,0,0,0,0,0,0,0)$$

where $a_6, a_7 \neq 0$. It is isomorphic over $\mathbb{C}$ to

$$(0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0)$$

To finish the proof we have to consider the case

$$2\alpha_{2,5} + \alpha_{3,7} \neq 0$$

If $\alpha_{2,5} = 0, \alpha_{3,7} \neq 0$ then P^{-1} exists if and only if D^{-1} exists. If $\alpha_{2,5} \neq 0, \alpha_{3,7} = 0$ then we obtain an algebra which is isomorphic to μ_{11}^{16}, given by

$$(1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0)$$

Assume in the following $\alpha_{2,5}, \alpha_{3,7} \neq 0$. Then P^{-1} exists if and only if D^{-1} exists except for the case where $7\alpha_{3,7} = 4\alpha_{2,5}, 5\alpha_{3,7}^2 - 6\alpha_{2,5}\alpha_{3,7} + 4\alpha_{2,5} \neq 0$. In that case we obtain an algebra being isomorphic to $\mu_{11}^4(0,0)$ given by

$$(1,0,0,0,0,0,1,\frac{4}{7},0,0,0,0,\frac{8}{21},0,0,0)$$

That concludes our proof. $\square$

We have also studied the existence question of non-singular prederivations for certain filiform algebras of dimension $n \geq 12$. We will consider the following conditions, which are isomorphism invariants of $\mathfrak{g}$:

(a) $\mathfrak{g}$ contains no one-codimensional subspace $U \supseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$.
(b) $\mathfrak{g}^{\frac{n-4}{2}}$ is abelian, if n is even.
(c) $[\mathfrak{g}^1, \mathfrak{g}^1] \subseteq \mathfrak{g}^6$.

We will focus on algebras satifying property (a):

3.3. **Definition.** Let $\mathcal{A}_n^1$ denote the set of n-dimensional filiform laws whose algebras satisfy the properties $(a),(b),(c)$. Denote by $\mathcal{A}_n^2$ the set of n-dimensional filiform laws whose algebras satisfy $(a),(b)$, but *not* (c).

The above properties of $\mathfrak{g}$ can be expressed in terms of the corresponding structure constants $\alpha_{k,s}$. It is easy to verify the following (use (2), (3)):

$\alpha_{2,5} \neq 0$, if and only $\mathfrak{g}$ satisfies property (a).
$\alpha_{\frac{n}{2},n} = 0$, if and only if $\mathfrak{g}$ satisfies property (b).
$\alpha_{3,7} = 0$, if and only if $\mathfrak{g}$ satisfies property (c).

If $\mathfrak{g}$ satisfies property (a) we may change the adpated basis so that it stays adapted and

$$\alpha_{2,5} = 1.$$

In fact, we may take $f \in GL(\mathfrak{g})$ defined by $f(e_1) = ae_1, f(e_2) = be_2$ and $f(e_i) = [f(e_1), f(e_{i-1})]$ for $3 \leq i \leq n$ with suitable nonzero constants a and b.
The following results follow by straightforward computation:

3.4. **Proposition.** *Let $\mathfrak{g}$ be a filiform Lie algebra with law in $\mathcal{A}_n^1$. Then $\mathfrak{g}$ admits a non-singular prederivation if and only if*

$$\alpha_{3,i} = 0, \quad i = 8, \dots, n$$
$$\alpha_{2,j} = \frac{1}{2^{j-5}(j-3)}\binom{2j-8}{j-4}\alpha_{2,6}^{j-5}, \quad j = 7, \dots, n-1$$

On the other hand, $\mathfrak{g}$ admits a non-singular derivation if and only if in addition

$$\alpha_{2,n} = \frac{1}{2^{n-5}(n-3)}\binom{2n-8}{n-4}\alpha_{2,6}^{n-5}$$

Note that the formula contains the Catalan numbers

$$C_m = \frac{1}{m+1}\binom{2m}{m}$$

3.5. **Proposition.** *Let $\mathfrak{g}$ be a filiform Lie algebra with law in $\mathcal{A}_n^2$. Then $\mathfrak{g}$ admits a non-singular prederivation if and only if it admits a non-singular derivation.*

Acknowledgement: I am grateful to the referee for helpful remarks.

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