ESTIMATES ON BINOMIAL SUMS OF PARTITION FUNCTIONS

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Abstract. Let $p(n)$ denote the partition function and define $p(n,k) = \sum_{j=0}^{k} \binom{n-j}{k-j} p(j)$ where $p(0) = 1$. We prove that $p(n,k)$ is unimodal and satisfies $p(n,k) < \frac{2.825}{\sqrt{n}} 2^n$ for fixed $n \geq 1$ and all $1 \leq k \leq n$. This result has an interesting application: the minimal dimension of a faithful module for a k-step nilpotent Lie algebra of dimension n is bounded by $p(n,k)$ and hence by $\frac{3}{\sqrt{n}} 2^n$, independently of k. So far only the bound n^{n-1} was known. We will also prove that $p(n,n-1) < \sqrt{n} \exp(\pi\sqrt{2n/3})$ for $n \geq 1$ and $p(n-1,n-1) < \exp(\pi\sqrt{2n/3})$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra of dimension n over a field K of characteristic zero. An invariant of $\mathfrak{g}$ is defined by

$$\mu(\mathfrak{g}) := \min\{\dim M \mid M \text{ is a faithful } \mathfrak{g}\text{–module}\}$$

Ado's theorem asserts that $\mu(\mathfrak{g})$ is finite. Following the details of the proof we see that $\mu(\mathfrak{g}) \leq f(n)$ for a function f only depending on n. It is an open problem to determine good upper bounds for $f(n)$ valid for a given class of Lie algebras of dimension n. Interest for such a refinement of Ado's theorem comes from a question of Milnor on fundamental groups of complete affine manifolds [6]. The existence of left-invariant affine structures on a Lie group G of dimension n implies $\mu(\mathfrak{g}) \leq n+1$ for its Lie algebra $\mathfrak{g}$. It is known that there exist nilpotent Lie algebras which do not satisfy this bound [5]. It is however difficult to prove good bounds for $\mu(\mathfrak{g})$ only depending on $\dim \mathfrak{g}$. In 1937 Birkhoff [3] proved $\mu(\mathfrak{g}) \leq 1 + n + n^2 + \cdots + n^{k+1}$ for all nilpotent Lie algebras $\mathfrak{g}$ of dimension n and nilpotency class k. His construction used the universal enveloping algebra of $\mathfrak{g}$. In 1969 this method was slightly improved by Reed [7] who proved $\mu(\mathfrak{g}) \leq 1 + n^k$. That yields the bound $\mu(\mathfrak{g}) \leq 1 + n^{n-1}$ only depending on n. We have improved the bound in [4] as follows:

1.1. Theorem. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n and nilpotency class k. Denote by $p(n)$ the number of partitions of n into positive integers with $p(0) = 1$ and set*

$$p(n,k) = \sum_{j=0}^{k} \binom{n-j}{k-j} p(j).$$

Then $\mu(\mathfrak{g}) \leq p(n,k)$.

The aim of this paper is to study the function $p(n,k)$ and to give upper bounds for it. We will show the following:

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1.2. **Theorem.** *The function $p(n,k)$ is unimodal for fixed $n \geq 4$. More precisely we have with $k(n) = \lfloor \frac{n+3}{2} \rfloor$*

$$p(n,1) < p(n,2) < \cdots < p(n,k(n)-1) < p(n,k(n)),$$
$$p(n,k(n)) > p(n,k(n)+1) > \cdots > p(n,n-1) > p(n,n).$$

1.3. **Theorem.** *There is the following estimate for $p(n,k)$:*

$$p(n,k) < \frac{2.825}{\sqrt{n}}\, 2^n \quad \textit{for fixed } n \geq 1 \textit{ and all } 1 \leq k \leq n$$

1.4. **Corollary.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. Then*

$$\mu(\mathfrak{g}) < \frac{3}{\sqrt{n}}\, 2^n$$

A nilpotent Lie algebra $\mathfrak{g}$ of dimension n and nilpotency class k is called *filiform* if $k = n-1$. In that case the estimate for $\mu(\mathfrak{g})$ can be improved. In fact it holds $\mu(\mathfrak{g}) \leq 1 + p(n-2, n-2)$ which was the motivation to prove the following propositions:

1.5. **Proposition.** *Let $\alpha = \sqrt{\frac{2}{3}}\pi$. Then*

$$p(n-1,n-1) < e^{\alpha\sqrt{n}} \quad \textit{for all } \; n \geq 1.$$

1.6. **Proposition.** *Let $\alpha = \sqrt{\frac{2}{3}}\pi$. Then*

$$p(n,n-1) < \sqrt{n} e^{\alpha\sqrt{n}} \quad \textit{for all } \; n \geq 1.$$

1.7. *Remark.* If $k, n \to \infty$ with $\frac{k}{n} \leq 1-\delta$ for some fixed $\delta > 0$ then one has asymptotically

$$p(n,k) \sim \binom{n}{k} \prod_{j=1}^{\infty} \frac{1}{1-(\frac{k}{n})^j}.$$

For $k/n = 1/2$ the infinite product is approximately 3.4627466194550636. The theorem shows that $\mu(\mathfrak{g}) \leq p(n,k)$ is a better estimate than $\mu(\mathfrak{g}) \leq 1 + n^k$, especially if k is not small in comparison to n. As for a bound for $\mu(\mathfrak{g})$ independent of k, the corollary yields a better one than n^{n-1}. Note that some of the estimates on $p(n,k)$ have been stated in [4], where the proof of Lemma 5 is not complete. In fact, the upper bound given there for $p(n,n-1)$ depends on a strong upper bound for $p(n)$ itself, which so far is not proved. Using the known upper bound for $p(n)$ in [2] however it is not difficult to prove the above estimates.

We have included a table which shows the values for $p(k)$ and $p(n,k)$ for $n = 50$ and $1 \leq k \leq 50$. I thank Michael Stoll for helpful discussions.

k	$p(k)$	$p(50,k)$
1	1	51
2	2	1276
3	3	20875
4	5	251126
5	7	2368708
6	11	18240890
7	15	117911248
8	22	652850403
9	30	3143939547
10	42	13327191287
11	56	50207862055
12	77	169422173829
13	101	515401493777
14	135	1421191021907
15	176	3568459118188
16	231	8190773240690
17	297	17243902126004
18	385	33393294003697
19	490	59630690096752
20	627	98399515067097
21	792	150323197512416
22	1002	212938456376977
23	1255	280067870621181
24	1575	342413939297475
25	1958	389526824102747
26	2436	412637434996367
27	3010	407312833046180
28	3718	374834739612319
29	4565	321717177399531
30	5604	257604118720316
31	6842	192465300826581
32	8349	134186828954271
33	10143	87302345518136
34	12310	52999252173708
35	14883	30018139013576
36	17977	15859467681399
37	21637	7814276022624
38	26015	3589870410395
39	31185	1537270615509
40	37338	613479208559
41	44583	228106170152
42	53174	79012160892
43	63261	25493798901
44	75175	7662394094
45	89134	2145558341
46	105558	559858427
47	124754	136194920
48	147273	30906004
49	173525	6547151
50	204226	1295971

2. Unimodality

2.1. Definition. Let f be a sequence and define

$$F(n,\ell) := \sum_{j=0}^{n} \binom{n-j}{\ell} f(j)$$

for $0 \le \ell \le n$, where the binomial coefficient is understood to be zero if $n-j<\ell$. Then $F(n,\ell)$ is called *unimodal*, if there exists a sequence K with $K(n) \le K(n+1) \le K(n)+1$ such that for all $n \ge 0$

$$F(n,0) < F(n,1) < F(n,2) < \cdots < F(n,K(n)-1) \le F(n,K(n)),$$
$$F(n,K(n)) > F(n,K(n)+1) > \cdots > F(n,n-1) > F(n,n) > F(n,n+1) = 0.$$

2.2. Example. *If $f(n)=1$ for all $n \ge 0$, then*

$$F(n,\ell) = \sum_{j=0}^{n} \binom{n-j}{\ell} = \binom{n+1}{\ell+1}$$

is unimodal. Setting $\ell = n-k$ and using $\binom{n-j}{k-j} = \binom{n-j}{n-k}$ we may rewrite the sum as

$$\sum_{j=0}^{n} \binom{n-j}{k-j} = \binom{n+1}{k}$$

In general $F(n,\ell)$ will only be unimodal if we impose a certain restriction on the growth of $f(n)$. Before we give a criterion we note that the recursion for the binomial coefficients implies the following lemma:

2.3. Lemma. *Let $F(n,n+1)=0$. For $1 \le \ell \le n$ it holds*

$$F(n+1,\ell) = F(n,\ell) + F(n,\ell-1) \tag{1}$$

$$F(n+1,\ell+1) - F(n+1,\ell) = F(n,\ell+1) - F(n,\ell-1) \tag{2}$$

Proof.

$$\begin{aligned} F(n,\ell) + F(n,\ell-1) &= \sum_{j=0}^{n} \left(\binom{n-j}{\ell} + \binom{n-j}{\ell-1} \right) f(j) \\ &= \sum_{j=0}^{n} \binom{n+1-j}{\ell} f(j) \\ &= F(n+1,\ell) \end{aligned}$$

Substituting $\ell+1$ for ℓ in (1) yields $F(n+1,\ell+1) = F(n,\ell+1) + F(n,\ell)$ so that the difference yields (2). □

2.4. Proposition. *Let f be a sequence satisfying*

(a) *$f(n) > 0$ for all $n \ge 0$ and $f(3) \le 2f(0) + f(1)$.*
(b) *$f(n+1) \ge f(n)$ for all $n \ge 0$.*
(c) *$f(n) < \sum_{j=0}^{n-1} f(j)$ for all $n \ge 3$.*

Then $F(n,\ell) = \sum_{j=0}^{n} \binom{n-j}{\ell} f(j)$ is unimodal.

Proof. The result follows by induction on n. For $n \leq 3$ one directly obtains $K(0) = K(1) = 0$, $K(2) = 0, 1$ and $K(3) = 1$ by $(a), (b), (c)$. For example, if $n = 3$ then $F(3,0) \leq F(3,1) > F(3,2) > F(3,3) > 0$ says

$$f(0) + f(1) + f(2) + f(3) \leq 3f(0) + 2f(1) + f(2) > 3f(0) + f(1) > f(0) > 0$$

which follows from the assumptions. Assuming for n

$$F(n,0) < F(n,1) < F(n,2) < \cdots < F(n,K(n)-1) \leq F(n,K(n)),$$
$$F(n,K(n)) > F(n,K(n)+1) > \cdots > F(n,n-1) > F(n,n) > F(n,n+1) = 0.$$

we obtain for $n+1$ using the recursion (2):

$$F(n+1,1) < F(n+1,2) < \cdots < F(n+1,K(n)),$$
$$F(n+1,K(n)+1) > F(n+1,K(n)+1) > \cdots > F(n+1,n) > F(n+1,n+1) > 0.$$

If $F(n+1,K(n)) \leq F(n+1,K(n)+1)$ we set $K(n+1) = K(n)+1$, and otherwise $K(n+1) = K(n)$. It remains to show that $F(n+1,0) < F(n+1,1)$. But since $F(n+1,0) = F(n,0) + f(n+1)$ and $K(n) \geq 1$ for $n \geq 3$ we have

$$\begin{aligned} F(n+1,1) - F(n+1,0) &= F(n,1) - f(n+1) \\ &\geq F(n,0) - f(n+1) \\ &= f(0) + f(1) + \cdots + f(n) - f(n+1) > 0 \end{aligned}$$

by assumption (c). □

2.5. **Corollary.**

$$P(n,n-k) = \sum_{j=0}^{n} \binom{n-j}{n-k} p(j) = p(n,k)$$

is unimodal with $0 \leq k \leq n$.

Proof. We can apply the proposition since the partition function $p(n)$ satisfies conditions $(a), (b), (c)$. Here only (c) is non-trivial. In fact, it is well known that

$$p(n) \leq p(n-1) + p(n-2)$$

for all $n \geq 2$, i.e., that $p(n)$ is a "sub-Fibonacci" sequence. If we set $\ell = n-k$, then $0 \leq k \leq n$ and $P(n,n-k)$ is unimodal. □

3. Lemmas on $p(n,k)$

For the proof of the theorems we need some lemmas.

3.1. **Lemma.** *Denote by* $p_k(j)$ *the number of those partitions of* j *in which each term does not exceed* k*. If* $|q| < 1$ *then*

$$\sum_{j=0}^{\infty} p_k(j) q^j = \prod_{j=1}^{k} \frac{1}{1-q^j} \tag{3}$$

$$\sum_{j=0}^{\infty} j p_k(j) q^j = \sum_{j=1}^{k} \frac{jq^j}{1-q^j} \cdot \prod_{j=1}^{k} \frac{1}{1-q^j} \tag{4}$$

Proof. The first identity is well known, see for example [1], Theorem 13-1. The product is the generating function for $p_k(n)$. The second identity follows from the first by differentiation. Denoting $F_k(q) = \prod_{j=1}^k \frac{1}{1-q^j}$ we have

$$q \cdot \frac{d}{dq} F_k(q) = \sum_{j=1}^{k} \frac{jq^j}{1-q^j} \cdot F_k(q)$$
$$\sum_{j=0}^{\infty} jp_k(j)q^j = q \cdot \frac{d}{dq} \sum_{j=0}^{\infty} p_k(j)q^j$$

□

In the following we will need good upper bounds for the infinite product

$$F(q) := \prod_{j=1}^{\infty} \frac{1}{1-q^j}.$$

$F(q)$ is directly related to the Dedekind eta-function, which is defined on the upper half plane $\mathbb{H}$ as

$$\eta(z) := q^{\frac{1}{24}} \prod_{j=1}^{\infty} (1-q^j),$$

where $q := e^{2\pi i z}$. To obtain that approximately $F(\frac{1}{2}) = 3.4627466194550636$, we could use $F(\frac{1}{2}) = (\frac{1}{2})^{\frac{1}{24}} \cdot \eta(z)^{-1}$ with $z = \frac{i \log 2}{2\pi}$. The eta-function can be computed by many computer algebra systems. On the other hand, it is not difficult to estimate the product directly.

3.2. **Lemma.** *For* $0 < q < 1$ *and* $\ell \geq 2$ *we have*

$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} < \exp\left(\frac{q^\ell}{(1-q)^2}\right) \cdot \prod_{j=1}^{\ell-1} \frac{1}{1-q^j} \tag{5}$$

$$\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} < \frac{q}{(1-q)^3} + \sum_{j=1}^{\ell-1} \frac{jq^j(q^j-q)}{(1-q^j)(1-q)} \tag{6}$$

Proof. By the mean value theorem there exists a τ_j with $1-q \leq 1-q^j < \tau_j < 1$ such that $\log \frac{1}{1-q^j} = -\log(1-q^j) = \tau_j^{-1} q^j$ for all $j \geq 1$. Hence

$$\log \frac{1}{1-q^j} < \frac{q^j}{1-q}$$

for all $j \geq 1$ and

$$\sum_{j=\ell}^{\infty} \log \frac{1}{1-q^j} < \sum_{j=\ell}^{\infty} \frac{q^j}{1-q} = \frac{q^\ell}{(1-q)^2}.$$

Taking exponentials on both sides yields

$$\prod_{j=\ell}^{\infty} \frac{1}{1-q^j} < \exp \frac{q^\ell}{(1-q)^2}.$$

This proves (5). To show the second inequality we again use $1-q \le 1-q^j$ and

$$\sum_{j=1}^{\infty} \frac{jq^j}{1-q} = \frac{q}{(1-q)^3}$$

so that

$$\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} = \sum_{j=1}^{\ell-1} \frac{jq^j}{1-q^j} + \sum_{j=\ell}^{\infty} \frac{jq^j}{1-q^j} < \sum_{j=1}^{\ell-1} \frac{jq^j}{1-q^j} - \sum_{j=1}^{\ell-1} \frac{jq^j}{1-q} + \frac{q}{(1-q)^3}$$

□

3.3. **Lemma.** *For* $n \ge 4$ *and* $k = \lfloor \frac{n+3}{2} \rfloor$ *it holds*

$$\sum_{j=0}^{k} (n+1-2k+j) \binom{n-j}{k-j} p(j) > 0 \tag{7}$$

Proof. If n is even, then $k = \frac{n+2}{2}$ and $n+1-2k+j = j-1$. Since for $j \ge 2$ all terms of the sum are positive, it is enough to estimate the sum of the first 4 terms:

$$\begin{aligned}\sum_{j=0}^{3} (j-1) \binom{n-j}{k-j} p(j) &= \binom{n}{k} \sum_{j=0}^{3} (j-1)\, a_{n,k,j}\, p(j) \\ &= \binom{n}{k} \left(-1 + 2 \cdot \frac{k(k-1)}{n(n-1)} + 6 \cdot \frac{k(k-1)(k-2)}{n(n-1)(n-2)} \right) \\ &= \binom{n}{k} \left(\frac{n+14}{4(n-1)} \right) > 0\end{aligned}$$

for $k \ge 3$ and $n \ge 4$, where

$$a_{n,k,j} := \binom{n-j}{k-j} \binom{n}{k}^{-1} = \frac{k!(n-j)!}{n!(k-j)!}$$

If n is odd, then $k = \frac{n+3}{2}$ and $n+1-2k+j = j-2$. Now the sum in (7) contains two negative terms and one needs the first 8 terms:

$$\sum_{j=0}^{7} (j-2) \binom{n-j}{k-j} p(j) = \binom{n}{k} \left(\frac{5(11n^4 + 120n^3 - 2966n^2 + 9864n + 10251)}{128n(n-2)(n-4)(n-6)} \right) > 0.$$

for $k \ge 7$, $n \ge 11$. For $4 \le n \le 10$ the sum in (7) is also positive. □

In the same way we obtain:

3.4. **Lemma.** *Let* $\ell = \lfloor \frac{n+5}{2} \rfloor$. *For all* $n \ge 4$ *and all* k *with* $\ell \le k \le n$ *it holds*

$$p(n,k) > \frac{1745}{512} \binom{n}{k}$$

Proof. Since $a_{n,k,j} \ge a_{n,\ell,j}$ for $k \ge \ell$ we have for $n \ge 18$

$$p(n,k) \ge \binom{n}{k} \sum_{j=0}^{\ell} a_{n,\ell,j} p(j) \ge \binom{n}{k} \sum_{j=0}^{11} a_{n,\ell,j} p(j) \ge \frac{1745}{512} \binom{n}{k}$$

For $4 \le n \le 18$ the lemma is true also. □

3.5. Lemma. *For* $n \geq 4$ *and* $k = \lfloor \frac{n+3}{2} \rfloor + 1$ *it holds*

$$\sum_{j=0}^{k} (n+1-2k+j) \binom{n-j}{k-j} p(j) < 0 \tag{8}$$

Proof. If n is even, then $k = \frac{n+4}{2}$. We can rewrite (8) as

$$\sum_{j=0}^{k} j \binom{n-j}{k-j} p(j) < 3p(n,k) \tag{9}$$

It can be checked by computer that the inequality holds for small n. We have used the computer algebra package Pari to verify it. So we may assume, let us say, $n \geq 500$. Then $\frac{k}{n} \leq q = \frac{252}{500}$ for all $n \geq 500$ and hence

$$a_{n,k,j} = \frac{k}{n} \cdot \frac{k-1}{n-1} \cdots \frac{k-j+1}{n-j+1} \leq \left(\frac{k}{n}\right)^j \leq q^j.$$

It follows

$$\begin{aligned} \sum_{j=0}^{k} j \binom{n-j}{k-j} p(j) &= \binom{n}{k} \sum_{j=0}^{k} j\, a_{n,k,j}\, p(j) < \binom{n}{k} \sum_{j=0}^{k} j p(j) q^j \\ &< \binom{n}{k} \sum_{j=0}^{\infty} j p_k(j) q^j < 9.96868 \binom{n}{k} \end{aligned}$$

The last inequality follows from Lemma 3.1 and Lemma 3.2: for $q = \frac{252}{500}$ we have

$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} < 3.54029829$$

$$\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} < 2.81577392$$

and therefore

$$\sum_{j=0}^{\infty} j p_k(j) q^j \leq \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} \cdot \prod_{j=1}^{\infty} \frac{1}{1-q^j} < 9.96867959.$$

On the other hand we have by Lemma 3.4

$$3p(n,k) > \frac{5235}{512} \binom{n}{k}$$

so that inequality (9) follows. If n is odd then $k = \frac{n+5}{2}$ and the proof works as before. □

3.6. Lemma. *For* $1 \leq k \leq n-1$ *it holds*

$$p(n,k) < \binom{n}{k} \prod_{j=1}^{\infty} \frac{1}{1-(\frac{k}{n})^j}. \tag{10}$$

Proof. Estimating as in the preceding lemma and applying Lemma 3.1 we have

$$p(n,k) \leq \binom{n}{k} \sum_{j=0}^{k} \left(\frac{k}{n}\right)^j p_k(j) < \binom{n}{k} \prod_{j=1}^{\infty} \frac{1}{1-(\frac{k}{n})^j}$$

□

4. Proof of the Theorems

Proof of Theorem 1.2: Assume that $n \geq 4$ is fixed. We have already proved that $p(n,k)$ is unimodal. What we must show is that $p(n,k)$ becomes maximal exactly for $k = \lfloor \frac{n+3}{2} \rfloor$. We formulate this as two lemmas:

4.1. **Lemma.** *For $n \geq 4$ and $1 \leq k \leq \lfloor \frac{n+3}{2} \rfloor$ we have*

$$p(n,k-1) < p(n,k)$$

Proof. Using $p(n+1,k) - p(n,k) = p(n,k-1)$ we see that we have to prove $2p(n,k) - p(n+1,k) > 0$ which is equivalent to the following inequality:

$$\sum_{j=0}^{k} (n+1-2k+j) \binom{n-j}{k-j} p(j) > 0.$$

But this is obvious for $1 \leq k \leq \lfloor \frac{n+3}{2} \rfloor - 1$, because in that case the sum has for $j \geq 1$ only positive terms and the first term with $j = 0$ is nonnegative. For $k = \lfloor \frac{n+3}{2} \rfloor$ there exist negative terms, but the claim follows from Lemma 3.3.

□

4.2. **Lemma.** *For $n \geq 4$ and $\lfloor \frac{n+3}{2} \rfloor + 1 \leq k \leq n$ we have*

$$p(n,k-1) > p(n,k) \tag{11}$$

Proof. The inequality is equivalent to

$$\sum_{j=0}^{k} (n+1-2k+j) \binom{n-j}{k-j} p(j) < 0.$$

For $k = \lfloor \frac{n+3}{2} \rfloor + 1$ it follows from Lemma 3.5. We can now apply the unimodality of $p(n,k)$, see Corollary 2.5, to obtain the lemma. □

Proof of Theorem 1.3: For $n < 500$ the theorem can be checked by computer. Using Sterling's formula we obtain

$$\binom{n}{\lfloor \frac{n+3}{2} \rfloor} < \frac{2^n}{\sqrt{\pi n/2}}$$

for all $n \geq 1$ and hence with $q = 252/500$, $k(n) = \lfloor \frac{n+3}{2} \rfloor$, $n \geq 500$

$$p(n,k) \leq p(n,k(n)) < \binom{n}{k(n)} \prod_{j=1}^{\infty} \frac{1}{1-q^j} < 3.54029829 \cdot \frac{2^n}{\sqrt{\pi n/2}} < \frac{2.825}{\sqrt{n}} 2^n.$$

For the proof of the propositions we need the following lemma.

4.3. **Lemma.** *Let $\alpha = \sqrt{\frac{2}{3}}\pi$. Then for $n \geq 3$ we have*

$$\frac{\sqrt{n}}{\sqrt{n+1}-1} < 1 + \frac{\pi}{\sqrt{6n}} < e^{\alpha\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-1\right)}$$

Proof. Using the inequality

$$1 + \frac{1}{2n} - \frac{1}{8n^2} < \sqrt{1+\frac{1}{n}}$$

and $\exp(x) > 1 + x + x^2/2$ for $x > 0$ we obtain

$$\begin{aligned} e^{\alpha\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-1\right)} &> \exp\Big(\alpha\sqrt{n}\Big(\frac{1}{2n} - \frac{1}{8n^2}\Big)\Big) = \exp\Big(\frac{\pi}{\sqrt{6n}}\Big(1 - \frac{1}{4n}\Big)\Big) \\ &> 1 + \frac{\pi}{\sqrt{6n}} - \frac{\pi}{4n\sqrt{6n}} + \frac{\pi^2}{12n} - \frac{\pi^2}{24n^2} + \frac{\pi^2}{192n^3} \\ &> 1 + \frac{\pi}{\sqrt{6n}} \end{aligned}$$

for $n \geq 1$. On the other hand we have for $n \geq 17$

$$\begin{aligned} \frac{1}{1+\frac{\pi}{\sqrt{6n}}} &< 1 - \frac{\pi}{\sqrt{6n}} + \frac{\pi^2}{6n} < 1 + \frac{1}{2n} - \frac{1}{8n^2} - \frac{1}{\sqrt{n}} \\ &< \sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n+1}-1}{\sqrt{n}} \end{aligned}$$

Taking reciprocal values yields the second part of the lemma. For $3 \leq n \leq 16$ one verifies the lemma directly. □

Proof of Proposition 1.5: Let $\alpha = \sqrt{\frac{2}{3}}\pi$. In [2], section 14.7 formula (11), the following upper bound for $p(n)$ is proved:

$$p(n) < \frac{\pi}{\sqrt{6n}} e^{\alpha\sqrt{n}} \quad \text{for all } \; n \geq 1$$

We want to prove the proposition by induction on n. By Lemma 4.3 we have

$$1 + \frac{\pi}{\sqrt{6n}} < e^{\alpha\sqrt{n+1}-\alpha\sqrt{n}}$$

which holds for all $n \geq 1$. Assuming the claim for $n-1$ it follows for n:

$$\begin{aligned} p(n,n) = p(n-1,n-1) + p(n) &< e^{\alpha\sqrt{n}} + \frac{\pi}{\sqrt{6n}} e^{\alpha\sqrt{n}} \\ &= \left(1 + \frac{\pi}{\sqrt{6n}}\right) e^{\alpha\sqrt{n}} < e^{\alpha\sqrt{n+1}} \end{aligned}$$

Proof of Proposition 1.6: It follows from Lemma 4.3 that

$$\sqrt{n} e^{\alpha\sqrt{n}} < \left(\sqrt{n+1} - 1\right) e^{\alpha\sqrt{n+1}}$$

By induction on n and Proposition 1.5 we have:

$$\begin{aligned} p(n+1,n) = p(n,n) + p(n,n-1) &< e^{\alpha\sqrt{n+1}} + \sqrt{n} e^{\alpha\sqrt{n}} \\ &< \sqrt{n+1} e^{\alpha\sqrt{n+1}} \end{aligned}$$

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