Left-invariant affine structures on nilpotent Lie groups

Dietrich Burde

Habilitationsschrift

Düsseldorf

1999

1991 *Mathematics Subject Classification.* 17B30, 17B10, 22E25, 22E60

I would like to thank Fritz Grunewald for introducing me to many interesting topics in mathematics and for helpful discussions.

Contents

Introduction

This monograph is devoted to affine structures on Lie algebras and representations of nilpotent Lie algebras. The origin of affine structures is the study of left–invariant affine structures on Lie groups. These structures play an important role in the study of fundamental groups of affine manifolds and in the theory of affine crystallographic groups. The beginning of this study goes back to Auslander [**3**] and Milnor [**64**], though many others have contributed important work. The subject embeds into the theory of compact manifolds with geometric structures. Euclidean, hyperbolic, projective or affine structures are well known examples for such geometric structures. We are mainly concerned with affine structures. The fundamental group of a compact complete affine manifold is an affine crystallographic group, ACG in short. This is a natural generalization of the classical Euclidean crystallographic groups, i.e., ECGs which are discrete subgroups of the isometry group of the Euclidean $\mathbb{R}^n$ with compact quotient.

Bieberbach presented around 1911 several important results for these groups. In particular he showed that every ECG contains an abelian subgroup of finite index consisting of parallel translations. A natural problem is to generalize Bieberbach's results to ACGs. In general, the theorems are no longer true. But weaker analogues do hold or are conjectured. A particular analogue, known as the *Auslander conjecture*, asserts that every ACG is virtually polycyclic, i.e., contains a polycyclic subgroup of finite index. In other words, the fundamental group of compact complete affine manifolds is conjectured to be virtually polycyclic. Milnor asked whether the conjecture is true without the compactness assumption, but this was answered negatively by Margulis [**61**] in 1983. Milnor also proved that any virtually polycyclic group appears as the fundamental group of some complete affine manifold, and asked whether the manifold could be chosen to be compact. An important source for examples of affine manifolds comes from complete left-invariant affine structures on Lie groups. Given a discrete subgroup of such a Lie group the quotient becomes a complete affine manifold. On the other hand, a virtually polycyclic group is virtually contained in a connected Lie group. Milnor asked the following:

Does every solvable Lie group admit a complete left-invariant affine structure, or equivalently, does every simply connected solvable Lie group admit a simply transitive operation by affine transformations on some $\mathbb{R}^n$ *?*

This question can be formulated in purely algebraic terms. It corresponds to the question whether any solvable Lie algebra satisfies a certain algebraic property which we will call affine structure. Milnor formulated the question in the seventies. What was known at that time were a few special cases and results in low dimensions, where the answer is positive. Auslander had proved that a Lie group with a complete left-invariant affine structure is solvable. Indeed many mathematicians believed that the answer should be positive in general. The problem became widely known as the *Milnor conjecture*. It had a remarkable history. After several attempts to prove the existence of affine structures, counterexamples in dimension 11 were discovered in 1993 by Benoist, and also by Grunewald and the author. Here we present new systematical counterexamples on the Lie algebra level with a shorter proof:

THEOREM. *There exist filiform nilpotent Lie algebras of dimension* $10 \le n \le 12$ *which admit no affine structures. On the other hand, all filiform Lie algebras of dimension* $n \le 9$ *admit an affine structure.*

A Lie algebra $\mathfrak{g}$ of dimension n over a field K is called *filiform* if it is nilpotent of nilindex $p = n - 1$. Here the integer p with $\mathfrak{g}^p = 0$ and $\mathfrak{g}^{p-1} \neq 0$ is called nilindex and $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ for $k \ge 1$. The counterexamples rely on the fact that there are certain filiform Lie algebras of dimension n without any *faithful* linear representation of dimension $n+1$. Since the Lie algebra of a Lie group with left-invariant affine structure has always such a representation, we obtain counterexamples to Milnors conjecture. However, it is not clear how to find such Lie algebras in general.

The present work is organized as follows. In Chapter 1 we give a survey on the background of Milnors problem. We will explain how the problem can be formulated in purely algebraic terms. Then we prove consequences of our counterexamples with respect to representation theory of Lie algebras and to finitely generated nilpotent groups.

In Chapter 2 we state the algebraic preliminaries which are important for our work. That includes Lie algebra cohomology, Betti numbers and deformation theory of Lie algebras. Moreover we discuss filiform Lie algebras and adapted bases for such algebras.

In Chapter 3 necessary and sufficient conditions for the existence of affine structures on Lie algebras and Lie groups are given. In particular we prove the following criteria:

THEOREM. *Let* $\mathfrak{g}$ *be a filiform Lie algebra and suppose that* $\mathfrak{g}$ *has an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with a Lie algebra $\mathfrak{h}$ *and its center* $\mathfrak{z}(\mathfrak{h})$*. Then* $\mathfrak{g}$ *admits an affine structure.*

THEOREM. *Let* $\mathfrak{g}$ *be a filiform Lie algebra over a field* K *and assume that there exists an affine cohomology class* $[\omega] \in H^2(\mathfrak{g}, K)$*. Then* $\mathfrak{g}$ *admits an affine structure.*

Here a 2–cocycle $\omega : \mathfrak{g} \wedge \mathfrak{g} \to K$ is called *affine* if it is nonzero on $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}$. It follows that all elements in $[\omega]$ are affine and the class $[\omega] \in H^2(\mathfrak{g}, K)$ is called affine. The converse of the preceding theorems is not true in general. To apply these criteria we compute explicitly the cohomology groups $H^2(\mathfrak{g}, K)$ for all filiform Lie algebras $\mathfrak{g}$ of dimension $n \le 11$. That is also useful for the study of Betti numbers of nilpotent Lie algebras. All computations are ckecked with the computer algebra package Reduce. In higher dimensions, that is for $n \ge 12$, there arise new phenomena with respect to affine structures. We study all filiform Lie algebras of dimension $n \ge 12$ satisfying the following properties: $\mathfrak{g}$ does not contain a one-codimensional subspace $U \supseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$, and $\mathfrak{g}^{(n-4)/2}$ is abelian, provided n is even. Here $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$. These algebras split in a natural way into two distinct classes depending on whether $[\mathfrak{g}^1, \mathfrak{g}^1] \subseteq \mathfrak{g}^6$ is satisfied or not. Denote these classes by $\mathfrak{A}_n^1(K)$ and $\mathfrak{A}_n^2(K)$ respectively. We study such algebras and prove:

THEOREM. *Let* $\mathfrak{g} \in \mathfrak{A}_n^1(K)$, $n \geq 12$. *Then* $\mathfrak{g}$ *has an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with a Lie algebra $\mathfrak{h} \in \mathfrak{A}_{n+1}^1(K)$ *and hence admits an affine structure.*

The theorem holds true also for $\mathfrak{g} \in \mathfrak{A}_{12}^2(K)$, but in general not for $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$. Here $H^2(\mathfrak{g}, K)$ has dimension 2 or 3. In particular if $b_2(\mathfrak{g}) = 2$ there cannot be an affine cohomology class.

For $n \leq 11$ we study the existence of affine structures for all filiform algebras of dimension n. For that purpose we apply various constructions of affine structures.

In Chapter 4 we discuss that the existence question of affine structures is intimately related to a very interesting question of refining *Ado's Theorem*: For a finite-dimensional Lie algebra $\mathfrak{g}$ let $\mu(\mathfrak{g})$ be the *minimal* dimension of a faithful $\mathfrak{g}$–module. That is an invariant of $\mathfrak{g}$ which is finite by Ado's Theorem and difficult to determine, especially for nilpotent and solvable Lie algebras. One is interested in finding general upper bounds for $\mu(\mathfrak{g})$, in particular linear bounds in the dimension of $\mathfrak{g}$. If the center of $\mathfrak{g}$ is zero then $\mu(\mathfrak{g}) \leq \dim \mathfrak{g}$. If $\mathfrak{g}$ admits an affine structure then $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$. It is not known whether $\mu(\mathfrak{g})$ grows polynomially with $\dim(\mathfrak{g})$ or not. If $\mathfrak{g}$ is nilpotent of dimension n then Reed showed $\mu(\mathfrak{g}) < 1 + n^n$ in 1969. He used Birkhoff's construction of a faithful representation by quotients of the universal enveloping algebra of $\mathfrak{g}$. Here we improve this bound by showing:

THEOREM. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of dimension* n. *Then* $\mu(\mathfrak{g}) < \frac{3}{\sqrt{n}} 2^n$.

However, the upper bound $\mu(\mathfrak{g}) \leq n + 1$ involved in the study of affine structures is much sharper. It will be more difficult to prove such a bound. We will determine the filiform Lie algebras of dimension $n \leq 11$ over $\mathbb{C}$ with $\mu(\mathfrak{g}) \leq n + 1$. For special classes of Lie algebras we are able to compute $\mu(\mathfrak{g})$ explicitly. A good example to illustrate that the results are not obvious, even not in simple cases, are abelian Lie algebras: Denote by $\lceil x \rceil$ the least integer greater or equal than x.

THEOREM. *Let* $\mathfrak{g}$ *be an abelian Lie algebra of dimension* n *over an arbitrary field* K. *Then* $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1} \rceil$.

In Chapter 5, finally, the counterexamples for $n \leq 11$ are determined. Also new counterexamples in dimension 12 are presented. These are certain filiform Lie algebras $\mathfrak{g}$ not containing a one-codimensional subspace $U \supseteq \mathfrak{g}^1$ with $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$, and where $\mathfrak{g}^4$ is not abelian. The proof uses the classification of faithful Δ–modules. That method, unfortunately, is not suitable to study the problem in more generality. It remains open how the Lie algebras $\mathfrak{g}$ satisfying $\mu(\mathfrak{g}) \geq \dim \mathfrak{g} + 2$ can be found. In the context of this work we think it is interesting to study first the following question, which is true at least for $n = 13$:

OPEN PROBLEM. *Does a Lie algebra* $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$ *satisfy* $\mu(\mathfrak{g}) \geq n + 2$ *if and only if there is no affine* $[\omega] \in H^2(\mathfrak{g}, K)$ *?*

CHAPTER 1

Background on Milnor's Conjecture

In this chapter a survey is given on the origin of Milnor's conjecture. It was first stated in the context of affine manifolds and affine crystallographic groups. We will explain how the conjecture can be formulated in purely algebraic terms. Working on the algebraic level we will construct counterexamples. Finally we will show the connections to representation theory of nilpotent Lie groups and Lie algebras and to finitely generated nilpotent groups.

1.1. Affine manifolds

Let G be a Lie group acting smoothly and transitively on a smooth manifold X. Let $U \subset X$ be an open set and let $f : U \to X$ be a smooth map. The map f is called *locally*–(X, G) if for each component $U_i \subset U$, there exists $g_i \in G$ such that the restriction of g_i to $U_i \subset X$ equals the restriction of f to $U_i \subset U$.

1.1.1. Definition. Let M be a smooth manifold of the same dimension as X. An (X, G)*–atlas* on M is a pair (U, Φ) where U is an open covering of M and $\Phi = \{\varphi_\alpha : U_\alpha \to X\}_{U_\alpha \in U}$ is a collection of coordinate charts such that for each pair $(U_\alpha, U_\beta) \in U \times U$ the restriction of $\varphi_\alpha \circ \varphi_\beta^{-1}$ to $\varphi_\beta(U_\alpha \cap U_\beta)$ is locally–(X, G). An (X, G)*–structure on* M is a maximal (X, G)–atlas and M together with an (X, G)–structure is called an (X, G)*–manifold.*

The notion of an (X, G)–diffeomorphism between two (X, G)–manifolds is defined in a straightforward manner. Note that an (X, G)–structure on M induces an (X, G)–structure on the universal covering manifold $\widetilde{M}$ of M.

We are mainly interested in affine manifolds which we will define now. Let $\mathrm{Aff}(\mathbb{R}^n)$ be the group of affine transformations which is given by

$$\left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL_n(\mathbb{R}),\ b \in \mathbb{R}^n \right\}.$$

It acts on the real affine space $\{(v, 1)^t \mid v \in \mathbb{R}^n\}$ by

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Av + b \\ 1 \end{pmatrix}$$

1.1.2. Definition. Let M be an n-dimensional manifold. An (X, G)–structure on M, where X is the real n–dimensional affine space, also denoted by $\mathbb{R}^n$ here, and $G = \mathrm{Aff}(\mathbb{R}^n)$ is called an *affine structure* on M and M is called an *affine manifold.*

(X, G)–manifolds are sometimes called geometric manifolds. Very often X will be a space with a geometry on it and G the group of transformations of X which preserve this geometry. The most important (X, G)–structures arise from the classical geometries. Examples are *Euclidean structures*, where X is the Euclidean $\mathbb{R}^n$ and $G = O_n(\mathbb{R}) \ltimes \mathbb{R}^n$, hyperbolic structures with the real hyperbolic space $X = \mathbb{H}^n$ and $G = PSO^+(n,1)$, projective structures with the real projective space $X = \mathbb{RP}^n$ and $G = PGL_{n+1}(\mathbb{R})$, flat conformal structures with the n–sphere $X = \mathbb{S}^n$ and $G = SO(n+1,1)$ and finally flat Lorentzian structures, where X is the n–dimensional Lorentz space and $G = O(n-1,1) \ltimes \mathbb{R}^n$.

We are mainly concerned with affine structures. Euclidean structures form a subclass of affine structures, and affine structures form a subclass of projective structures.

Affine structures on a smooth manifold M are in correspondance with a certain class of connections on the tangent bundle of M. The following result can be found in [**57**]:

1.1.3. Proposition. *There is a natural correspondence of affine structures on a manifold M and flat torsionfree affine connections ∇ on M.*

For the notion of an affine connection see [**57**]. The covariant differentiation of a vector field Y in the direction of a vector field X is denoted by $\nabla_X : Y \mapsto \nabla_X(Y)$. For later reference we recall that the connection ∇ is called *torsionfree*, or *symmetric* if

$$\nabla_X(Y) - \nabla_Y(X) - [X,Y] = 0 \tag{1}$$

and *flat* or of *curvature zero*, if

$$\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = 0 \tag{2}$$

1.2. Existence of affine structures

A Euclidean structure on a manifold automatically gives an affine structure. It is well known that the torus and the Klein bottle are the only compact two-dimensional manifolds that can be given Euclidean structures [**78**]. When do affine structures exist on a manifold M ? The first interesting case is that M is a closed 2–manifold. If M is a 2–torus, then there exist many affine structures, among them *non-Euclidean* ones as the following example from [**57**] shows:

1.2.1. Example. *Let Γ be the set of transformations of $\mathbb{R}^2$ given by*

$$(x,y) \mapsto (x+ny+m, y+n), \quad n,m \in \mathbb{Z}$$

Then Γ is a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^2)$ acting properly discontinuously on $\mathbb{R}^2$. The quotient space $\mathbb{R}^2/\Gamma$ is diffeomorphic to a torus and has a non-Euclidean affine structure inherited from the affine space $\mathbb{R}^2$.

A classification of all affine structures on the 2–torus is given in [**59**],[**66**]. If M is a closed 2–manifold, i.e., compact and without boundary, different from a 2–torus or the Klein bottle, then there exist no affine structures. This follows from Benzecri's result [**11**] of 1955:

1.2.2. Theorem. *A closed surface admits affine structures if and only if its Euler characteristic vanishes.*

Note that in contrast every surface admits a real projective structure, because it admits a Riemannian metric of constant curvature. The theorem implies in particular that, if the surface has genus $g \geq 2$, there exists no torsionfree affine connection with curvature zero. This was generalized by Milnor [**63**] in 1958:

1.2.3. Theorem. *A closed surface of genus $g = 0$ or $g \geq 2$ does not possess any affine connection with curvature zero.*

1.2.4. Remark. In higher dimensions there is no such criterion for the existence of an affine structure. However, Smillie [**76**] proved that a closed manifold does not admit an affine structure if its fundamental group is built up out of finite groups by taking free products, direct products and finite extensions. In particular, a connected sum of closed manifolds with finite fundamental groups admits no affine structure. It is also known [**29**] that certain Seifert fiber spaces admit no affine structure: Let M be a Seifert fiber space with vanishing first Betti number. Then M does not admit any affine structure.

1.2.5. Remark. The following sections will show how to obtain many examples of affine manifolds.

1.3. Fundamental groups of affine manifolds

1.3.1. Definition. A subgroup $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ is called an *affine crystallographic group* or ACG, if Γ acts properly discontinuously on $\mathbb{R}^n$ and if the quotient space $\mathbb{R}^n/\Gamma$ is compact.

This is a natural generalization of the classical Euclidean crystallographic groups (ECGs), which are discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ with compact quotient. While discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ act properly discontinuously on $\mathbb{R}^n$ this is not true in general for discrete subgroups of $\mathrm{Aff}(\mathbb{R}^n)$, e.g., for infinite discrete subgroups of $GL_n(\mathbb{R})$.

The study of ACGs is strongly connected with the study of fundamental groups of affine manifolds: Torsionfree ACGs are just the fundamental groups of complete compact affine manifolds [**81**]:

1.3.2. Proposition. *If Γ is a torsionfree ACG then the quotient $\mathbb{R}^n/\Gamma$ is a complete compact affine manifold with fundamental group isomorphic to Γ. Conversely any connected complete compact affine manifold is the quotient $\mathbb{R}^n/\Gamma$ with some subgroup $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ acting freely and properly discontinuously on $\mathbb{R}^n$.*

Here an affine manifold M is called *complete* if its universal covering $\widetilde{M}$ is affinely diffeomorphic to $\mathbb{R}^n$.

The structure of ECGs is well known. Bieberbach proved around 1911 the following results: two ECGs of dimension n are isomorphic as abstract groups iff they are conjugate in $\mathrm{Aff}(\mathbb{R}^n)$, and the number of nonisomorphic ECGs for given n is finite. That answered a part of Hilbert's 18^{th} problem. ECGs have been classified up to isomorphism in small dimension. The number of nonisomorphic ECGs for $n = 2$ is 17. These are the symmetry groups of certain ornaments in the plane. For $n = 3$ there are 219 different ECGs. This is very important for crystallography. Indeed, all 219 groups are realized as the symmetry groups of genuine crystals. The number for $n = 4$ is 4783. A further result of Bieberbach is that every ECG is virtually abelian, i.e., contains an abelian subgroup of finite index. A similar result for Lorentz–flat manifolds has been proved in [**43**].

The results from the Euclidean case do not generalize to the affine case, but weaker analogues survive [**44**], at least conjecturally. A famous conjecture here is the Auslander conjecture. Auslander [**3**] claimed in 1964, that every ACG is virtually polycyclic. As he later discovered his proof contained a gap. An equivalent formulation of his conjecture is:

1.3.3. Auslander Conjecture. *The fundamental group of a compact complete affine manifold is virtually polycyclic.*

A group Γ is *virtually polycyclic* if it has a subgroup of finite index which is *polycyclic*, that is, admits a finite composition series with cyclic quotients. The number of infinite cyclic quotients is called the *rank* of Γ. Examples of polycyclic groups are finitely generated nilpotent groups. A polycyclic group is solvable but the converse is not always true. However any discrete solvable subgroup of a Lie group with finitely many components is polycyclic. Since $\mathrm{Aff}(\mathbb{R}^n)$ is such a Lie group it suffices to prove that an ACG is virtually solvable. Also one may assume that the ACG is torsionfree because of Selberg's lemma.

The Auslander conjecture seems to be a very hard problem. In dimension 3 it was proved by Fried and Goldman [**36**]. For the Lorentz case it was proved in [**41**]:

1.3.4. Theorem. *The fundamental group of a manifold with compact complete flat Lorentz structure is virtually polycyclic.*

Auslander's conjecture is proved for dimension $n \leq 6$ and other cases, see [**1**],[**43**].

John Milnor [**64**] studied the fundamental groups of affine manifolds and proved in 1977:

1.3.5. Theorem. *Any torsion–free virtually polycyclic group appears as the fundamental group of a complete affine manifold.*

He also conjectured the converse, i.e., an even stronger version of Auslander's conjecture. However, this was disproved by Margulis [**61**] in 1983 with very surprising counterexamples in dimension 3:

1.3.6. Theorem. *There exist non-compact complete affine manifolds with a free non-abelian fundamental group of rank* 2.

Note that a free nonabelian group cannot be virtually polycyclic. The Lorentz-flat manifolds constructed by Margulis are not compact and do not have vanishing Euler characteristic. The examples can be generalized to higher dimensions.

Milnor asked in [**64**] whether a torsion-free and virtually polycyclic group Γ can be the fundamental group of a complete *compact* affine manifold. If the manifold is not required to be compact then the answer is positive by Theorem 1.3.5. In general however the answer is negative as I will explain in section 1.6. We will prove the following theorem:

1.3.7. Theorem. *There are finitely generated torsionfree nilpotent groups of rank* $10, 11$ *and* 12 *which are not the fundamental group of any compact complete affine manifold.*

1.4. Left–invariant affine structures on Lie groups

In this section we state Milnor's conjecture as he published it in [**64**]. Then we will explain how to formulate it in purely algebraic terms.

1.4.1. Definition. Let G be an n–dimensional real Lie group. An affine structure on G is called *left–invariant*, if each left-multiplication map $L(g) : G \to G$ is an affine diffeomorphism.

It is well known that many examples of affine manifolds can be constructed via left–invariant affine structures on Lie groups.

There is a close relationship between the above question of Milnor on a torsion-free and virtually polycyclic group Γ of rank k and the corresponding question for a k–dimensional Lie group. It is known that such a Γ can be the fundamental group of a compact manifold of dimension k with universal covering diffeomorphic to $\mathbb{R}^k$ [**5**], and Γ may be embedded up to finite index as a discrete subgroup of some Lie group G. Milnor himself proposed the corresponding question as follows [**64**]:

1.4.2. Milnor's Question. *Does every solvable Lie group G admit a complete left–invariant affine structure, or equivalently, does the universal covering group $\widetilde{G}$ operate simply transitively by affine transformations of $\mathbb{R}^k$?*

Note that if G admits such a structure, then the coset space G/Γ for any discrete subgroup Γ is a complete affine manifold. In many cases, Γ can be chosen so that G/Γ is also compact. Milnor's question has a very remarkable history. When he asked this question in 1977, there was some evidence for the existence of such structures. After that many articles appeared proving some special cases, see for example [**53**], [**72**], [**4**]. However, the general question was still open and it was rather a conjecture than a question by the time. Many mathematicians believed that Milnor's question should have a positive answer. In fact, around 1990 there appeared articles in the literature which claimed to prove the conjecture, e.g., [**15**] and [**68**]. However, in 1993 the conjecture was disproved by Benoist [**10**] and then more generally by Grunewald and the author [**18**],[**21**]: The conjecture is not true, not even for nilpotent Lie groups. We will prove the following theorem:

1.4.3. Theorem. *There are filiform nilpotent Lie groups of dimension* $10, 11$ *and* 12 *which do not admit any left-invariant affine structure.*

We will discuss the counterexamples, old and new ones, in Chapter 5 on the Lie algebra level. Although Milnor's conjecture is not true it is still a challenging problem to determine which Lie groups do admit such a structure.

It is possible to formulate Milnor's problem in purely algebraic terms. Therefore we make the following definitions: Let G denote a finite-dimensional connected 1–connected Lie group with Lie algebra $\mathfrak{g}$.

1.4.4. Definition. Let A be a finite-dimensional vector space over a field K and let $A \times A \to A, (x, y) \mapsto x \cdot y$ be a K–bilinear product which satisfies

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \tag{3}$$

for all $x, y, z \in A$. Then A together with the product is called *left-symmetric algebra* or LSA. The product is also called left-symmetric.

The name left-symmetric becomes evident, if we rewrite condition (3) as $(x, y, z) = (y, x, z)$, where $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ denotes the *associator* of three elements $x, y, z \in A$. Sometimes these algebras are called *Vinberg-algebras* or *semi-associative algebras.* There is a large literature on LSAs, see for example [**48**], [**65**], [**75**], [**62**], [**22**] and the references cited therein. Vinberg [**80**] used LSAs to classify convex homogeneous cones. A left-symmetric algebra is Lie-admissible. That means, $[x, y] = x \cdot y - y \cdot x$ defines a Lie bracket on the vector space A. The underlying Lie algebra will be denoted by $\mathfrak{g}$.

1.4.5. Definition. An *affine structure* or LSA–structure on a Lie algebra $\mathfrak{g}$ is a K–bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is left-symmetric and satisfies

$$[x, y] = x \cdot y - y \cdot x \tag{4}$$

Denote the left-multiplication in the LSA by $L(x)y = x \cdot y$, and the right multiplication by $R(x)y = y \cdot x$. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

1.4.6. Proposition. *There are canonical one-to-one correspondences between the following classes of objects, up to suitable equivalence:*

(a) $\{$*Left-invariant affine structures on* $G\}$
(b) $\{$*Flat torsion-free left-invariant affine connections on* $G\}$
(c) $\{$*Affine structures on* $\mathfrak{g}\}$

Under the bijection, bi-invariant affine structures correspond to associative LSA–structures.

Proof. The details of the correspondence are given in [**21**] and [**32**]. We will recall how a left-invariant affine structure induces an affine structure on its Lie algebra. Suppose G admits such an structure. Then there exists a left-invariant flat torsionfree affine connection ∇ on G. Since ∇ is left-invariant, for any two left-invariant vector fields $X, Y \in \mathfrak{g}$, the covariant derivative $\nabla_X(Y) \in \mathfrak{g}$ is also left-invariant. Hence covariant differentiation defines a bilinear multiplication

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},\ (X, Y) \mapsto XY = \nabla_X(Y)$$

The conditions that ∇ has zero torsion and zero curvature amounts via (1), (2) to

$$XY - YX = [X, Y] \tag{5}$$

$$X(YZ) - Y(XZ) = [X, Y]Z = (XY)Z - (YX)Z \tag{6}$$

This multiplication is an affine structure on $\mathfrak{g}$ by definition. $\square$

The completeness of left-invariant affine structures on G also can be expressed in algebraic terms. For the following see [**75**]:

1.4.7. Definition. Let A be an LSA and $T(A) = \{x \in A \mid \operatorname{tr} R(x) = 0\}$. The largest left ideal of A contained in $T(A)$ is called the *radical* of A and is denoted by $\operatorname{rad}(A)$. An LSA is called *complete* if $A = \operatorname{rad}(A)$.

We now have the following addition to Proposition 1.4.6:

1.4.8. Proposition. *Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. A left-invariant affine structure on G is complete if and only if the corresponding LSA–structure on $\mathfrak{g}$ is complete. Complete left-invariant affine structures on G correspond bijectively to simply transitive actions of G by affine transformations on $\mathbb{R}^n$.*

Let K be a field of characteristic zero. In [**75**] it is proved:

1.4.9. Proposition. *Let A be an LSA over K. Then there are equivalent:*

(a) *A is complete.*
(b) *$R(x)$ is nilpotent for all $x \in A$.*
(c) $\operatorname{tr}(R(x)) = 0$ *for all $x \in A$.*
(d) *$Id + R(x)$ is bijective for all $x \in A$.*

Now we will state the algebraic version of Milnor's question:

1.4.10. Milnor's Question. *Does every solvable Lie algebra admit a (complete) affine structure ?*

In this algebraic version the question makes sense for arbitrary Lie algebras over arbitrary fields. In fact, if the field has prime characteristic, then the results differ very much from the characteristic zero case, see [**19**]. Auslander [**4**] proved:

1.4.11. Proposition. *A Lie group admitting a complete left-invariant affine structure is solvable.*

Lie groups with incomplete left-invariant affine structures need not be solvable. There are many examples of reductive Lie groups with left-invariant affine structures. A basic example is $GL_n(\mathbb{C})$. All left-invariant affine structures on $GL_n(\mathbb{C})$ have been classified, see [**20**], [**8**]. Every such structure can be obtained as a certain deformation of the standard bi-invariant affine structure corresponding to the associative matrix algebra structure of $\mathfrak{gl}_n(\mathbb{C})$. It is not known in general which reductive Lie groups admit left-invariant affine structures. If a reductive Lie algebra $\mathfrak{g}$ has 1–dimensional center and is the direct sum of just two factors, i.e., $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}(\mathfrak{g})$ where $\mathfrak{s}$ is split simple, then $\mathfrak{g}$ has an affine structure if and only if $\mathfrak{s}$ is of type A_n, i.e., if $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}_{n+1}(K)$ [**20**].

1.5. Affine structures and representation theory

The basic observations which finaly relate Milnors question and his problem mentioned at the end of section 1.3 to problems of representation theory of Lie algebras are the following two propositions.

1.5.1. Proposition. *Let $\mathfrak{g}$ be an n–dimensional Lie algebra over a field K of characteristic zero. If $\mathfrak{g}$ admits an affine structure then $\mathfrak{g}$ possesses a faithful Lie algebra module of dimension $n+1$.*

1.5.2. Proposition. *Let Γ be a torsionfree finitely generated nilpotent group of rank n and G_Γ its real Malcev-completion with Lie algebra $\mathfrak{g}_\Gamma$. If Γ is the fundamental group of a compact complete affine manifold then $\mathfrak{g}_\Gamma$ has a faithful module of dimension $n+1$.*

The real Malcev-completion of Γ is the uniquely determined connected and simply connected real Lie group which contains an isomorphic copy of Γ as a discrete and cocompact subgroup.

For the proof of the first proposition see 3.1.18. The second one is well known, see for example [**44**]. In fact one proves that if Γ is a torsionfree finitely generated nilpotent group of rank n then G_Γ acts simply transitively on the affine space of dimension n. Then one can apply Proposition 1.4.8.

Theorem 1.4.3 is now proved as follows. Consider the examples of n–dimensional nilpotent Lie algebras which do not have a faithful module of dimension $n+1$ given in Chapter 5. Then the corresponding connected and simply connected Lie groups provide the desired counterexamples: one uses Propositions 1.4.6 and 1.5.1.

For the proof of Theorem 1.3.7 we start with the examples of n–dimensional nilpotent Lie algebras over $\mathbb{Q}$ which do not have a faithful module of dimension $n+1$. Then the connected and simply connected real Lie group corresponding to $\mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{g}$ is the real

Malcev-completion of a suitable finitely generated nilpotent group Γ. The claim follows from Proposition 1.5.2.

Proposition 1.5.1 directly leads to the refinement of Ado's theorem which we will treat in Chapter 4: We will study how the minimal dimension of a faithful $\mathfrak{g}$–module for a Lie algebra $\mathfrak{g}$ depends on the dimension of $\mathfrak{g}$.

1.6. Affine and projective nilmanifolds

1.6.1. Definition. A *nilmanifold* is a compact manifold which is diffeomorphic to a quotient of a simply–connected nilpotent Lie group G by a discrete subgroup Γ. It is called *affine* resp. *projective nilmanifold*, if it admits an affine resp. projective structure. A nilmanifold is called *filiform*, if the Lie algebra of G is filiform, see Definition 2.4.1.

First examples of affine nilmanifolds are the torus $\mathbb{T}^2$, or the Heisenberg nilmanifolds G/Γ where G is a Heisenberg Lie group. An example of an affine filiform nilmanifold is the Heisenberg nilmanifold of dimension 3. Fried, Goldman and Hirsch [**37**] showed:

1.6.2. Theorem. *Let M be a compact complete affine manifold with nilpotent fundamental group. Then M is an affine nilmanifold.*

An affine or projective structure on a nilmanifold $N = G/\Gamma$ is called *left-invariant*, if the corresponding structure on the universal cover $\widetilde{N} \simeq G$ is left-invariant. The following results are proved in [**9**]:

1.6.3. Theorem. *Let $N = G/\Gamma$ be a filiform nilmanifold. Then any affine or projective structure on N is left–invariant.*

Note that the theorem is false if N is not filiform [**9**].

1.6.4. Theorem. *Let $N = G/\Gamma$ be a nilmanifold of dimension $n \geq 2$ such that the Lie algebra of G has a 1–dimensional center and has no faithful module of dimension $n+1$. Then N admits no projective structures.*

Proceeding as in the argument for Theorem 1.3.7 we obtain from the above theorem:

1.6.5. Theorem. *There exist filiform nilvarieties of dimension $10, 11$ and 12 which do not admit any affine or projective structure.*

CHAPTER 2

Algebraic preliminaries

2.1. Affine structures on Lie algebras

Let A be a left-symmetric algebra over K with underlying Lie algebra $\mathfrak{g}$. By definition the product $x \cdot y$ in A satisfies the two conditions

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z$$
$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y, z \in A$. The left-multiplication L in A is given by $L(x)(y) = x \cdot y$. The two conditions are equivalent to

$$L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \text{ is a Lie algebra homomorphism} \tag{7}$$

$$\mathbf{1} : \mathfrak{g} \to \mathfrak{g}_L \text{ is a 1–cocycle in } Z^1(\mathfrak{g}, \mathfrak{g}_L) \tag{8}$$

where $\mathfrak{g}_L$ denotes the $\mathfrak{g}$–module with action given by L, and $\mathbf{1}$ is the identity map. $Z^1(\mathfrak{g}, \mathfrak{g}_L)$ is the space of 1–cocycles with respect to $\mathfrak{g}_L$, see section 2.2. Note that the right-multiplication R is in general not a Lie algebra representation of $\mathfrak{g}$.

Let $\mathfrak{g}$ be of dimension n and identify $\mathfrak{g}$ with K^n by choosing a K–basis. Then $\mathfrak{gl}(\mathfrak{g})$ gets identified with $\mathfrak{gl}_n(K)$.

2.1.1. Definition. The Lie algebra of the Lie group $\mathrm{Aff}(G)$ is called the Lie algebra of *affine transformations* and is denoted by $\mathfrak{aff}(\mathfrak{g})$. It can be identified as a vector space with $\mathfrak{gl}_n(K) \oplus K^n$.

Given an affine structure on $\mathfrak{g}$, define a map $\alpha : \mathfrak{g} \to \mathfrak{aff}(K^n)$ by $\alpha(x) = (L(x), x)$. That is a Lie algebra homomorphism:

2.1.2. Lemma. *The linear map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ satisfies* (7), (8) *iff $\alpha : \mathfrak{g} \to \mathfrak{aff}(K^n)$ is a Lie algebra homomorphism.*

Proof. Let more generally $\alpha(x) = (L(x), t(x)) \in \mathfrak{gl}_n(K) \oplus K^n$ with a bijective linear map $t : \mathfrak{g} \to \mathfrak{g}$. We have

$$\alpha([x, y]) = [\alpha(x), \alpha(y)] \iff \begin{cases} L([x, y]) = [L(x), L(y)] \\ L(x)(t(y)) - L(y)(t(x)) = t([x, y]) \end{cases} \tag{9}$$

To see this, use the identification of $\alpha(x)$ with

$$\alpha(x) = \begin{pmatrix} L(x) & t(x) \\ 0 & 0 \end{pmatrix}$$

Hence the Lie bracket in $\mathfrak{aff}(K^n)$ is given by

$$\begin{aligned}[\alpha(x), \alpha(y)] &= [(L(x), t(x)), (L(y), t(y))] \\ &= (L(x)L(y) - L(y)L(x), L(x)(t(y)) - L(y)(t(x)))\end{aligned}$$

It follows that α is a Lie algebra homomorphism iff L is and t is a bijective 1–cocycle in $Z^1(\mathfrak{g}, \mathfrak{g}_L)$. The lemma follows with $t = \mathbf{1}$, the identity map on $\mathfrak{g}$. $\square$

2.2. Cohomology of Lie algebras

In the following let $\mathfrak{g}$ be a Lie algebra of dimension n over K. Denote by M an $\mathfrak{g}$–module with action $\mathfrak{g} \times M \to M$, $(x, m) \mapsto x \bullet m$. The space of p–cochains is defined by

$$C^p(\mathfrak{g}, M) = \begin{cases} \operatorname{Hom}_K(\Lambda^p \mathfrak{g}, M) & \text{if } p \geq 0, \\ 0 & \text{if } p < 0. \end{cases}$$

The *coboundary operators* $d_p : C^p(\mathfrak{g}, M) \to C^{p+1}(\mathfrak{g}, M)$ are defined by

$$\begin{aligned}(d_p\omega)(x_1 \wedge \cdots \wedge x_{p+1}) = & \sum_{1 \leq r < s \leq p+1} (-1)^{r+s} \omega([x_r, x_s] \wedge x_1 \wedge \cdots \wedge \widehat{x_r} \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge x_{p+1}) \\ & + \sum_{t=1}^{p+1} (-1)^{t+1} x_t \bullet \omega(x_1 \wedge \cdots \wedge \widehat{x_t} \cdots \wedge x_{p+1}),\end{aligned}$$

for $p \geq 0$ and $\omega \in C^p(\mathfrak{g}, M)$. If $p < 0$ then we set $d_p = 0$. A standard computation shows $d_p \circ d_{p-1} = 0$, hence the definition

$$H^p(\mathfrak{g}, M) = \ker d_p / \operatorname{im} d_{p-1} = Z^p(\mathfrak{g}, M)/B^p(\mathfrak{g}, M)$$

makes sense. This space is called the p^{th} *cohomology group* of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$–module M. The elements from $Z^p(\mathfrak{g}, M)$ are called *p–cocycles*, and from $B^p(\mathfrak{g}, M)$ *p–coboundaries*. The sequence

$$0 \to C^0(\mathfrak{g}, M) \xrightarrow{d_0} C^1(\mathfrak{g}, M) \xrightarrow{d_1} C^2(\mathfrak{g}, M) \to \cdots$$

yields a cochain complex, which is called *standard* cochain complex and is denoted by $\{C^\bullet(\mathfrak{g}, M), d\}$. For details of Lie algebra cohomology see [**54**]. Sometimes the following formula is useful: let $\dim \mathfrak{g} = n$ and $\dim M = m$, then

$$\dim H^p(\mathfrak{g}, M) = \dim(\ker d_{p-1}) + \dim(\ker d_p) - m \binom{n}{p-1}$$

2.2.1. Betti numbers. There are important special cases of Lie algebra cohomology. If $M = K$ denotes the 1–dimensional trivial module, i.e., $x \bullet m = 0$ for all $x \in \mathfrak{g}$, then the numbers $\dim H^p(\mathfrak{g}, K)$ are of special interest.

2.2.1. Definition. The number $b_p(\mathfrak{g}) = \dim H^p(\mathfrak{g}, K)$ is called the p^{th} *Betti number*.

The Betti number can also be defined as the dimension of $H_p(\mathfrak{g}, K)$, the p^{th} *homology group* with trivial coefficients.

2.2.2. Definition. A Lie algebra $\mathfrak{g}$ is called *unimodular*, if $\operatorname{tr}(\operatorname{ad}(x)) = 0$ for all $x \in \mathfrak{g}$.

For example, nilpotent Lie algebras are unimodular. We have the following theorem which can be found in [**54**] in the context of Poincaré's duality:

2.2.3. Theorem. *Let $\mathfrak{g}$ be a n–dimensional unimodular Lie algebra. Then it holds $H_p(\mathfrak{g}, K) \cong H^{n-p}(\mathfrak{g}, K)$.*

The theorem holds in particular for nilpotent Lie algebras. There are many questions regarding the Betti numbers of nilpotent Lie algebras. In some cases, explicit Betti numbers have been computed: if $\mathfrak{h}_n$ denotes the Heisenberg Lie algebra of dimension $2n+1$ over K, then Santharoubane [**71**] proved

$$b_p(\mathfrak{h}_n) = \binom{2n}{p} - \binom{2n}{p-2}$$

for all $0 \le i \le n$. See also [**6**] for the explicit Betti numbers of 2–step nilpotent Lie algebras related to the Heisenberg algebra. If $\mathfrak{g}$ is a complex nilpotent Lie algebra containing an abelian ideal of codimension 1, then there is a combinatorical formula for $b_p(\mathfrak{g})$ using partitions [**7**]. A special case of such a Lie algebra is the standard graded filiform Lie algebra $L(n+1)$. It is defined by

$$[e_1, e_i] = e_{i+1}, \ 2 \le i \le n$$

where $(e_1, e_2, \dots, e_{n+1})$ is a basis of $L(n+1)$, the undefined brackets being zero.

2.2.4. Proposition. *The p^{th} Betti number of $L(n+1)$ is given by*

$$b_p(L(n+1)) = P_{p,n} + P_{p-1,n}$$

for $1 \le p \le n+1$, where $P_{0,n} = 1$ and

$$P_{p,n} = \#\Big\{(a_1, \dots, a_p) \in \mathbb{Z}^p \mid 1 \le a_1 < \dots < a_p \le n, \quad \sum_{j=1}^{p} a_j = \Big\lceil \frac{p(n+1)}{2} \Big\rceil \Big\}$$

For small p this yields explicit formulas for the Betti numbers b_p, e.g.,

$$\begin{aligned} b_1(L(n)) &= 2 \\ b_2(L(n)) &= \left\lfloor \frac{n+1}{2} \right\rfloor, \\ b_3(L(n)) &= \left\lfloor \binom{\frac{n+1}{2}}{2} + \frac{1}{8} \right\rfloor, \\ b_4(L(n)) &= \left\lfloor \frac{4}{3}\binom{\frac{n+1}{2}}{3} + \frac{4n+13}{36} \right\rfloor \end{aligned}$$

In general one cannot expect to obtain an explicit formula. Another problem is to obtain upper and lower bounds for the Betti numbers of nilpotent Lie algebras. Dixmier [**33**] proved a lower bound $b_p(\mathfrak{g}) \ge 2$ for $0 < p < \dim \mathfrak{g}$. An upper bound is given by $b_p(\mathfrak{g}) \le \binom{n}{p} - \binom{n-2}{p-1}$ for $p = 1, \dots, n-1$ where $n = \dim \mathfrak{g} \ge 3$ and $\mathfrak{g}$ is non-abelian [**25**].

The following question on the Betti numbers comes from topology. Let M be a smooth n–manifold. The *toral rank* $\mathrm{rk}(M)$ of a smooth manifold M is the dimension of the largest torus which acts almost freely on M. Halperin [**47**] conjectured in 1968:

2.2.5. Toral-Rank-Conjecture. *It holds* $\dim H^*(M) \ge 2^{\mathrm{rk}(M)}$.

In particular, if $M = N/\Gamma$ is a nilmanifold, where N is a nilpotent Lie group with Lie algebra $\mathfrak{n}$ and Γ is a discrete cocompact subgroup of N, then $\mathrm{rk}(M) = \dim \mathfrak{z}(\mathfrak{n})$ and $H^*(N/\Gamma) = H^*(\mathfrak{n}, \mathbb{R})$. If the toral rank conjecture would be true, then $\dim H^*(\mathfrak{n}, K) \geq 2^{\dim \mathfrak{z}(\mathfrak{n})}$ for such Lie algebras $\mathfrak{n}$. There are, however, nilpotent Lie algebras which are not models of nilmanifolds. For these algebras there is no change of basis yielding rational structure constants. In dimension 7 there are already infinitely many of such nilpotent Lie algebras [**74**]. Nevertheless the estimate on the Betti numbers seems to be true for all nilpotent Lie algebras. There is the toral rank conjecture for nilpotent Lie algebras, which says

2.2.6. Toral-Rank-Conjecture. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over K. Then*

$$\dim H^*(\mathfrak{g}, K) \geq 2^{\dim \mathfrak{z}(\mathfrak{g})}$$

The conjecture for Lie algebras is proved in the following cases [**25**], [**30**]:

2.2.7. Proposition. *Let $\mathfrak{g}$ be a nilpotent Lie algebra which satisfies one of the following conditions:*

(a) $\mathfrak{g}$ *is* 2*–step nilpotent*
(b) $\dim \mathfrak{z}(\mathfrak{g}) \leq 5$
(c) $\dim \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \leq 7$
(d) $\dim \mathfrak{g} \leq 14$

Then $\dim H^*(\mathfrak{g}, K) \geq 2^{\dim \mathfrak{z}(\mathfrak{g})}$.

This implies, writing TRC for toral rank conjecture:

2.2.8. Corollary. *The TRC is true for nilmanifolds of dimension $n \leq 14$.*

Finally we mention the b_2–conjecture for nilpotent Lie algebras. It says that

$$b_2(\mathfrak{g}) > \frac{b_1(\mathfrak{g})^2}{4}.$$

This is discussed in [**24**]. But in fact, the conjecture was already proved in 1977 by H. Koch [**58**].

2.2.2. First and second cohomology group. We will give a short interpretation of the groups $H^p(\mathfrak{g}, M)$ for $p = 1, 2$. For more details see [**54**]. First of all, $H^0(\mathfrak{g}, M) = \{m \in M \mid x \bullet m = 0 \ \forall x \in \mathfrak{g}\} = M^{\mathfrak{g}}$ is called the space of $\mathfrak{g}$*–invariants* in M. Let us investigate $H^1(\mathfrak{g}, M)$. We have

$$(d_1\omega)(x \wedge y) = x \bullet \omega(y) - y \bullet \omega(x) - \omega([x, y])$$

Hence the space of 1–cocycles and 1–coboundaries is given by

$$Z^1(\mathfrak{g}, M) = \{\omega \in \mathrm{Hom}(\mathfrak{g}, M) \mid \omega([x, y]) = x \bullet \omega(y) - y \bullet \omega(x)\}$$
$$B^1(\mathfrak{g}, M) = \{\omega \in \mathrm{Hom}(\mathfrak{g}, M) \mid \omega(x) = x \bullet m \text{ for some } m \in M\}$$

Suppose that M equals the trivial $\mathfrak{g}$–module K. Then $d_0 = 0$ and $(d_1\omega)(x \wedge y) = \omega([y, x])$. Hence

$$\begin{aligned} H^1(\mathfrak{g}, K) &= \{\omega \in \mathfrak{g}^* \mid \omega([\mathfrak{g}, \mathfrak{g}]) = 0\} \\ &\cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \end{aligned}$$

The first Betti number is given by $b_1(\mathfrak{g}) = \dim \mathfrak{g} - \dim[\mathfrak{g},\mathfrak{g}]$. In particular, $b_1(\mathfrak{g})$ is nonzero for nilpotent Lie algebras. If $M = \mathfrak{g}$ is the adjoint module, then we obtain the space $H^1(\mathfrak{g},\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ of outer derivations of $\mathfrak{g}$. Here $\mathrm{Int}(\mathfrak{g}) = \{\mathrm{ad}\, x \mid x \in \mathfrak{g}\}$. The space of 2–cocycles and 2–coboundaries is given by

$$\begin{aligned} Z^2(\mathfrak{g},M) = \{\omega \in \mathrm{Hom}(\Lambda^2\mathfrak{g},M) \mid & x_1 \bullet \omega(x_2\wedge x_3) - x_2 \bullet \omega(x_1 \wedge x_3) + x_3 \bullet \omega(x_1\wedge x_2) \\ & - \omega([x_1,x_2]\wedge x_3) + \omega([x_1,x_3]\wedge x_2) - \omega([x_2,x_3]\wedge x_1) = 0\} \\ B^2(\mathfrak{g},M) = \{\omega \in \mathrm{Hom}(\Lambda^2\mathfrak{g},M) \mid & \omega(x_1\wedge x_2) = x_1\bullet f(x_2) - x_2 \bullet f(x_1) - f([x_1,x_2]) \\ & \text{for some } f \in \mathrm{Hom}(\mathfrak{g},M)\} \end{aligned}$$

2.2.9. Definition. A Lie algebra $\mathfrak{h}$ is called an *extension of* $\mathfrak{g}$ *by* $\mathfrak{a}$ if there is an *exact sequence* of Lie algebras $0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$. Two such extensions are called *equivalent*, if there exists a Lie algebra isomorphism σ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & \mathfrak{h} \arrow[r, "\pi"] \arrow[d, "\sigma"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] & \mathfrak{h} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

commutes. The extension is called *abelian*, if $\mathfrak{a}$ is abelian.

If $\mathfrak{a}$ is abelian then an extension as above makes $\mathfrak{a}$ into a $\mathfrak{g}$–module in a well defined way by

$$x \bullet y = \iota^{-1}([\pi^{-1}(x), \iota(y)]_{\mathfrak{h}}),\ x \in \mathfrak{g}, y \in \mathfrak{a} \tag{10}$$

Since $\iota(\mathfrak{a})$ is an ideal in $\mathfrak{h}$ we see that ι^{-1} is defined on $[\pi^{-1}(x),\iota(y)]$. Hence we obtain an action $\mathfrak{g}\times\mathfrak{a}\to\mathfrak{a}$. A computation shows that this is a $\mathfrak{g}$–module action. We have the following proposition, see [**54**]:

2.2.10. Proposition. *Let* $\mathfrak{a}$ *be an abelian Lie algebra with the above* $\mathfrak{g}$*–module action. There is a one-to-one correspondence between* $H^2(\mathfrak{g},\mathfrak{a})$ *and the set of equivalence classes of abelian extensions of* $\mathfrak{g}$ *by* $\mathfrak{a}$*. The zero class* $[0] \in H^2(\mathfrak{g},\mathfrak{a})$ *corresponds to the semi-direct product of* $\mathfrak{g}$ *and* $\mathfrak{a}$*.*

Proof. Let $\mathfrak{h}$ be an abelian extension of $\mathfrak{g}$ by $\mathfrak{a}$. Then we have an exact sequence

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

Then (10) defines a $\mathfrak{g}$–module structure on $\mathfrak{a}$. Fix a linear map $\tau : \mathfrak{g} \to \mathfrak{h}$ for the above extension with $\pi\circ\tau = \mathbf{1}$ on $\mathfrak{g}$. Then we may attach to this situation a map $\omega \in \mathrm{Hom}(\Lambda^2\mathfrak{g},\mathfrak{a})$ by

$$\omega(x\wedge y) = \iota^{-1}([\tau(x),\tau(y)] - \tau([x,y])) \tag{11}$$

A computation shows that ω is a 2–cocycle. Its cohomology class $[\omega] \in H^2(\mathfrak{g},\mathfrak{a})$ does not depend on the choice of τ. Moreover two equivalent abelian extensions $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ by $\mathfrak{a}$ define the same cohomology class and the same $\mathfrak{g}$–action. Hence we have obtained a unique $[\omega] \in H^2(\mathfrak{g},\mathfrak{a})$.

We can go backwards as well. Fix an $[\omega] \in H^2(\mathfrak{g},\mathfrak{a})$ and define a Lie bracket on $\mathfrak{h} = \mathfrak{a}\oplus\mathfrak{g}$ by

$$[(a,x),(b,y)]_{\mathfrak{h}} = (x\bullet b - y\bullet a + \omega(x\wedge y), [x,y]_{\mathfrak{g}}) \tag{12}$$

This yields a short exact sequence $0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$ with $\iota(a) = (a, 0)$ and $\pi(a, x) = x$. With $\tau : \mathfrak{g} \to \mathfrak{h}$, $x \mapsto (0, x)$ we get back the given ω via (11).

The zero class $[0] \in H^2(\mathfrak{g}, \mathfrak{a})$ corresponds to the equivalence class of the semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_\theta \mathfrak{a}$ with $\theta(x)y = x \bullet y$ as in (10). In that case τ is an injective Lie algebra homomorphism, called *Lie algebra splitting.* We also say then that the above exact sequence *splits.* $\square$

As a special case where $\mathfrak{a} = K$ is the trivial $\mathfrak{g}$–module we obtain:

2.2.11. Corollary. *The elements of $H^2(\mathfrak{g}, K)$ classify the equivalence classes of central extensions of $\mathfrak{g}$ by K.*

In that case the space of 2–cocycles and 2–coboundaries is given by

$$\begin{aligned} Z^2(\mathfrak{g}, K) &= \{\omega \in \operatorname{Hom}(\Lambda^2\mathfrak{g}, K) \mid \omega([x_1, x_2] \wedge x_3) - \omega([x_1, x_3] \wedge x_2) \\ &\qquad + \omega([x_2, x_3] \wedge x_1) = 0\} \\ B^2(\mathfrak{g}, K) &= \{\omega \in \operatorname{Hom}(\Lambda^2\mathfrak{g}, K) \mid \omega(x_1 \wedge x_2) = f([x_1, x_2]) \\ &\qquad \text{for some } f \in \operatorname{Hom}(\mathfrak{g}, K)\} \end{aligned}$$

2.3. Deformations of Lie algebras

The theory of deformations of algebraic structures parallels the theory of deformations of complex analytic structures, initiated by Kodaira and Spencer. Algebraic deformations were first introduced by Gerstenhaber [**38**] for arbitrary rings and associative algebras, and for Lie algebras by Nijenhuis and Richardson [**67**]. They studied one-parameter deformations of Lie algebras and established the connection between Lie algebra cohomology and infinitesimal deformations.

2.3.1. Definition. Let $\mathfrak{g}$ be a Lie algebra over K with bracket $[\, , \,]$ and $g, h \in \mathfrak{g}$, $\varphi_k \in \operatorname{Hom}(\Lambda^2\mathfrak{g}, \mathfrak{g}) = C^2(\mathfrak{g}, \mathfrak{g})$. A *formal one-parameter deformation* of $\mathfrak{g}$ is a power series

$$[g, h]_t := [g, h] + \sum_{k \geq 1} \varphi_k(g, h) t^k,$$

such that the Jacobi identity for $[\, , \,]_t$ holds. Here $C^2(\mathfrak{g}, \mathfrak{g})$ refers to the standard cochain complex $\{C^\bullet(\mathfrak{g}, \mathfrak{g}), d\}$ of $\mathfrak{g}$ with coefficients in the adjoint $\mathfrak{g}$–module $\mathfrak{g}$.

Note that $\mathfrak{g}$ is not required to be finite-dimensional. Indeed, deformations of infinite-dimensional Lie algebras have been intensively studied because of the applications in physics. The conditions imposed by the Jacobi identity for $[\, , \,]_t$ are usually described by the following product in the differential graded Lie algebra structure of the complex $\{C^\bullet(\mathfrak{g}, \mathfrak{g}), d\}$: if $\alpha \in C^p(\mathfrak{g}, \mathfrak{g})$ and $\beta \in C^q(\mathfrak{g}, \mathfrak{g})$, then the product $[\alpha, \beta] \in C^{p+q-1}(\mathfrak{g}, \mathfrak{g})$ is defined by

$$\begin{aligned} &[\alpha, \beta](g_1, \ldots, g_{p+q-1}) \\ &= \sum_{i_1 < \cdots < i_q} (-1)^{\sum_s (i_s - s)} \alpha(\beta(g_{i_1}, \ldots, g_{i_q}), g_1, \ldots, \widehat{g_{i_1}}, \ldots, \widehat{g_{i_q}}, \ldots, g_{p+q-1}) \\ &- (-1)^{(p-1)(q-1)} \sum_{j_1 < \cdots < j_q} (-1)^{\sum_t (j_t - t)} \beta(\alpha(g_{j_1}, \ldots, g_{j_p}), g_1, \ldots, \widehat{g_{j_1}}, \ldots, \widehat{g_{j_p}}, \ldots, g_{p+q-1}). \end{aligned}$$

where the summation is taken over indices i_r, j_r with $1 \leq i_r, j_r \leq p+q-1$.
The Jacobi identity for $[\ ,\ ]_t$ is equivalent to the sequence of relations

$$d\varphi_k = -\frac{1}{2}\sum_{i=1}^{k-1}[\varphi_i, \varphi_{k-i}], \quad k = 1, 2, 3, \dots$$

For $k = 1$ we obtain $d\varphi_1 = 0$ which means $\varphi_1 \in Z^2(\mathfrak{g}, \mathfrak{g})$. The cohomology class of φ_1 is called the *differential* of the formal deformation $[\ ,\ ]_t$ and depends only on the equivalence class of the deformation. Here two formal deformations of $\mathfrak{g}$ are called *equivalent*, if the resulting Lie algebras are isomorphic. A cohomology class from $H^2(\mathfrak{g}, \mathfrak{g})$ is called *linear* or *infinitesimal* deformation of $\mathfrak{g}$. Note however that an infinitesimal deformation $\varphi \in H^2(\mathfrak{g}, \mathfrak{g})$ is not necessarily the differential of any formal deformation. The above equations for $k = 2, 3, 4, \dots$ are necessary and sufficient conditions for it. They are usually formulated in terms of higher Massey–Lie products. If they hold, φ_1 is also called *integrable*.

In the literature there is sometimes used a more general definition of a Lie algebra deformation, see [**35**]: Let $\mathfrak{g}$ be a Lie algebra over a field K of characteristic zero and A be a K–algebra with a fixed augmentation $\varepsilon : A \to K$ satisfying $\varepsilon(1) = 1$. Assume furthermore that A is a complete local algebra. A is called *local*, if it has a unique maximal ideal $\mathfrak{m}$. It is called *complete*, if A equals the inverse limit $\varprojlim A/\mathfrak{m}^n$. A typical example of a complete local K–algebra is the power series ring $A = K[[X_1, \dots, X_n]]$ in n variables.

2.3.2. Definition. A formal deformation of $\mathfrak{g}$ with base A is a Lie A–algebra structure on the completed tensor product $A\widehat{\otimes}_K\mathfrak{g} = \varprojlim(A/\mathfrak{m}^n \otimes \mathfrak{g})$ such that the map

$$\varepsilon\widehat{\otimes}\mathrm{id} \,:\, A\widehat{\otimes}_K\mathfrak{g} \to K \otimes_K \mathfrak{g}$$

is a Lie algebra homomorphism.

If the parametrization algebra is $A = K[[t]]$ we obtain the first definition of a formal one-parameter deformation, which we will use here in this work.

2.4. Filiform Lie algebras

In the study of nilpotent Lie algebras, the Lie algebras with maximal nilindex with respect to their dimension play an important role. They are called *filiform algebras*.

2.4.1. Definition. Let $\mathfrak{n}$ be a Lie algebra over a field K. The lower central series $\{\mathfrak{n}^k\}$ of $\mathfrak{n}$ is defined by

$$\mathfrak{n}^0 = \mathfrak{n},$$
$$\mathfrak{n}^k = [\mathfrak{n}^{k-1}, \mathfrak{n}],\ k \geq 1.$$

The integer p is called *nilindex* of $\mathfrak{n}$ if $\mathfrak{n}^p = 0$ and $\mathfrak{n}^{p-1} \neq 0$. In that case $\mathfrak{n}$ is called p–step nilpotent. A nilpotent Lie algebra $\mathfrak{n}$ of dimension n and nilindex $p = n-1$ is called *filiform*.

2.4.2. Remark. If we denote the *type* of a nilpotent Lie algebra $\mathfrak{n}$ by $\{p_1, p_2, \dots, p_r\}$ where $\dim(\mathfrak{n}^{i-1}/\mathfrak{n}^i) = p_i$ for all $i = 1, \dots, r$, then the filiform Lie algebras are just the algebras of type $\{2, 1, 1, \dots, 1\}$. That explains the name "filiform" which means threadlike.

2.4.3. EXAMPLE. *Let $L = L(n)$ be the n-dimensional Lie algebra defined by*

$$[e_1, e_i] = e_{i+1},\ i = 2, \dots, n-1$$

where $(e_1, \dots, e_n)$ is a basis of $L(n)$ and the undefined brackets are zero. This is called the standard graded filiform.

If the Lie algebra $\mathfrak{n}$ is filiform then the lower central series

$$\mathfrak{n} = \mathfrak{n}^0 \supseteq \mathfrak{n}^1 \supseteq \cdots \supseteq \mathfrak{n}^{n-2} \supseteq \mathfrak{n}^{n-1} = 0$$

is of maximal length. The *upper* central series for $\mathfrak{n}$ is also of length $n-1$:

$$0 = \mathfrak{n}_0 \subseteq \mathfrak{n}_1 \subseteq \cdots \subseteq \mathfrak{n}_{n-2} \subseteq \mathfrak{n}_{n-1} = \mathfrak{n}$$

Here $\mathfrak{n}_0 = 0$, $\mathfrak{n}_k = \{x \in \mathfrak{n} \mid \mathrm{ad}(x)(\mathfrak{n}) \subseteq \mathfrak{n}_{k-1}\}$. In particular, $\mathfrak{n}_1 = \mathfrak{z}(\mathfrak{n})$ is the center of $\mathfrak{n}$.

The classification of filiform Lie algebras is a difficult problem which is so far known up to dimension 11, see [**16**],[**40**].

2.4.1. Adapted bases for filiform algebras. In the paper of Vergne [**79**] the following is proved: for any filiform Lie algebra there exist a so called *adapted basis* $(e_1, \dots, e_n)$ such that

$$\begin{aligned}
[e_1, e_i] &= e_{i+1},\ i = 2, \dots, n-1 \\
[e_i, e_j] &\in \mathrm{span}\{e_{i+j}, \dots, e_n\},\ i, j \geq 2,\ i+j \leq n \\
[e_{i+1}, e_{n-i}] &= (-1)^i \alpha e_n,\ 1 \leq i < n-1
\end{aligned}$$

with a certain scalar α, which is automatically zero if n is odd, and the undefined brackets being zero. Moreover the brackets $[e_i, e_j]$ for $i, j \geq 2$ are completely determined by the brackets

$$[e_k, e_{k+1}] = \sum_{s=2k+1}^{n} \alpha_{k,s} e_s,\ 2 \leq k \leq [n/2]$$

Conversely one can define a filiform Lie algebra by the above brackets, which however in general do not satisfy the Jacobi identity. Rather the Jacobi identity is equivalent to certain polynomial equations in the parameters $\alpha_{k,s}$.

The existence of an adapted basis has a deformation theoretic interpretation: any n–dimensional filiform Lie algebra is isomorphic to an infinitesimal deformation of the algebra $L = L(n)$. The above $\alpha_{k,s}$ can be viewed as deformation parameters:

2.4.4. DEFINITION. Let $\mathcal{I}_n$ be an index set given by

$$\mathcal{I}_n^0 = \{(k, s) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq k \leq [n/2],\ 2k+1 \leq s \leq n\},$$

$$\mathcal{I}_n = \begin{cases} \mathcal{I}_n^0 & \text{if } n \text{ is odd,} \\ \mathcal{I}_n^0 \cup \{(\frac{n}{2}, n)\} & \text{if } n \text{ is even.} \end{cases}$$

Let $(e_1, \ldots, e_n)$ be a basis of $L(n)$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$. For any element $(k, s) \in \mathcal{I}_n$ we can associate a 2–cocycle $\psi_{k,s} \in Z^2(L, L)$ for the Lie algebra cohomology with coefficients in the adjoint module L as follows:

$$\psi_{k,s}(e_1 \wedge e_i) = 0,$$
$$\psi_{k,s}(e_k \wedge e_{k+1}) = e_s$$

for $1 \leq i \leq n$, $2 \leq k \leq n-1$. Then the condition $\psi_{k,s} \in Z^2(L, L)$ for basis vectors e_1, e_i, e_j with $2 \leq i, j$ is given by

$$[e_1, \psi_{k,s}(e_i \wedge e_j)] = \psi_{k,s}([e_1, e_i] \wedge e_j) + \psi_{k,s}(e_i \wedge [e_1, e_j])$$

and we obtain the following formula:

$$\psi_{k,s}(e_i \wedge e_j) = \begin{cases} (-1)^{k-i}\binom{j-k-1}{k-i}(\operatorname{ad} e_1)^{(j-k-1)-(k-i)} e_s & \text{if } 2 \leq i \leq k < j \leq n \ , \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

The $\psi_{k,s}$ defined by (13) in fact lie in $Z^2(L, L)$, and Vergne's result can be formulated as follows:

2.4.5. Lemma. *Any n–dimensional filiform Lie algebra over $\mathbb{C}$ is isomorphic to a Lie algebra L_ψ with basis $(e_1, \ldots, e_n)$ whose Lie brackets are given by*

$$[e_i, e_j] = [e_i, e_j]_L + \psi(e_i \wedge e_j), \ 1 \leq i, j \leq n. \tag{14}$$

Here ψ is a 2–cocycle which can be expressed by

$$\psi = \sum_{(k,s) \in \mathcal{I}_n} \alpha_{k,s} \psi_{k,s}$$

with $\alpha_{k,s} \in \mathbb{C}$ and satisfies

$$\psi(a \wedge \psi(b \wedge c)) + \psi(b \wedge \psi(c \wedge a)) + \psi(c \wedge \psi(a \wedge b)) = 0 \tag{15}$$

The 2–cocycle $\psi \in Z^2(L, L)$ defines an infinitesimal deformation L_ψ of L.

2.4.6. Definition. A basis $(e_1, \ldots, e_n)$ of an n–dimensional filiform Lie algebra is called *adapted*, if the brackets relative to this basis are given by (14) with a 2–cocycle $\psi = \sum_{(k,s) \in \mathcal{I}_n} \alpha_{k,s} \psi_{k,s}$.

2.4.7. Lemma. *All brackets of an n–dimensional filiform Lie algebra in an adapted basis $(e_1, \ldots, e_n)$ are determined by the brackets*

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, n-1$$
$$[e_k, e_{k+1}] = \alpha_{k,2k+1} e_{2k+1} + \ldots + \alpha_{k,n} e_n, \quad 2 \leq k \leq [(n-1)/2]$$
$$[e_{\frac{n}{2}}, e_{\frac{n+2}{2}}] = \alpha_{\frac{n}{2},n} e_n, \quad \textit{if } n \equiv 0(2)$$

From (13) follows:

2.4.8. Lemma. *The brackets of an n–dimensional filiform Lie algebra in an adapted basis are given by:*

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \dots, n-1 \tag{16}$$

$$[e_i, e_j] = \sum_{r=1}^{n} \left(\sum_{\ell=0}^{[(j-i-1)/2]} (-1)^{\ell} \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,\, r-j+i+2\ell+1} \right) e_r, \quad 2 \le i < j \le n. \tag{17}$$

where the constants $\alpha_{k,s}$ are zero for all pairs (k, s) not in $\mathcal{I}_n$.

Note that an adapted basis for a filiform Lie algebra is not unique. Nevertheless we can associate a matrix of coefficients to a filiform Lie algebra with respect to an adapted basis:

$$\begin{pmatrix} \alpha_{2,5} & \alpha_{2,6} & \alpha_{2,7} & \alpha_{2,8} & \cdots & \alpha_{2,n} \\ & & \alpha_{3,7} & \alpha_{3,8} & \cdots & \alpha_{3,n} \\ & & & & \cdots & \cdots \\ & & & & & \alpha_{[\frac{n}{2}],n} \end{pmatrix}$$

The matrix has $\frac{1}{4}(n-3)^2$ parameters if n is odd, and $\frac{1}{4}(n^2 - 6n + 12)$ if n is even. The Jacobi identity defines certain equations with polynomials $f_i \in K[\alpha_{k,s}]$. If $n < 8$, there are no equations, i.e., the Jacobi identity is satisfied automatically. In general, with respect to an adapted basis, the polynomial equations are much simpler than usual.

2.4.9. Example. *Any filiform nilpotent Lie algebra of dimension 9 has an adapted basis $(e_1, \dots e_9)$ such that the Lie brackets are given by:*

$$\begin{aligned}
[e_1, e_i] &= e_{i+1}, \ i \ge 2 \\
[e_2, e_3] &= \alpha_{2,5} e_5 + \alpha_{2,6} e_6 + \alpha_{2,7} e_7 + \alpha_{2,8} e_8 + \alpha_{2,9} e_9 \\
[e_2, e_4] &= \alpha_{2,5} e_6 + \alpha_{2,6} e_7 + \alpha_{2,7} e_8 + \alpha_{2,8} e_9 \\
[e_2, e_5] &= (\alpha_{2,5} - \alpha_{3,7}) e_7 + (\alpha_{2,6} - \alpha_{3,8}) e_8 + (\alpha_{2,7} - \alpha_{3,9}) e_9 \\
[e_2, e_6] &= (\alpha_{2,5} - 2\alpha_{3,7}) e_8 + (\alpha_{2,6} - 2\alpha_{3,8}) e_9 \\
[e_2, e_7] &= (\alpha_{2,5} - 3\alpha_{3,7} + \alpha_{4,9}) e_9 \\
[e_3, e_4] &= \alpha_{3,7} e_7 + \alpha_{3,8} e_8 + \alpha_{3,9} e_9 \\
[e_3, e_5] &= \alpha_{3,7} e_8 + \alpha_{3,8} e_9 \\
[e_3, e_6] &= (\alpha_{3,7} - \alpha_{4,9}) e_9 \\
[e_4, e_5] &= \alpha_{4,9} e_9
\end{aligned}$$

The Jacobi identity is satisfied if and only if the following equation holds:

$$\alpha_{4,9}(2\alpha_{2,5} + \alpha_{3,7}) - 3\alpha_{3,7}^2 = 0.$$

That follows from a short computation. The matrix of coefficients is given by

$$\begin{pmatrix} \alpha_{2,5} & \alpha_{2,6} & \alpha_{2,7} & \alpha_{2,8} & \alpha_{2,9} \\ & & \alpha_{3,7} & \alpha_{3,8} & \alpha_{3,9} \\ & & & & \alpha_{4,9} \end{pmatrix}$$

There are 9 deformation parameters $\{\alpha_{k,s} \mid (k, s) \in \mathcal{I}_9\}$.

2.5. The variety of Lie algebra laws

Let V be a vector space of dimension n over an algebraically closed field K of characteristic zero, with fixed basis $(e_1, \ldots, e_n)$. A Lie algebra structure on V determines a multiplication table relative to the basis. If

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$$

then the point $(c_{ij}^k) \in K^{n^3}$ is called a *Lie algebra law.* The constants c_{ij}^k are subject to two sets of conditions:

$$c_{ij}^k + c_{ji}^k = 0 \tag{18}$$

$$\sum_{l=1}^{n} (c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0 \tag{19}$$

They correspond to the skew-symmetry and the Jacobi identity of the Lie bracket. Equations (18), (19) determine a certain Zariski-closed set in n^3–dimensional affine space with coordinates c_{ij}^k, $1 \leq i, j, k \leq n$. It is often called an affine algebraic variety, although it need not be irreducible.

2.5.1. Definition. The set of all Lie algebra laws of dimension n is denoted by $\mathcal{L}_n(K)$. It is called the *variety of Lie algebra laws*, or the variety of structure constants of n–dimensional Lie algebras.

A set $\mathcal{L}_n(K)$ may also be considered as a closed subset of the affine algebraic variety $\mathrm{Hom}(\Lambda^2 V, V)$, via the bilinear skew–symmetric map defining the Lie bracket. The linear reductive group $GL(V) = GL_n(K)$ acts on $\mathrm{Hom}(\Lambda^2 V, V)$ by

$$(g * \mu)(x \wedge y) = g(\mu(g^{-1}(x) \wedge g^{-1}(y))), \quad g \in GL_n(K),\, \mu \in \mathrm{Hom}(\Lambda^2 V, V)$$

It induces an action of $GL(V)$ on $\mathcal{L}_n(K)$. The orbits under this action correspond to isomorphism classes of n–dimensional Lie algebras: two sets of structure constants generate isomorphic Lie algebras if and only if they are transformed into each other under the action of the group $GL(V)$. We obtain an "orbit space". In the theory of algebraic transformation groups there is a concept of how to define such a quotient. Let G be a linear reductive algebraic group over an algebraically closed field K and X be an affine variety X with G–action and ring of coordinates $K[X]$. Then there exists an affine variety $G \backslash X$ such that $K[G \backslash X] = K[X]^G$, see [**77**]. Here

$$K[X]^G = \{f \in K[X] \mid f(g * x) = f(x) \quad \forall\, g \in G, x \in X\}$$

denotes the ring of invariants of $K[X]$. It is generated by finitely many elements. The affine variety $G \backslash X$ is called *algebraic quotient* of X by G. The inclusion $K[X]^G \hookrightarrow K[X]$ defines a morphism $\pi_X : X \to G \backslash X$, which is called *quotient map.* The topology on $G \backslash X$ is the quotient topology with respect to π_X. Any fiber $\pi_X^{-1}(\eta)$, $\eta \in G \backslash X$ contains exactly one closed orbit: The set $G \backslash X$ algebraically parametrizes the *closed* orbits of G in X. In general $G \backslash X$ does not parametrize the set of *all* orbits of G in X.

In our case we obtain an algebraic quotient $G \backslash X$ with $G = GL(V)$ and $X = \mathcal{L}_n(K)$. In the following however, we will only study the affine variety X.

2.5.2. Example. *Let $n = 2$ and $K = \mathbb{C}$. Then the algebraic set $\mathcal{L}_2(\mathbb{C})$ is the affine plane $\mathbb{C}^2$: since condition* (19) *is satisfied automatically we have only the linear condition* (18)*. Relative to the action of the group $GL(2, \mathbb{C})$ this plane splits into two orbits: the origin $\{0\}$ and its complement $\mathbb{C}^2 \setminus \{0\}$. The orbit space consists of two points.*

The situation becomes much more difficult for larger n. There are several questions about the varieties $\mathcal{L}_n(K)$. One is interested in the irreducible components and the dimensions and degrees of these. What are their generic points ? It is also interesting to know the open orbits in $\mathcal{L}_n(K)$.

2.5.3. Remark. A Lie algebra law is called *rigid*, if its $GL_n(K)$–orbit in $\mathcal{L}_n(K)$ is open. If the field K is algebraically closed of characteristic zero, then a Lie algebra law is rigid iff its corresponding Lie algebra has no non-trivial formal one-parameter deformation. The vanishing of the second Lie algebra cohomology with adjoint coefficients is a sufficient, but not necessary condition for rigidity.

If $\lambda \in \mathcal{L}_n(K)$ then we denote the corresponding Lie algebra by $\mathfrak{g}_\lambda$:

2.5.4. Proposition. *Let $\lambda \in \mathcal{L}_n(K)$ be a Lie algebra law such that $H^2(\mathfrak{g}_\lambda, \mathfrak{g}_\lambda) = 0$. Then λ is rigid, i.e., has an open orbit.*

In $\mathcal{L}_n(K)$ we have the following closed subsets: the set of Lie algebra laws which corresponds to solvable respectively nilpotent Lie algebras.

2.5.5. Definition. Denote by $\mathcal{N}_n(K)$ the laws of n–dimensional nilpotent Lie algebras and by $\mathcal{R}_n(K)$ the laws of n–dimensional solvable Lie algebras.

It is known that the number of nonisomorphic rigid Lie algebra laws of a given dimension is finite. However, this number is growing very fast with the dimension. The following is proved in [**26**]:

2.5.6. Proposition. *The number of rigid laws $\lambda \in \mathcal{R}_n(K)$ with pairwise nonisomorphic solvable Lie algebras $\mathfrak{g}_\lambda$ and $H^2(\mathfrak{g}_\lambda, \mathfrak{g}_\lambda) = 0$ is not less than $\Gamma(\sqrt{n})$, provided $n \geq 81$.*

Here $\Gamma(x)$ denotes Euler's Gamma function. A classification $\mathcal{L}_n(K)$ is known up to dimension 7 [**56**],[**27**]. The number $s(n)$ of irreducible components of $\mathcal{L}_n(K)$ is growing rapidly. In fact, the following estimate is known [**56**] :

$$e^{\sqrt{n}} < s(n) < 2^{n^4/6}$$

The following table shows the number of components and the number of open orbits of $\mathcal{L}_n(K)$ for $1 \leq n \leq 7$:

Dimension	Components	Open orbits
1	1	1
2	1	1
3	2	1
4	4	2
5	7	3
6	17	6
7	49	14

The variety $\mathcal{N}_n(K)$ has also been been studied by many authors, see for example [**46**],[**79**]. It is known that the variety $\mathcal{N}_n(K)$ is irreducible for $n \leq 6$ and reducible for $n = 7, 8$ and $n \geq 11$.

2.5.7. DEFINITION. Let $\mathcal{F}_n(K)$ be the subset of $\mathcal{N}_n(K)$ consisting of those elements which define a filiform Lie algebra. Let $\mathcal{A}_n(K)$ denote the subset of $\mathcal{F}_n(K)$ consisting of elements which are the structure constants of a filiform Lie algebra with respect to an adapted basis. If $\lambda \in \mathcal{F}_n(K)$ then we denote the corresponding Lie algebra by $\mathfrak{g}_\lambda$. Denote the class of n–dimensional filiform Lie algebras over K by $\mathfrak{F}_n(K)$.

The result of Vergne [**79**] implies:

2.5.8. LEMMA. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$. Then there exists a basis $(e_1, \dots, e_n)$ such that the corresponding Lie algebra law belongs to $\mathcal{A}_n(K)$.*

The notion of a filiform algebra law appears naturally: $\mathcal{F}_n(K)$ is a Zariski–open subset of $\mathcal{N}_n(K)$, see [**42**].

Let $\mathfrak{g}$ be a filiform Lie algebra of dimension $n \geq 7$ and $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ for $k \geq 2$. The following properties are isomorphism invariants of $\mathfrak{g}$:

(a) $\mathfrak{g}$ contains a one-codimensional subspace $U \supseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$.
(b) $\mathfrak{g}$ contains *no* one-codimensional subspace $U \supseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$.
(c) $\mathfrak{g}^{\frac{n-4}{2}}$ is abelian, where n is even.
(d) $[\mathfrak{g}^1, \mathfrak{g}^1] \subseteq \mathfrak{g}^6$.

A Lie algebra law in $\mathcal{A}_n(K)$ is the vector of structure constants depending on the parameters $\alpha_{k,s}$. Properties (a)–(d) can be expressed by relations of these parameters.

2.5.9. LEMMA. *If $\mathfrak{g}$ is a filiform Lie algebra of dimension $n \geq 7$ then there is a basis for $\mathfrak{g}$ such that its Lie algebra law belongs to $\mathcal{A}_n(K)$ and satisfies*

(i1) $\alpha_{2,5} = 0$, *if and only if $\mathfrak{g}$ satisfies property (a).*
(i2) $\alpha_{2,5} = 1$, *if $\mathfrak{g}$ satisfies property (b).*
(i3) $\alpha_{\frac{n}{2},n} = 0$, *if and only if $\mathfrak{g}$ satisfies property (c).*
(i4) $\alpha_{3,7} = 0$, *if and only if $\mathfrak{g}$ satisfies property (d).*

PROOF. With respect to an adapted basis $(e_1, \dots, e_n)$ the space $\mathfrak{g}^4$ is spanned by $\{e_6, \dots, e_n\}$ and we have $[e_2, e_3] = \alpha_{2,5}e_5 + \ldots + \alpha_{2,n}e_n$. The brackets (17) show that property (a) just means $\alpha_{2,5} = 0$. Property (b) means $\alpha_{2,5} \neq 0$. In that case we change the basis so that it stays adapted and satisfies $\alpha_{2,5} = 1$. Let the base change $f \in GL(V)$ be as follows:

$$
\begin{aligned}
f(e_1) &= ae_1 \\
f(e_2) &= be_2 \\
f(e_i) &= [f(e_1), f(e_{i-1})], \quad 3 \leq i \leq n,
\end{aligned}
$$

where $a, b \in K^*$. A suitable choice of a, b yields an adapted basis such that the corresponding Lie algebra law satisfies $\alpha_{2,5} = 1$. With respect to an adapted basis, $\mathfrak{g}^{\frac{n-4}{2}} = \operatorname{span}\{e_{\frac{n}{2}}, \dots, e_n\}$ and $\mathfrak{g}^6$ is spanned by $\{e_8, \dots, e_n\}$. By lemma 2.4.7, property (c) means $\alpha_{\frac{n}{2},n} = 0$. The brackets (17) show that property (d) means $\alpha_{3,7} = 0$. $\square$

CHAPTER 3

Construction of affine structures

In general it is difficult to decide whether or not there exists a left-invariant affine structure on a given Lie group. By Proposition 1.4.6 we may pass to the Lie algebra level. Here we will prove some necessary and sufficient conditions for the existence of affine structures. Among nilpotent Lie algebras it has many advantages to consider *filiform* Lie algebras, in particular for the study of Milnor's conjecture. It also turns out that some irreducible components of the affine variety $\mathcal{N}_n(K)$ are the closure of sets of filiform Lie algebra laws. Our construction of affine structures on filiform Lie algebras splits into two parts. First we try to construct affine structures on all filiform Lie algebras of dimension $n \leq 11$. That will be successful for $n \leq 9$, whereas in dimension 10 and 11 there appear counterexamples. In that context we compute explicitly the cohomology groups $H^2(\mathfrak{g}, K)$ for all filiform Lie algebras $\mathfrak{g}$ of dimension $n \leq 11$. Secondly we study all filiform Lie algebras of dimension $n \geq 12$ satisfying property (b) and (c). They split naturally into two distinct classes depending on whether property (d) is satisfied or not. We denote the class of n–dimensional filiform Lie algebras satisfying property (b),(c),(d) by $\mathfrak{A}_n^1(K)$, and those satisfying property (b),(c) but not (d) by $\mathfrak{A}_n^2(K)$. There exists an adapted basis such that the corresponding Lie algebra laws form subsets of $\mathcal{A}_n(K)$, see Lemma 2.5.9. We prove that all Lie algebras $\mathfrak{g} \in \mathfrak{A}_n^1(K)$ possess a canonical affine structure which is induced by the *extension property*: for all $\mathfrak{g} \in \mathfrak{A}_n^1(K)$ there is a Lie algebra $\mathfrak{h} \in \mathfrak{A}_{n+1}^1(K)$ such that there is an exact sequence of Lie algebras:

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

Hence all $\mathfrak{g} \in \mathfrak{A}_n^1(K)$, $n \geq 12$ possess a natural affine structure given by formula (20) of Proposition 3.1.8. The situation for $\mathfrak{A}_n^2(K)$ is more complicated. Although many Lie algebras $\mathfrak{g} \in \mathfrak{A}_n^2(K)$ do possess such an extension by an $\mathfrak{h} \in \mathfrak{A}_{n+1}^2(K)$, there are infinitely many which do not. It is interesting to study the obstructions. These are given by a polynomial which can be obtained by computing the cohomology classes $[\omega] \in H^2(\mathfrak{g}, K)$.

3.1. Conditions for the existence of affine structures

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic zero. We give some necessary and sufficient conditions for the existence of affine structures on $\mathfrak{g}$.

3.1.1. Sufficient conditions for affine structures.

3.1.1. PROPOSITION. *A Lie algebra $\mathfrak{g}$ admits an affine structure if and only if there is a $\mathfrak{g}$–module M of dimension $\dim \mathfrak{g}$ such that the vector space $Z^1(\mathfrak{g}, M)$ contains a nonsingular 1–cocycle.*

PROOF. Let $\varphi \in Z^1(\mathfrak{g}, M)$ be a nonsingular 1-cocycle with inverse transformation φ^{-1}. The module M corresponds to a linear representation $\theta : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Then

$$L(x) := \varphi^{-1} \circ \theta(x) \circ \varphi$$

defines a $\mathfrak{g}$–module N such that $\varphi^{-1} \circ \varphi = \mathbf{1} \in Z^1(\mathfrak{g}, N)$. It follows that $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation and $\mathbf{1}([x, y]) = \mathbf{1}(x)y - \mathbf{1}(y)x$ is a bijective 1–cocycle in $Z^1(\mathfrak{g}, \mathfrak{g}_L)$. Hence $L(x)y = x \cdot y$ defines a left-symmetric structure on $\mathfrak{g}$. Conversely, $\mathbf{1}$ is a nonsingular 1–cocycle if $\mathfrak{g}$ admits a left-symmetric structure. □

3.1.2. COROLLARY. *If the Lie algebra $\mathfrak{g}$ admits a nonsingular derivation, then there exists an affine structure on $\mathfrak{g}$.*

PROOF. Let d be a nonsingular derivation and $\mathfrak{g}$ the adjoint module of $\mathfrak{g}$. Since $Z^1(\mathfrak{g}, \mathfrak{g})$ equals the space $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$, d is a nonsingular 1–cocycle. □

3.1.3. COROLLARY. *If the Lie algebra $\mathfrak{g}$ is graded by positive integers, then there exists an affine structure on $\mathfrak{g}$.*

PROOF. Suppose that $\mathfrak{g} = \oplus_{i \in \mathbb{N}} \mathfrak{g}_i$ is a graduation, i.e., $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. Then there is a nonsingular derivation defined by $d(x_i) = ix_i$ for $x_i \in \mathfrak{g}_i$. □

3.1.4. COROLLARY. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra or a nilpotent Lie algebra of dimension $n \leq 6$. Then $\mathfrak{g}$ admits an affine structure.*

PROOF. It is well known that in both cases $\mathfrak{g}$ can be graded by positive integers. □

The existence of a nonsingular derivation is a strong condition on the Lie algebra. In fact, such a Lie algebra is necessarily nilpotent [**50**]. However the converse does not hold, see [**34**]. We will also present many nilpotent Lie algebras without any nonsingular derivation.

3.1.5. DEFINITION. A Lie algebra $\mathfrak{g}$ is called *characteristically nilpotent*, if all its derivations are nilpotent.

By Engel's theorem, such a Lie algebra is nilpotent, since all inner derivations $\mathrm{ad}(x)$ are nilpotent. One might think that characteristically nilpotent Lie algebras form a small subclass of nilpotent Lie algebras. This is not true, see [**42**]. The example of a characteristically nilpotent Lie algebra, given in [**34**], is 3–step nilpotent. Although there is no nonsingular derivation there exists an affine structure. That follows from a theorem of Scheuneman [**72**]:

3.1.6. PROPOSITION. *Let $\mathfrak{g}$ be a 3–step nilpotent Lie algebra. Then $\mathfrak{g}$ admits an affine structure.*

There have been attempts to generalize this result for 4–step nilpotent Lie algebras. However, only in special cases a positive result could be proved [**31**]. The general case is still open.

There is the following result about extending affine structures to a semi-direct product:

3.1.7. LEMMA. *Let $\mathfrak{g}$ be a Lie algebra with affine structure and $\mathfrak{h} = \mathfrak{g} \ltimes_\theta \mathfrak{a}$ be the semi-direct product of $\mathfrak{g}$ and an abelian Lie algebra $\mathfrak{a}$ with Lie homomorphism $\theta : \mathfrak{g} \to \mathrm{Der}(\mathfrak{a})$. Then $\mathfrak{h}$ admits an affine structure.*

Proof. We construct a left-symmetric product on $\mathfrak{h} = \mathfrak{g} \ltimes_\theta \mathfrak{a}$ as follows:

$$(x,a)\cdot(y,b) = (x\cdot y, \theta(x)b)$$

for $x, y \in \mathfrak{g}$, $a, b \in \mathfrak{a}$. Here $x \cdot y$ denotes the left-symmetric product on $\mathfrak{g}$. Let $u = (x,a)$, $v = (y,b)$, $w = (z,c)$ elements of $\mathfrak{h}$. Then

$$\begin{aligned} u\cdot v - v\cdot u &= (x\cdot y, \theta(x)b) - (y\cdot x, \theta(y)a) \\ &= ([x,y], \theta(x)b - \theta(y)a) \\ &= [u,v]_{\mathfrak{h}} \\ u\cdot(v\cdot w) - (u\cdot v)\cdot w &= (x\cdot(y\cdot z) - (x\cdot y)\cdot z, \theta(x)\theta(y)c - \theta(x\cdot y)c) \\ &= (y\cdot(x\cdot z) - (y\cdot x)\cdot z, \theta(y)\theta(x)c - \theta(y\cdot x)c) \\ &= v\cdot(u\cdot w) - (v\cdot u)\cdot w \end{aligned}$$

□

Let $0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$ be a split exact sequence of Lie algebras. Then, by the preceding lemma, an affine structure on $\mathfrak{g}$ induces an affine structure on $\mathfrak{h}$. The following important proposition yields an affine structure on $\mathfrak{g}$ itself.

3.1.8. Proposition. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and suppose that $\mathfrak{g}$ has an extension*

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with $\iota(\mathfrak{a}) = \mathfrak{z}(\mathfrak{h})$. Then $\mathfrak{g}$ admits an affine structure.

Proof. If $\mathfrak{g}$ has such an extension then $\mathfrak{h}$ must be nilpotent. We first show that we may assume $\dim \mathfrak{a} = 1$. For that choose elements $h_1, \dots, h_n \in \mathfrak{h}$ so that their residue classes form an adapted basis of $\mathfrak{g}$. Since $\iota(\mathfrak{a}) = \mathfrak{z}(\mathfrak{h})$ there is a $j \in \{1, \dots, n\}$ such that $v := [h_j, h_n] \neq 0$, otherwise $h_n \in \mathfrak{z}(\mathfrak{h})$. Then the residue class of v in $\mathfrak{g}$ is zero, hence $v \in \iota(\mathfrak{a}) = \mathfrak{z}(\mathfrak{h})$. Choose a subspace $U \subseteq \iota(\mathfrak{a})$ such that $\dim(\iota(\mathfrak{a})/U) = 1$ and $v \notin U$. Since U is an ideal of $\mathfrak{h}$ we can define $\mathfrak{h}' = \mathfrak{h}/U$. Then π induces a Lie algebra surjection $\pi' : \mathfrak{h}' \to \mathfrak{g}$ with 1–dimensional kernel, since $U \subseteq \ker(\pi)$. To obtain the desired extension we must show $\ker(\pi') = \mathfrak{z}(\mathfrak{h}')$: Clearly $\ker(\pi') \subseteq \mathfrak{z}(\mathfrak{h}')$. Conversely let $x \in \mathfrak{z}(\mathfrak{h}')$ and let h'_j be the residue class of h_j in $\mathfrak{h}'$. Then $\pi'(x) \in \mathfrak{z}(\mathfrak{g})$ and hence x can be written as $x = \lambda h'_n + u$ with $\lambda \in K$, $u \in \ker(\pi')$. It follows $0 = [h'_j, x] = \lambda[h'_j, h'_n] + [h'_j, u]$ where $[h'_j, u] = 0$ and $[h'_j, h'_n] \neq 0$, hence $\lambda = 0$ and $x = u \in \ker(\pi')$.

Now we assume that $\dim \mathfrak{a} = 1$. Then the center $\mathfrak{z}(\mathfrak{h})$ is 1–dimensional and it follows that $\mathfrak{h} \in \mathfrak{F}_{n+1}(K)$. We will construct an affine structure on $\mathfrak{g}$ as follows. Let $(f_1, \dots, f_{n+1})$ be an adapted basis for $\mathfrak{h}$ and let $e_i := f_i \bmod \mathfrak{z}(\mathfrak{h})$ for $i = 1, \dots, n$. Then $(e_1, \dots, e_n)$ is an adapted basis of $\mathfrak{g}$. Let $\mathfrak{h}_3 = \operatorname{span}\{f_3, \dots, f_{n+1}\}$ and $\mathfrak{g}_2 = \operatorname{span}\{e_2, \dots, e_n\}$. There is a uniquely determined linear map $\varphi : \mathfrak{g} \to \mathfrak{h}_3$ satisfying $\varphi(x) = [f_1, \overline{x}]_{\mathfrak{h}}$ for all $x \in \mathfrak{g}$ where $\overline{x} \in \mathfrak{h}$ is any element with $\pi(\overline{x}) = x$. The restriction of φ to $\mathfrak{g}_2$ is bijective since it is evidently injective. Denote its inverse by $\psi : \mathfrak{h}_3 \to \mathfrak{g}_2$. Now set for all $x, y \in \mathfrak{g}$

$$x \bullet y := \psi([\overline{x}, \varphi(y)]_{\mathfrak{h}}) \tag{20}$$

The formula is well defined since $[\overline{x}, \varphi(y)]_{\mathfrak{h}} = [\overline{x}, [f_1, \overline{y}]] \in \mathfrak{h}_3$. As we will show it satisfies conditions (3) and (4) of Definition 1.4.5 and hence defines an affine structure on $\mathfrak{g}$:

$$
\begin{aligned}
x \bullet y - y \bullet x &= \psi([\overline{x},[f_1,\overline{y}]] - [\overline{y},[f_1,\overline{x}]]) \\
&= \psi([\overline{y},[f_1,\overline{x}]] - [f_1,[\overline{y},\overline{x}]] - [\overline{y},[f_1,\overline{x}]]) \\
&= \psi([f_1,[\overline{x},\overline{y}]]) = \psi([f_1,\overline{[x,y]_{\mathfrak{g}}}]) = \psi(\varphi([x,y]_{\mathfrak{g}})) \\
&= [x,y]_{\mathfrak{g}}
\end{aligned}
$$

where the brackets are taken in $\mathfrak{h}$ if not otherwise denoted. Using the identity $[f_1, \overline{\psi(w)}] = w$ for all $w \in \mathfrak{h}_3$ and again the Jacobi identity we obtain for all $x, y, z \in \mathfrak{g}$:

$$
\begin{aligned}
x \bullet (y \bullet z) - y \bullet (x \bullet z) &= \psi([\overline{x},[\overline{y},\varphi(z)]] - [\overline{y},[\overline{x},\varphi(z)]]) \\
&= [x,y]_{\mathfrak{g}} \bullet z \\
&= (x \bullet y) \bullet z - (y \bullet x) \bullet z
\end{aligned}
$$

□

3.1.9. Corollary. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and suppose that $\mathfrak{g}$ has an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with a Lie algebra $\mathfrak{h} \in \mathfrak{F}_{n+1}(K)$. Then $\mathfrak{g}$ admits an affine structure.

3.1.10. Remark. The converse of Proposition 3.1.8 is not always true. There are examples of filiform Lie algebras $\mathfrak{g} \in \mathfrak{F}_n(K)$ which admit an affine structure but no extension $0 \to \mathfrak{z}(\mathfrak{h}) \to \mathfrak{h} \to \mathfrak{g} \to 0$, see Remark 3.4.10.

3.1.11. Definition. Let $\mathfrak{g} \in \mathfrak{F}_n(K)$. A 2–cocycle $\omega \in Z^2(\mathfrak{g}, K)$ is called *affine*, if $\omega : \mathfrak{g} \wedge \mathfrak{g} \to K$ does not vanish on $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}$. A class $[\omega] \in H^2(\mathfrak{g}, K)$ is called affine if every representative is affine.

3.1.12. Lemma. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and $\omega \in Z^2(\mathfrak{g}, K)$ be an affine 2–cocycle. Then its cohomology class $[\omega] \in H^2(\mathfrak{g}, K)$ is affine and nonzero.*

Proof. If $\mathfrak{z}(\mathfrak{g}) = \operatorname{span}\{z\}$, then ω is affine iff $\omega(z \wedge y) \neq 0$ for some $y \in \mathfrak{g}$. For $\xi \in B^2(\mathfrak{g}, K)$ we have $\xi(z \wedge y) = f([z, y]) = f(0) = 0$ for some linear form $f \in \mathfrak{g}^*$. Hence ω is not a 2–coboundary and $[\omega]$ is affine. □

Using the interpretation of $H^2(\mathfrak{g}, K)$ with trivial coefficients we obtain:

3.1.13. Proposition. *A Lie algebra $\mathfrak{g} \in \mathfrak{F}_n(K)$ has an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{21}$$

with $\mathfrak{h} \in \mathfrak{F}_{n+1}(K)$ if and only if there exists an affine $[\omega] \in H^2(\mathfrak{g}, K)$.

Proof. Let $\mathfrak{z}(\mathfrak{g}) = \operatorname{span}\{z\}$ and suppose that $\mathfrak{g}$ has such an extension. Then $\mathfrak{z}(\mathfrak{h})$ is a trivial $\mathfrak{g}$–module equal to K. The extension determines a unique class $[\omega] \in H^2(\mathfrak{g}, K)$ and we may assume that the Lie bracket is given by

$$[(a, x), (b, y)]_{\mathfrak{h}} := (\omega(x \wedge y), [x, y]_{\mathfrak{g}}) \tag{22}$$

on the vector space $\mathfrak{h} := K \oplus \mathfrak{g}$. Suppose that $\omega(z \wedge y) = 0$ for all $y \in \mathfrak{g}$. Then $(a, 0)$ and (a, z) are contained in $\mathfrak{z}(\mathfrak{h})$. This contradicts $\mathfrak{z}(\mathfrak{h}) \cong K$. Hence ω is affine.

Conversely an affine $[\omega] \in H^2(\mathfrak{g}, K)$ determines an extension

$$0 \to K \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

via the Lie bracket (22) on $\mathfrak{h} := K \oplus \mathfrak{g}$. Let $(a, x) \in \mathfrak{z}(\mathfrak{h})$. Then $x \in \mathfrak{z}(\mathfrak{g})$ and it follows that x is a multiple of z. Since $\omega(z, y) \neq 0$ for some $y \in \mathfrak{g}$ it follows that (a, z) is not in $\mathfrak{z}(\mathfrak{h})$. Hence $x = 0$ and $\mathfrak{z}(\mathfrak{h})$ is the trivial one-dimensional $\mathfrak{g}$–module K. □

An immediate consequence is the following:

3.1.14. Corollary. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and assume that there exists an affine class $[\omega] \in H^2(\mathfrak{g}, K)$. Then $\mathfrak{g}$ admits an affine structure.*

If we know that the second Betti number of a filiform Lie algebra equals two, i.e., if it is minimal, then we can conclude:

3.1.15. Corollary. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$, $n \geq 6$ be a filiform Lie algebra with $b_2(\mathfrak{g}) = 2$. Then there exists no affine $[\omega] \in H^2(\mathfrak{g}, K)$.*

Proof. If $\mathfrak{g}$ is filiform of dimension $n \geq 5$, there exist two linear independent classes $[\omega_1], [\omega_2] \in H^2(\mathfrak{g}, K)$: let $(e_1, \dots, e_n)$ be an adapted basis for $\mathfrak{g}$ and define ω_1, ω_2 by

$$\begin{aligned} &\omega_1(e_2 \wedge e_3) = 1, \\ &\omega_2(e_2 \wedge e_5) = 1, \ \omega_2(e_3 \wedge e_4) = -1 \end{aligned}$$

where the undefined values are zero. A short calculation shows that $\omega_1, \omega_2 \in Z^2(\mathfrak{g}, K)$: the condition for ω to be a 2–cocycle is

$$\omega([e_i, e_j] \wedge e_k) - \omega([e_i, e_k] \wedge e_j) + \omega([e_j, e_k] \wedge e_i) = 0 \quad \text{for } i < j < k \tag{23}$$

Let $\omega = \omega_1$. The calculation is clear, except maybe for $i = 1, j = 2$. Then it reduces to $\omega_1(e_3 \wedge e_k) = \omega_1(e_{k+1} \wedge e_2)$ for $k \geq 3$.

The cohomology class of ω_1 is nonzero: assume that $\omega_1 \in B^2(\mathfrak{g}, K)$. Then there is an $f \in \mathfrak{g}^*$ with $f([e_i, e_j]) = \omega_1(e_i \wedge e_j)$. This implies $0 = \omega_1(e_1 \wedge e_{k-1}) = f([e_1, e_{k-1}]) = f(e_k)$ for $k \geq 3$. Hence we obtain a contradiction:

$$1 = \omega(e_2 \wedge e_3) = f([e_2, e_3]) = f\Big(\sum_{k=5}^{n} \alpha_{2,k} e_k\Big) = \sum_{k=5}^{n} \alpha_{2,k} f(e_k) = 0$$

Similarly we see that the class of ω_2 and of any linear combination of ω_1 and ω_2 is nonzero. Because of $\dim H^2(\mathfrak{g}, K) = 2$ we know that $([\omega_1], [\omega_2])$ must be a basis of the cohomology. It is clear that for $n \geq 6$ any linear combination of ω_1, ω_2 is zero on $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}$. Hence the corollary is proved. □

3.1.16. Proposition. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ be a filiform Lie algebra with abelian commutator subalgebra. Then $\mathfrak{g}$ admits an affine structure.*

Proof. If $[\mathfrak{g}, \mathfrak{g}]$ is abelian then there exists a basis $(e_1, \dots, e_n)$ such that the Lie brackets are given by

$$\begin{aligned} [e_1, e_i] &= e_{i+1}, \quad i = 2, \dots, n-1 \\ [e_2, e_i] &= \sum_{k=i+2}^{n} \alpha_{2,k-i+3}\, e_k, \quad i = 3, \dots, n-2. \end{aligned}$$

with parameters $\alpha_{2,s} \in K$, where $5 \leq s \leq n$. That is proved in [**17**]. The Jacobi identity is satisfied automatically. Define a left-symmetric product on $\mathfrak{g}$ by

$$\begin{aligned} e_1 \bullet e_i &= e_{i+1}, \quad i = 2, \dots, n-1 \\ e_2 \bullet e_i &= \sum_{k=i+2}^{n} \alpha_{2,k-i+3}\, e_k, \quad i = 2, \dots, n-2 \end{aligned}$$

where $\alpha_{2,n+1} = 0$. A short computation shows that this product satisfies $[e_i, e_j] = e_i \bullet e_j - e_j \bullet e_i$ and $(e_i, e_j, e_k) = (e_j, e_i, e_k)$, where $(e_i, e_j, e_k) = e_i \bullet (e_j \bullet e_k) - (e_i \bullet e_j) \bullet e_k$. □

Let $\omega \in Z^2(\mathfrak{g}, K)$ be a 2–cocycle. We may view ω as a skew-symmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to K$. Another result is:

3.1.17. PROPOSITION. *Let $\mathfrak{g}$ be a Lie algebra such that there exists an $\omega \in Z^2(\mathfrak{g}, K)$ which is nondegenerate as skew-symmetric bilinear form. Then $\mathfrak{g}$ has even dimension and admits an affine structure.*

PROOF. If $\dim \mathfrak{g}$ is odd then ω must be degenerate. Let $\mathfrak{g}^*$ be the dual of the adjoint module $\mathfrak{g}$. The action $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $(x, f) \mapsto x \bullet f$ for $x \in \mathfrak{g}, f \in \mathfrak{g}^*$ where $(x \bullet f)(y) = -f([x, y])$. Define a map $C^2(\mathfrak{g}, K) \to \operatorname{Hom}_K(\mathfrak{g}, \mathfrak{g}^*)$, $\xi \mapsto \varphi_\xi$ with $\varphi_\xi(x) \in \mathfrak{g}^*$, $\varphi_\xi(x)(y) = \xi(x, y)$. That map induces an injection

$$H^2(\mathfrak{g}, K) \hookrightarrow H^1(\mathfrak{g}, \mathfrak{g}^*), \ [\xi] \mapsto [\varphi_\xi]$$

see [**12**]. In fact, φ_ξ is a 1–cocycle:

$$\begin{aligned} \varphi_\xi([x, y])(z) = \xi([x, y], z) &= \xi(y, [x, z]) - \xi(x, [y, z]) \\ &= -\varphi_\xi(y)([x, z]) + \varphi_\xi(x)([y, z]) \\ &= (x \bullet \varphi_\xi(y))(z) - (y \bullet \varphi_\xi(x))(z) \end{aligned}$$

Since ω is nondegenerate, $\ker(\varphi_\omega) = \{x \in \mathfrak{g} \mid \omega(x, y) = 0 \ \forall y\} = 0$. This implies that the linear map $\varphi_\omega : \mathfrak{g} \to \mathfrak{g}^*$ is invertible. We obtain an invertible 1–cocycle $\varphi_\omega \in Z^1(\mathfrak{g}, \mathfrak{g}^*)$. By Proposition 3.1.1, $\mathfrak{g}$ admits an affine structure. □

3.1.2. Necessary conditions for affine structures. There is the following important *necessary* condition for the existence of affine structures:

3.1.18. LEMMA. *Let $\mathfrak{g}$ be a Lie algebra admitting an affine structure. Then $\mathfrak{g}$ has a faithful $\mathfrak{g}$–module of dimension* $\dim \mathfrak{g} + 1$.

PROOF. Since $\mathfrak{g}$ has an affine structure there exists a faithful Lie homomorphism $h : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$ into the Lie algebra of affine transformations. This is the *affine holonomy representation* on the Lie algebra level. If $\dim \mathfrak{g} = n$, then $\mathfrak{aff}(\mathfrak{g}) \subseteq \mathfrak{gl}_{n+1}(K)$ and we obtain a faithful linear representation of dimension $n + 1$. □

3.2. Two classes of filiform Lie algebras

Investigating affine structures on filiform Lie algebras we find that the filiform algebras of dimension $n \geq 12$ satisfying property (b) and (c) are of importance. Recall that these properties for a filiform Lie algebra $\mathfrak{g}$ are:

(b) $\mathfrak{g}$ does not contain a one-codimensional subspace $U \supseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$.
(c) $\mathfrak{g}^{\frac{n-4}{2}}$ is abelian, provided n is even.

By lemma 2.5.9, such Lie algebras have an adapted basis $(e_1, \dots, e_n)$ such that the Lie algebra law is in $\mathcal{A}_n(K)$ and satisfies

$$\alpha_{2,5} = 1, \tag{24}$$

$$\alpha_{\frac{n}{2},n} = 0, \quad \text{if } n \equiv 0(2) \tag{25}$$

If $\mathfrak{g}$ moreover satisfies property (d), i.e., $[\mathfrak{g}^1, \mathfrak{g}^1] \subseteq \mathfrak{g}^6$, then we have also

$$\alpha_{3,7} = 0, \tag{26}$$

see lemma 2.5.9. Define the following two classes of Lie algebras:

3.2.1. Definition. Let $\mathfrak{A}_n^1(K)$ denote the class of filiform Lie algebras of dimension $n \geq 12$ satisfying property (b),(c),(d). There is a basis such that the corresponding Lie algebra laws belong to $\mathcal{A}_n(K)$ and satisfy (24), (25), (26). Denote the set of such Lie algebra laws by $\mathcal{A}_n^1(K)$. Let $\mathfrak{A}_n^2(K)$ denote the class of filiform Lie algebras of dimension $n \geq 12$ satisfying property (b),(c), but not property (d). There is a basis such that the corresponding Lie algebra laws belong to $\mathcal{A}_n(K)$ and satisfy (24), (25) and $\alpha_{3,7} \neq 0$. Denote the set of such Lie algebra laws by $\mathcal{A}_n^2(K)$.

It is clear that these two classes are disjoint, in the sense that a Lie algebra from the first class cannot be isomorphic to one of the second class. Our first result is:

3.2.2. Proposition. *Suppose that $\lambda \in \mathcal{A}_n(K)$, $n \geq 12$ satisfies conditions (24), (25). Then it follows $(\alpha_{3,7}, \alpha_{4,9}, \alpha_{5,11}) = (0,0,0)$ or $(\alpha_{3,7}, \alpha_{4,9}, \alpha_{5,11}) = \left(\frac{1}{10}, \frac{1}{70}, \frac{1}{420}\right)$.*

Proof. The claim follows from the Jacobi identity for λ. If $(e_1, \dots, e_n)$ is an adapted basis, then the Lie brackets with respect to that basis are given by (16), (17), that is

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \dots, n-1$$

$$[e_i, e_j] = \sum_{k=1}^{n} \left(\sum_{\ell=0}^{[(j-i-1)/2]} (-1)^\ell \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,\, k-j+i+2\ell+1} \right) e_k, \quad 2 \leq i < j \leq n.$$

with parameters $\alpha_{k,s} \in K$, where (k,s) runs through the index set $\mathcal{I}_n$ and $\alpha_{k,s} = 0$ for (k,s) not in $\mathcal{I}_n$. Let $J(e_i, e_j, e_k) = 0$ denote the Jacobi identity with e_i, e_j, e_k. Let $J(i,j,k,l)$ be the coefficient of e_l in $J(e_i, e_j, e_k)$. If $n \geq 12$ then we have the conditions $J(2,3,4,9) = J(2,4,5,11) = J(3,4,5,12) = 0$ which consist of the following equations:

$$\alpha_{4,9}(2 + \alpha_{3,7}) - 3\alpha_{3,7}^2 = 0 \tag{27}$$

$$\alpha_{5,11}(2 - \alpha_{3,7} - \alpha_{4,9}) + 2\alpha_{4,9}(3\alpha_{4,9} - 2\alpha_{3,7}) = 0 \tag{28}$$

$$3\alpha_{5,11}(\alpha_{3,7} + \alpha_{4,9}) - 4\alpha_{4,9}^2 = 0 \tag{29}$$

There are precisely two solutions:

$$(\alpha_{3,7}, \alpha_{4,9}, \alpha_{5,11}) = (0,0,0)$$

$$(\alpha_{3,7}, \alpha_{4,9}, \alpha_{5,11}) = \left(\frac{1}{10}, \frac{1}{70}, \frac{1}{420}\right)$$

From (27) we obtain $\alpha_{4,9} = 3\alpha_{3,7}^2/(2+\alpha_{3,7})$. If we substitute that in (29), we have $\alpha_{3,7} = 0$ or $\alpha_{5,11} = 6\alpha_{3,7}^3/((\alpha_{3,7}+2)(2\alpha_{3,7}+1))$. The denominator cannot be zero. Finally, (28) yields $\alpha_{3,7}(10\alpha_{3,7}-1) = 0$. $\square$

3.2.3. Definition. Let $\lambda \in \mathcal{A}_n^1(K)$, $n \geq 12$. Then by property (c) λ depends on the parameters $\alpha_{k,s} \in K$ where the index set for (k,s) is $\mathcal{I}_n^0$. We write this index set $\mathcal{I}_n^0$ as the disjoint union of three subsets as follows:

$$
\begin{aligned}
I_{1,n} &= \{(k,s) \mid 3 \leq k \leq [\frac{n-1}{2}],\ 2k+1 \leq s \leq \min(n, 3k-2)\} \\
I_{2,n} &= \{(k,s) \mid 2 \leq k \leq 3,\ 2k+2 \leq s \leq n\} \\
I_{3,n} &= \{(k,s) \mid 4 \leq k \leq [\frac{n+1}{3}],\ 3k-1 \leq s \leq n\}
\end{aligned}
$$

We use the following notation for λ:

$$\lambda = (\alpha_{k,s} \mid (k,s) \in I_{2,n} \cup I_{1,n} \cup I_{3,n}) \in \mathcal{A}_n^1(K)$$

The next proposition shows that the polynomial equations in the parameters $\alpha_{k,s}$ given by the Jacobi identity for $\lambda \in \mathcal{A}_n^1(K)$ can be solved so that the $2n+12$ parameters $\alpha_{k,s}$, $(k,s) \in I_{2,n}$ remain arbitrary and the others are polynomials in these parameters.

3.2.4. Proposition. *Let $n \geq 12$ and $(k,s) \in I_{2,n}$, $(l,r) \in I_{1,n} \cup I_{3,n}$. Then there exist polynomials $P_{(l,r)} \in K[\alpha_{k,s}]$ such that the following map is bijective:*

$$\Psi : \mathbb{A}^{2n-12}(K) \to \mathcal{A}_n^1(K), \quad (\alpha_{k,s})_{(k,s)\in I_{2,n}} \mapsto (\alpha_{k,s}, P_{(l,r)}((\alpha_{k,s})))_{(l,r)\in I_{1,n}\cup I_{3,n}}$$

Proof. For the proof we have to skip some details which are clear in principle but very lenghty. The Jacobi identity for $\lambda \in \mathcal{A}_n^1(K)$ is equivalent to polynomial equations in the parameters $\alpha_{k,s}$. We use induction on n to show that there is a solution such that:

$$\alpha_{l,r} = 0 \text{ for } (l,r) \in I_{1,n} \tag{30}$$

$$\alpha_{k,s} \text{ are free parameters for } (k,s) \in I_{2,n} \tag{31}$$

$$\alpha_{l,r} = P_{(l,r)}((\alpha_{k,s})) \text{ for } (l,r) \in I_{3,n} \tag{32}$$

Here all parameters with index set $\mathcal{I}_n^0$ have been assigned. The polynomials $P_{(l,r)}$ for $(l,r) \in I_{1,n}$ are just the zero polynomials. If the result holds in dimension $n-1$, we consider the Jacobi equations in dimension n over the field $K(\alpha_{k,s})$ with $(k,s) \in \mathcal{I}_{n-1}^0$. It is then obvious that we obtain linear equations in the variables $x = (\alpha_{4,n}, \ldots, \alpha_{r,n})$, where $r = [\frac{n-1}{2}]$. The system reduces to $r-3$ equations given by $J(2,k,k+1,n) = 0$ for $k = 3, \ldots, r-1$. The corresponding matrix has upper-triangular form with diagonal entries equal to 2. Hence there is a unique solution to express the $\alpha_{k,s}$ with $(k,s) \in \mathcal{I}_n^0 \setminus \mathcal{I}_{n-1}^0$ as polynomials in the free parameters. Therefore the claim holds also in dimension n. It follows that $\mathcal{A}_n^1(K)$ is polynomially isomorphic to the affine space $\mathbb{A}^{2n-12}(K)$. The map Ψ sending the $(2n-12)$–tuple of free parameters to the law $\lambda = (\alpha_{l,r})_{(l,r)\in\mathcal{I}_n^0}$ is bijective. $\square$

3.2.5. Example. *Let $n = 13$ and $\lambda \in \mathcal{A}^1_{13}(K)$. The matrix of coefficients contains 25 parameters*

$$\begin{pmatrix} \alpha_{2,5} & \alpha_{2,6} & \alpha_{2,7} & \alpha_{2,8} & \cdots & \cdots & \alpha_{2,13} \\ & & \alpha_{3,7} & \alpha_{3,8} & \cdots & \cdots & \alpha_{3,13} \\ & & & & \alpha_{4,9} & \cdots & \alpha_{4,13} \\ & & & & & \cdots & \cdots \\ & & & & & & \alpha_{6,13} \end{pmatrix}$$

The 14 parameters $\alpha_{2,6}, \dots, \alpha_{2,13}$ and $\alpha_{3,8}, \dots, \alpha_{3,13}$ can be chosen arbitrary and the remaining $\alpha_{l,r}$ are given by the following polynomials $P_{(l,r)}$:

$$\begin{aligned} \alpha_{3,7} &= \alpha_{4,9} = \alpha_{4,10} = \alpha_{5,11} = \alpha_{5,12} = \alpha_{5,13} = \alpha_{6,13} = 0 \\ \alpha_{2,5} &= 1 \\ \alpha_{4,11} &= 2\alpha_{3,8}^2 \\ \alpha_{4,12} &= -\frac{1}{2}\left[3\alpha_{4,11}(\alpha_{2,6} + \alpha_{3,8}) - 9\alpha_{3,9}\alpha_{3,8}\right] \\ \alpha_{4,13} &= -\frac{1}{2}\left[3\alpha_{4,12}(\alpha_{2,6} + \alpha_{3,8}) - 10\alpha_{3,10}\alpha_{3,8} - 5\alpha_{3,9}^2 + 4\alpha_{3,8}^2(2\alpha_{2,7} + 3\alpha_{3,9})\right] \end{aligned}$$

3.2.6. Definition. Let $\lambda \in \mathcal{A}^2_n(K)$, $n \geq 12$. Write the index set $\mathcal{I}^0_n$ as the disjoint union of three subsets as follows:

$$\begin{aligned} I_{4,n} &= \{(k,s) \mid 3 \leq k \leq [\frac{n-1}{2}],\ s = 2k+1\} \\ I_{5,n} &= \{(k,s) \mid k = 2,\ 6 \leq s \leq n \text{ and } k = 3,\ n-4 \leq s \leq n\} \\ I_{6,n} &= \mathcal{I}^0_n \setminus \{I_{4,n} \cup I_{5,n}\} \end{aligned}$$

We use the following notation for λ:

$$\lambda = (\alpha_{k,s} \mid (k,s) \in I_{5,n} \cup I_{4,n} \cup I_{6,n}) \in \mathcal{A}^2_n(K)$$

3.2.7. Proposition. *Let $n \geq 12$ and $(k,s) \in I_{5,n}$, $(l,r) \in I_{4,n} \cup I_{6,n}$. Then there exist polynomials $P_{(l,r)} \in K[\alpha_{k,s}]$ such that for $n = 12$ and $n \geq 16$ the following map is bijective:*

$$\Psi : \mathbb{A}^n(K) \to \mathcal{A}^2_n(K), \quad (\alpha_{k,s})_{(k,s)\in I_{5,n}} \mapsto (\alpha_{k,s}, P_{(l,r)}((\alpha_{k,s})))_{(l,r)\in I_{4,n}\cup I_{6,n}}$$

For $n = 13$, 14, 15 we have one additional free parameter, i.e., the corresponding map $\Psi : \mathbb{A}^{n+1}(K) \to \mathcal{A}^2_n(K)$ is bijective.

Proof. For $12 \leq n \leq 15$ the result follows by an explicit computation: For $n = 12$ see example 3.2.8. For $n = 13$ the free parameters can be chosen as $\alpha_{2,6}, \dots, \alpha_{2,13}$ and $\alpha_{3,8}, \dots, \alpha_{3,12}$. For $n = 14$ the free parameters are $\alpha_{2,6}, \dots, \alpha_{2,14}$ and $\alpha_{3,8}, \alpha_{3,10}, \dots, \alpha_{3,14}$, but not $\alpha_{3,9}$. For $n = 15$ the free parameters are $\alpha_{2,6}, \dots, \alpha_{2,15}$ and $\alpha_{3,8}, \alpha_{3,11}, \dots, \alpha_{3,15}$, but not $\alpha_{3,9}, \alpha_{3,10}$. That continues, but for $n \geq 16$ the Jacobi identity also implies a condition involving $\alpha_{3,8}$. We have one less free parameter for $n \geq 16$. We use induction on n to show that the following holds for $n \geq 16$:

$$\alpha_{l,2l+1} = \frac{(l-2)\,!}{2^{l-2}\cdot 5\cdot 7\cdots(2l-1)} \text{ for } (l,r)\in I_{4,n} \tag{33}$$

$$\alpha_{k,s} \text{ are free parameters for } (k,s)\in I_{5,n} \tag{34}$$

$$\alpha_{l,r} = P_{(l,r)}((\alpha_{k,s})) \text{ for } (l,r)\in I_{6,n} \tag{35}$$

There are n free parameters. If the result holds in dimension $n-1$, we consider the Jacobi equations in dimension n over the field $K(\alpha_{k,s})$ with $(k,s)\in \mathcal{I}^0_{n-1}$. We obtain linear equations in the $r-2$ variables $x = (\alpha_{3,n-5}, \alpha_{4,n}, \dots, \alpha_{r,n})$, where $r = [\frac{n-1}{2}]$. Writing down all equations we see that the corresponding matrix has full rank, so that there is a unique solution to express the $\alpha_{k,s}$ with $(k,s)\in \mathcal{I}^0_n \setminus \mathcal{I}^0_{n-1}$ as polynomials in the free parameters. Hence the result holds in dimension n. It follows that $\mathcal{A}^2_n(K)$, $n\geq 16$ is polynomially isomorphic to the affine space $\mathbb{A}^n(K)$. □

3.2.8. Example. *Let $n = 12$ and $\lambda \in \mathcal{A}^2_{12}(K)$. The matrix of coefficients contains 21 parameters. The 12 parameters $\alpha_{2,6},\dots,\alpha_{2,12}$ and $\alpha_{3,8},\dots,\alpha_{3,12}$ can be chosen arbitrary. The remaining $\alpha_{l,r}$ are given by the following polynomials $P_{(l,r)}$:*

$$\begin{aligned}
\alpha_{2,5} &= 1\\
\alpha_{3,7} &= 1/10\\
\alpha_{4,9} &= 1/70\\
\alpha_{5,11} &= 1/420\\
\alpha_{6,12} &= 0\\
\alpha_{4,10} &= (46\alpha_{3,8} - 3\alpha_{2,6})\,/147\\
\alpha_{5,12} &= (9250\alpha_{3,8} - 795\alpha_{2,6})\,/116424\\
\alpha_{4,11} &= \left(2107\alpha_{3,9} + 9000\alpha_{3,8}^2 - 2580\alpha_{3,8}\alpha_{2,6} - 154\alpha_{2,7} + 180\alpha_{2,6}^2\right)/6174\\
\alpha_{4,12} &= (3124044\alpha_{3,10} + 25174730\alpha_{3,9}\alpha_{3,8} - 3700725\alpha_{3,9}\alpha_{2,6} - 17820000\alpha_{3,8}^3\\
&\quad - 12711600\alpha_{3,8}^2\alpha_{2,6} - 4148060\alpha_{3,8}\alpha_{2,7} + 4752000\alpha_{3,8}\alpha_{2,6}^2 - 242550\alpha_{2,8}\\
&\quad + 581910\alpha_{2,7}\alpha_{2,6} - 356400\alpha_{2,6}^3)/8557164
\end{aligned}$$

3.2.9. Definition. If $\mathfrak{g}$ is a filiform Lie algebra of even dimension $n\geq 12$ satisfying property (b), but not property (c), there is a basis such that the Lie algebra law belongs to $\mathcal{A}_n(K)$ and satisfies $\alpha_{2,5} = 1$ and $\alpha_{\frac{n}{2},n}\neq 0$. Denote the set of such Lie algebra laws by $\mathcal{A}^3_n(K)$ and the class of such Lie algebras by $\mathfrak{A}^3_n(K)$.

We will study that class of Lie algebras only for $n = 12$. Algebras from $\mathfrak{A}^3_{12}(K)$ play a role for the extensions of 11–dimensional filiform Lie algebras. In general, for $n\geq 12$ it is difficult to obtain a characterization of $\mathcal{A}^3_n(K)$. An easy consequence of the Jacobi identity is the following:

3.2.10. Lemma. *Let $\lambda\in\mathcal{A}^3_{12}(K)$ with $\alpha_{2,5} = 1$. Then it follows $2\alpha_{2,5} + \alpha_{3,7}\neq 0$ and*

$$(2\alpha_{2,5}^2 - 5\alpha_{3,7}^2)(4\alpha_{2,5}^2 - 4\alpha_{2,5}\alpha_{3,7} + 3\alpha_{3,7}^2) = 0 \tag{36}$$

3.3. Affine structures induced by extensions

In this section we study the extension property for filiform Lie algebras of the classes $\mathfrak{A}_n^1(K)$ and $\mathfrak{A}_n^2(K)$. If the extension property for $\mathfrak{g}$ is satisfied, we obtain an affine structure on $\mathfrak{g}$, see Proposition 3.1.8.

3.3.1. THEOREM. *Let $\mathfrak{g} \in \mathfrak{A}_n^1(K)$. Then $\mathfrak{g}$ has an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with a Lie algebra $\mathfrak{h} \in \mathfrak{A}_{n+1}^1(K)$.

PROOF. Define maps

$$\Theta_n^1 : \mathcal{A}_{n+1}^1(K) \to \mathcal{A}_n^1(K), \quad (\alpha_{k,s})_{(k,s)\in\mathcal{I}_{n+1}^0} \mapsto (\alpha_{k,s})_{(k,s)\in\mathcal{I}_n^0}$$
$$\Phi_n^1 : \mathcal{A}_n^1(K) \to \mathcal{A}_{n+1}^1(K), \quad (\alpha_{k,s})_{(k,s)\in\mathcal{I}_n^0} \mapsto (\beta_{k,s})_{(k,s)\in\mathcal{I}_{n+1}^0}$$

where

$$\beta_{k,s} = \begin{cases} \alpha_{k,s} & \text{if } (k,s) \in \mathcal{I}_n^0 = I_{1,n} \cup I_{2,n} \cup I_{3,n}, \\ \text{free} & \text{if } (k,s) \in I_{2,n+1} \setminus I_{2,n} = \{(2,n+1),(3,n+1)\}, \\ P_{(k,s)}((\beta_{i,j})) & \text{if } (k,s) \in I_{1,n+1} \cup I_{3,n+1},\ (i,j) \in I_{2,n+1} \end{cases}$$

where $P_{(k,s)}$ are the polynomials from Proposition 3.2.4. That is well defined since the polynomials for $\mathcal{A}_n^1(K)$ are a subset of the polynomials for $\mathcal{A}_{n+1}^1(K)$. Let $\lambda = (\alpha_{k,s}) \in \mathcal{A}_n^1(K)$ and $\lambda' = (\beta_{k,s})$. It follows that $\lambda' \in \mathcal{A}_{n+1}^1(K)$: We have assigned the $\beta_{k,s}$ so that the Jacobi identity for λ' holds without any restrictions on the $\alpha_{k,s}$, see the proof of Proposition 3.2.4. Since $I_{2,n} \subset I_{2,n+1}$ the free parameters of $\lambda \in \mathcal{A}_n^1(K)$ remain free for $\lambda' \in \mathcal{A}_n^2(K)$. Note that this is not true in the case of Lie algebra laws from $\mathcal{A}_n^2(K)$, because $I_{5,n}$, the index set of free parameters, is in general *not* contained in $I_{5,n+1}$.

It follows $\Theta_n^1 \circ \Phi_n^1 = \mathrm{id}_{|\mathcal{A}_n^1(K)}$. Denote by $\mathfrak{g}$ the Lie algebra corresponding to $\lambda \in \mathcal{A}_n^1(K)$ with adapted basis $(e_1,\dots,e_n)$ and $\mathfrak{h}$ the Lie algebra corresponding to $\lambda' \in \mathcal{A}_{n+1}^1(K)$ with adapted basis $(f_1,\dots,f_{n+1})$. Let $\pi : \mathfrak{h} \to \mathfrak{g}$ by $\pi(f_i) = e_i$ for $i = 1,\dots,n$ and $\pi(f_{n+1}) = 0$. By definition, π is a surjective Lie algebra homomorphism with $\ker\pi = \mathfrak{z}(\mathfrak{h})$. Moreover $\tau : \mathfrak{g} \to \mathfrak{h}$, $\tau(e_i) = f_i$ is an injective map with $\pi \circ \tau = \mathbf{1}$. □

3.3.2. COROLLARY. *All Lie algebras $\mathfrak{g} \in \mathfrak{A}_n^1(K)$, $n \geq 12$ admit an affine structure induced by an affine cohomology class.*

In the case of $\mathcal{A}_n^2(K)$ we define the maps

$$\Theta_n^2 : \mathcal{A}_{n+1}^2(K) \to \mathcal{A}_n^2(K), \quad (\alpha_{k,s})_{(k,s)\in\mathcal{I}_{n+1}^0} \mapsto (\alpha_{k,s})_{(k,s)\in\mathcal{I}_n^0}$$
$$\Phi_n^2 : \mathcal{A}_n^2(K) \to K^{|\mathcal{I}_{n+1}^0|}, \quad (\alpha_{k,s})_{(k,s)\in\mathcal{I}_n^0} \mapsto (\beta_{k,s})_{(k,s)\in\mathcal{I}_{n+1}^0}$$

where

$$\beta_{k,s} = \begin{cases} \alpha_{k,s} & \text{if } (k,s) \in \mathcal{I}_n^0 = I_{4,n} \cup I_{5,n} \cup I_{6,n}, \\ \text{free} & \text{if } (k,s) \in \{(2,n+1),(3,n+1)\}, \\ P_{(k,s)}((\beta_{i,j})) & \text{if } (k,s) \in I_{4,n+1} \cup I_{6,n+1},\ (i,j) \in I_{5,n+1} \end{cases}$$

Let $\lambda = (\alpha_{k,s}) \in \mathcal{A}_n^2(K)$. For the map Φ_n^2 to be well defined and for $\Phi_n^2(\lambda) \in \mathcal{A}_{n+1}^2(K)$ to hold, λ has to satisfy one additional polynomial condition, namely

$$\alpha_{3,n-4} = P_n((\alpha_{k,s}))$$

with a certain polynomial P_n in the free variables. The reason is the following: Let $n \geq 16$. We have $I_{5,n} \subsetneq I_{5,n+1}$. The only element of $I_{5,n}$ which is not contained in $I_{5,n+1}$ is $\alpha_{3,n-4}$. That is a free parameter in dimension n but not in dimension $n+1$. The Jacobi identity for $\Phi_n^2(\lambda)$ imposes a polynomial condition on $\alpha_{3,n-4}$.

3.3.3. Definition. $\lambda = (\alpha_{k,s}) \in \mathcal{A}_n^2(K)$ and its corresponding Lie algebra $\mathfrak{g}_\lambda$ are said to satisfy property (L), if $\Phi_n^2(\lambda) \in \mathcal{A}_{n+1}^2(K)$, or equivalently if $\alpha_{3,n-4} = P_n((\alpha_{k,s}))$ holds for a certain polynomial P_n.

3.3.4. Remark. There are two different methods to determine the polynomial P_n. First we may compute the Jacobi identity for $\Phi_n^2(\lambda)$. Secondly we can compute $H^2(\mathfrak{g}_\lambda, K)$. It contains an affine cohomology class iff property (L) is satisfied.

The preceding discussion shows that the extension property does not hold in general for Lie algebras from $\mathfrak{A}_n^2(K)$. More precisely we have obtained:

3.3.5. Proposition. *Let $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$. Then there is an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with $\mathfrak{h} \in \mathfrak{A}_{n+1}^2(K)$ iff property (L) holds.

In particular for $n = 13$ we have:

3.3.6. Proposition. *A Lie algebra $\mathfrak{g} \in \mathfrak{A}_{13}^2(K)$ has an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with an $\mathfrak{h} \in \mathfrak{F}_{14}(K)$ if and only if

$$\alpha_{3,9} = P_{13}((\alpha_{k,s})) = \frac{1}{30030}\left(4290\alpha_{2,7} + 3321\alpha_{2,6}^2 - 92100\alpha_{2,6}\alpha_{3,8} + 514300\alpha_{3,8}^2\right) \tag{37}$$

Proof. As remarked above we have two possibilities to determine the polynomial P_{13}. Here we compute $H^2(\mathfrak{g}, K)$, see also Proposition 3.4.5, and apply Proposition 3.1.13. The result is: If (37) does not hold then $\dim H^2(\mathfrak{g}, K) = 2$ and hence $\mathfrak{g}$ cannot have such an extension. Otherwise $\dim H^2(\mathfrak{g}, K) = 3$ and there is an affine cohomology class. □

For $n = 12$ we have:

3.3.7. Proposition. *Let $n = 12$. Then all Lie algebras $\mathfrak{g} \in \mathfrak{A}_{12}^2(K)$ have an extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with a Lie algebra $\mathfrak{h} \in \mathfrak{A}_{13}^2(K)$. Hence these Lie algebras possess an affine structure.

Proof. We determine again $H^2(\mathfrak{g}, K)$. A straightforward computation shows that $b_2(\mathfrak{g}) = 3$ and that $H^2(\mathfrak{g}, K)$ contains an affine cohomology class, see Proposition 3.4.4. Hence the result follows. □

We also study the extension property for $\mathfrak{A}_{12}^3(K)$:

3.3.8. PROPOSITION. *A Lie algebra* $\mathfrak{g} \in \mathfrak{A}^3_{12}(K)$ *has no extension*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with an $\mathfrak{h} \in \mathfrak{F}_{13}(K)$.

PROOF. A computation shows that the space $H^2(\mathfrak{g}, K)$ is 4–dimensional and does not contain an affine cohomology class, see Proposition 3.4.7 $\square$

3.4. Computation of $H^2(\mathfrak{g}, K)$

In this section we compute the cohomology groups $H^2(\mathfrak{g}, K)$ for all $\mathfrak{g} \in \mathfrak{F}_n(K)$ with $n \leq 11$, and for some cases with $n = 12, 13$. All computations are done with the symbolic algebra package Reduce. If there exists an affine cohomology class, we obtain a central extension and an affine structure on $\mathfrak{g}$. Let $(e_1, \ldots, e_n)$ be an adapted basis for $\mathfrak{g}$.

3.4.1. LEMMA. *Let* $\omega \in \operatorname{Hom}(\Lambda^2\mathfrak{g}, K)$. *Then* ω *is an affine* 2*–cocycle iff* $\omega(e_1 \wedge e_n)$ *or* $\omega(e_2 \wedge e_n)$ *is nonzero.*

PROOF. By definition, ω is affine iff $\omega(e_j \wedge e_n) \neq 0$ for some $j \in \{1, \ldots, n\}$. But equation (23) for $i = 1, k = n$ implies $\omega(e_j \wedge e_n) = 0$ for $3 \leq j \leq n$. $\square$

3.4.2. DEFINITION. Define $\omega_\ell \in \operatorname{Hom}(\Lambda^2\mathfrak{g}, K)$ as follows:

$$\omega_\ell(e_k \wedge e_{2\ell+3-k}) = (-1)^k \quad \text{for } 1 \leq \ell \leq [(n-1)/2],\ 2 \leq k \leq [(2\ell+3)/2] \tag{38}$$

In the following we will mainly use $\omega_1, \ldots, \omega_4$. They are defined by

$$\begin{aligned}
&\omega_1(e_2 \wedge e_3) = 1\\
&\omega_2(e_2 \wedge e_5) = 1,\ \omega_2(e_3 \wedge e_4) = -1\\
&\omega_3(e_2 \wedge e_7) = 1,\ \omega_3(e_3 \wedge e_6) = -1,\ \omega_3(e_4 \wedge e_5) = 1\\
&\omega_4(e_2 \wedge e_9) = 1,\ \omega_4(e_3 \wedge e_8) = -1,\ \omega_4(e_4 \wedge e_7) = 1,\ \omega_4(e_5 \wedge e_6) = -1
\end{aligned}$$

3.4.3. LEMMA. *We have* $\omega_1, \omega_2 \in Z^2(\mathfrak{g}, K)$, *whereas* ω_ℓ, $\ell \geq 3$ *need not be* 2*–cocycles. If* $\ell < [(n-1)/2]$, *then* ω_ℓ *cannot be an affine* 2*–cocycle.*

PROOF. The first part follows from the proof of Corollary 3.1.15. Secondly we have $\omega_\ell(e_i \wedge e_n) = 0$ for $\ell < [(n-1)/2]$. $\square$

3.4.4. PROPOSITION. *Let* $\mathfrak{g} \in \mathfrak{A}^2_{12}(K)$. *Then* $H^2(\mathfrak{g}, K) = \operatorname{span}\{[\omega_1], [\omega_2], [\omega]\}$, *where* ω_1, ω_2 *are as in* (38) *and* ω *is an affine* 2*–cocycle with*

$$\begin{aligned}
\omega(e_1 \wedge e_{12}) &= 1\\
\omega(e_2 \wedge e_4) &= \alpha_{2,12}\\
\omega(e_2 \wedge e_5) &= \alpha_{2,11}\\
\omega(e_2 \wedge e_6) &= \alpha_{2,10} - 2\alpha_{3,12}\\
\vdots \quad &= \quad \vdots\\
\omega(e_6 \wedge e_7) &= \frac{1}{2310}
\end{aligned}$$

3.4.5. Proposition. *Let* $\mathfrak{g} \in \mathfrak{A}_{13}^2(K)$*. Then*

$$H^2(\mathfrak{g}, K) = \begin{cases} \operatorname{span}\{[\omega_1], [\omega_2], [\omega]\} & \text{if } \mathfrak{g} \text{ satisfies property (L)}, \\ \operatorname{span}\{[\omega_1], [\omega_2]\} & \text{otherwise.} \end{cases}$$

where ω *is an affine* 2*–cocycle with*

$$\begin{aligned} \omega(e_1 \wedge e_{13}) &= 1 \\ \omega(e_2 \wedge e_4) &= \alpha_{2,13} \\ \omega(e_2 \wedge e_6) &= \alpha_{2,11} - 2\alpha_{3,13} \\ \vdots \quad &= \quad \vdots \\ \omega(e_6 \wedge e_8) &= \frac{1}{2310} \end{aligned}$$

3.4.6. Remark. The last result can be generalized for all $n \geq 13$. In particular, $b_2(\mathfrak{g}) = 2$ for all algebras $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$ not satisfying property (L). These algebras have minimal second Betti numbers (among nilpotent Lie algebras) and are candidates for Lie algebras without affine structures.

3.4.7. Proposition. *Let* $\mathfrak{g} \in \mathfrak{A}_{12}^3(K)$*. Then* $H^2(\mathfrak{g}, K) = \operatorname{span}\{[\omega_1], \dots, [\omega_4]\}$*. In particular,* $H^2(\mathfrak{g}, K)$ *does not contain an affine cohomology class.*

In the following we determine the spaces $H^2(\mathfrak{g}, K)$ for all filiform Lie algebras $\mathfrak{g}$ of dimension $n \leq 11$. The numbers $\dim H^2(\mathfrak{g}, K)$ are called the second Betti numbers. The cohomology spaces give important information on $\mathfrak{g}$. In our case, we will use it to determine the existence of affine cohomology classes. For the computations we choose an adapted basis for $\mathfrak{g}$ so that its Lie algebra law lies in $\mathcal{A}_n(K)$. We divide $\mathcal{A}_n(K)$, $6 \leq n \leq 11$ into the following subsets depending on certain equalities or inequalities of the structure constants. These subsets correspond to well defined classes of filiform Lie algebras:

$$\begin{array}{ll}
\mathcal{A}_{6,1} & \text{if } \alpha_{3,6} \neq 0 \\
\mathcal{A}_{6,2} & \text{if } \alpha_{3,6} = 0 \\
\mathcal{A}_{7,1} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} \neq 0 \\
\mathcal{A}_{7,2} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} = 0 \\
\mathcal{A}_{8,1} & \text{if } \alpha_{4,8} \neq 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0 \\
\mathcal{A}_{8,2} & \text{if } \alpha_{4,8} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} \neq 0 \\
\mathcal{A}_{8,3} & \text{if } \alpha_{4,8} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{2,5} \neq 0 \\
\mathcal{A}_{8,4} & \text{if } \alpha_{4,8} = \alpha_{3,7} = \alpha_{2,5} = 0 \\
\mathcal{A}_{9,1} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ \alpha_{3,7}^2 \neq \alpha_{2,5}^2 \\
\mathcal{A}_{9,2} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ \alpha_{3,7}^2 = \alpha_{2,5}^2 \\
\mathcal{A}_{9,3} & \text{if } \alpha_{2,5} = \alpha_{3,7} = 0,\ \alpha_{4,9} \neq 0,\ \alpha_{2,6} + \alpha_{3,8} \neq 0 \\
\mathcal{A}_{9,4} & \text{if } \alpha_{2,5} = \alpha_{3,7} = 0,\ \alpha_{4,9} \neq 0,\ \alpha_{2,6} + \alpha_{3,8} = 0 \\
\mathcal{A}_{9,5} & \text{if } \alpha_{2,5} = \alpha_{3,7} = \alpha_{4,9} = 0,\ 2\alpha_{2,7} + \alpha_{3,9} \neq 0 \\
\mathcal{A}_{9,6} & \text{if } \alpha_{2,5} = \alpha_{3,7} = \alpha_{4,9} = 0,\ 2\alpha_{2,7} + \alpha_{3,9} = 0
\end{array}$$

$\mathcal{A}_{10,1}$ if $\alpha_{5,10} \neq 0,\ 2\alpha_{2,5} + \alpha_{3,7} \neq 0$
$\mathcal{A}_{10,2}$ if $\alpha_{5,10} \neq 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0$
$\mathcal{A}_{10,3}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ \alpha_{3,7}^2 \neq \alpha_{2,5}^2$
$\mathcal{A}_{10,4}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ \alpha_{3,7}^2 = \alpha_{2,5}^2$
$\mathcal{A}_{10,5}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} \neq 0,\ \alpha_{2,6}^2 + 2\alpha_{2,7}\alpha_{4,9} \neq 0$
$\mathcal{A}_{10,6}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} \neq 0,\ \alpha_{2,6}^2 + 2\alpha_{2,7}\alpha_{4,9} = 0$
$\mathcal{A}_{10,7}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} = 0,\ 2\alpha_{2,7} + \alpha_{3,9} \neq 0$
$\mathcal{A}_{10,8}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} = 0,\ 2\alpha_{2,7} + \alpha_{3,9} = 0,\ \alpha \neq 0$
$\mathcal{A}_{10,9}$ if $\alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} = 0,\ 2\alpha_{2,7} + \alpha_{3,9} = 0,\ \alpha = 0$
$\mathcal{A}_{11,1}$ if $2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ \alpha_{3,7} \neq 0,\ 10\alpha_{3,7} - \alpha_{2,5} \neq 0,\ \beta \neq 0$
$\mathcal{A}_{11,2}$ if $2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ 10\alpha_{3,7} - \alpha_{2,5} \neq 0;\ \beta = 0$ or $\alpha_{3,7} = 0$
$\mathcal{A}_{11,3}$ if $2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ 10\alpha_{3,7} - \alpha_{2,5} = 0$
$\mathcal{A}_{11,4}$ if $2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} \neq 0$
$\mathcal{A}_{11,5}$ if $2\alpha_{2,5} + \alpha_{3,7} = 0,\ \alpha_{4,9} = 0$

where $\alpha = 3\alpha_{4,10}(\alpha_{2,6} + \alpha_{3,8}) - 4\alpha_{3,8}^2$ and $\beta = (2\alpha_{2,5}^2 - 5\alpha_{3,7}^2)(4\alpha_{2,5}^2 - 4\alpha_{2,5}\alpha_{3,7} + 3\alpha_{3,7}^2)$.

3.4.8. PROPOSITION. *For $\lambda \in \mathcal{A}_n(K)$, $4 \leq n \leq 11$ the corresponding Lie algebra $\mathfrak{g}_\lambda$ admits an affine cohomology class as follows:*

$\dim \mathfrak{g}_\lambda$	*Class*	$H^2(\mathfrak{g}_\lambda, K)$	*affine* ω	$b_2(\mathfrak{g}_\lambda)$
4	$\mathcal{A}_4$	ω_1, ω	✓	2
5	$\mathcal{A}_5$	$\omega_1, \omega_2, \omega$	✓	3
6	$\mathcal{A}_{6,1}$	ω_1, ω_2	—	2
6	$\mathcal{A}_{6,2}$	$\omega_1, \omega_2, \omega$	✓	3
7	$\mathcal{A}_{7,1}$	$\omega_1, \omega_2, \omega$	✓	3
7	$\mathcal{A}_{7,2}$	$\omega_1, \omega_2, \omega_3, \omega$	✓	4
8	$\mathcal{A}_{8,1}$	$\omega_1, \omega_2, \omega_3$	—	3
8	$\mathcal{A}_{8,2}$	$\omega_1, \omega_2, \omega$	✓	3
8	$\mathcal{A}_{8,3}$	$\omega_1, \omega_2, \omega_3$	—	3
8	$\mathcal{A}_{8,4}$	$\omega_1, \omega_2, \omega_3, \omega$	✓	4
9	$\mathcal{A}_{9,1}$	$\omega_1, \omega_2, \omega$	✓	3
9	$\mathcal{A}_{9,2}$	$\omega_1, \omega_2, \omega, \omega'$	✓	4
9	$\mathcal{A}_{9,3}$	$\omega_1, \omega_2, \omega_3$	—	3
9	$\mathcal{A}_{9,4}$	$\omega_1, \omega_2, \omega_3, \omega$	✓	4
9	$\mathcal{A}_{9,5}$	$\omega_1, \omega_2, \omega_3, \omega$	✓	4
9	$\mathcal{A}_{9,6}$	$\omega_1, \omega_2, \omega_3, \omega, \omega'$	✓	5
10	$\mathcal{A}_{10,1}$	$\omega_1, \omega_2, \omega_3$	—	3
10	$\mathcal{A}_{10,2}$	$\omega_1, \omega_2, \omega_3, \omega_4$	—	4
10	$\mathcal{A}_{10,3}$	$\omega_1, \omega_2, \omega$	✓	3
10	$\mathcal{A}_{10,4}$	$\omega_1, \omega_2, \omega_3$	—	3
10	$\mathcal{A}_{10,5}$	$\omega_1, \omega_2, \omega_3$	—	3
10	$\mathcal{A}_{10,6}$	$\omega_1, \omega_2, \omega_3, \omega$	✓	4
10	$\mathcal{A}_{10,7}$	$\omega_1, \omega_2, \omega_3, \omega$	✓	4
10	$\mathcal{A}_{10,8}$	$\omega_1, \omega_2, \omega_3, \omega_4$	—	4
10	$\mathcal{A}_{10,9}$	$\omega_1, \omega_2, \omega_3, \omega_4, \omega$	✓	5

$\dim \mathfrak{g}_\lambda$	*Class*	$H^2(\mathfrak{g}_\lambda, K)$	*affine* ω	$b_2(\mathfrak{g}_\lambda)$
11	$\mathcal{A}_{11,1}$	ω_1, ω_2	$-$	2
11	$\mathcal{A}_{11,2}$	$\omega_1, \omega_2, \omega$	✓	3
11	$\mathcal{A}_{11,3}$	$\omega_1, \omega_2, \omega$	✓	3
11	$\mathcal{A}_{11,4}$	$\omega_1, \omega_2, \omega_3$	$-$	3

The notations here are as follows. By $\omega_1, \dots, \omega_4$ we denote always the 2–cocycles defined by (38). ω stands for an affine 2–cocycle, which might be different for distinct classes of Lie algebras. The same holds for ω'. For the cohomology the table shows representing 2–cocycles for a basis. So $\omega_1, \omega_2, \omega$ in the table means that $([\omega_1], [\omega_2], [\omega])$ is a basis of $H^2(\mathfrak{g}_\lambda, K)$. Note that for $n \geq 5$ the two-dimensional subspace spanned by $[\omega_1], [\omega_2]$ is always contained in $H^2(\mathfrak{g}_\lambda, K)$. A checkmark denotes the existence and a minus sign the absence of an affine 2–cocycle.

3.4.9. Remark. For $\lambda \in \mathcal{A}_{11,5}$ the Jacobi identity implies $(\alpha_{2,5}, \alpha_{3,7}, \alpha_{4,9}) = (0, 0, 0)$. In that case we have also determined the affine 2–cocycles for $\mathfrak{g}_\lambda$. However, this requires to introduce quite a lot of subclasses. On the other hand, it is not difficult to show that all such algebras admit an affine structure. To avoid unnecessary complicated notations we will omit the result here in the table.

3.4.10. Remark. Let $\lambda \in \mathcal{A}_{6,1}$. Then $\mathfrak{g}_\lambda$ does not admit an affine $[\omega] \in H^2(\mathfrak{g}_\lambda, K)$. However, $\mathfrak{g}_\lambda$ admits an affine structure since there exists a nonsingular derivation.

3.4.11. Example. *Let $\lambda \in \mathcal{A}_9(K)$ and $(e_1, \dots, e_9)$ be an adapted basis, see example 2.4.9. If $\lambda \in \mathcal{A}_{9,5}$ then the space $H^2(\mathfrak{g}_\lambda, K)$ is spanned by the classes of $\omega_1, \omega_2, \omega_3, \omega$, where ω is an affine 2–cocycle defined by:*

$$
\begin{aligned}
\omega(e_1 \wedge e_9) &= 1 \\
\omega(e_2 \wedge e_4) &= \alpha_{2,9} \\
\omega(e_2 \wedge e_6) &= \alpha_{2,7} - 2\alpha_{3,9} \\
\omega(e_2 \wedge e_7) &= \alpha_{2,6} - 3\alpha_{3,8} \\
\omega(e_3 \wedge e_4) &= \alpha_{2,8} \\
\omega(e_3 \wedge e_5) &= \alpha_{3,9} \\
\omega(e_3 \wedge e_6) &= \alpha_{3,8}
\end{aligned}
$$

In particular $\mathfrak{g}_\lambda$ admits an affine structure.

3.5. Affine structures for $\mathfrak{g} \in \mathfrak{F}_n(K)$, $n \leq 11$

In this section we treat the existence problem of affine structures for all filiform Lie algebras $\mathfrak{g} \in \mathfrak{F}_n(K)$, $n \leq 11$. Here we use the information of the previous sections so that we do not consider the algebras which admit an affine structure by the existence of an affine cohomology class. The answer will be affirmative for $n \leq 9$, but not for $n = 10, 11$. In fact, the counterexamples to the Milnor conjecture show that there are algebras without affine structures for $n = 10$ and $n = 11$. Nevertheless we can construct affine structures in many cases, using modifications of the adjoint representation. We will choose an adapted basis $(e_1, \dots, e_n)$ for $\mathfrak{g}$. It is necessary to refine the sets of structure constants for $n = 10, 11$. We split the sets $\mathcal{A}_{10,1}$, $\mathcal{A}_{10,4}$, $\mathcal{A}_{11,1}$ as follows:

$$\begin{array}{ll}
\mathcal{A}^1_{10,1} & \text{if } \alpha_{5,10} \ne 0,\ 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{3,7} = -\alpha_{2,5} \\
\mathcal{A}^2_{10,1} & \text{if } \alpha_{5,10} \ne 0,\ 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{3,7} = \alpha_{2,5} \\
\mathcal{A}^1_{10,4} & \text{if } \alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{3,7} = -\alpha_{2,5} \\
\mathcal{A}^2_{10,4} & \text{if } \alpha_{5,10} = 0,\ 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{3,7} = \alpha_{2,5} \\
\mathcal{A}^1_{11,1} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{2,5} = 0 \\
\mathcal{A}^2_{11,1} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{3,7} = 0 \\
\mathcal{A}^3_{11,1} & \text{if } 2\alpha_{2,5} + \alpha_{3,7} \ne 0,\ \alpha_{2,5}, \alpha_{3,7} \ne 0,\ 10\alpha_{3,7} - \alpha_{2,5} \ne 0,\ \beta \ne 0
\end{array}$$

where $\beta = (2\alpha^2_{2,5} - 5\alpha^2_{3,7})(4\alpha^2_{2,5} - 4\alpha_{2,5}\alpha_{3,7} + 3\alpha^2_{3,7})$. Note that the Jacobi identity for $\lambda \in \mathcal{A}_{10,1}$ implies $\alpha^2_{3,7} = \alpha^2_{2,5}$.

3.5.1. Nonsingular derivations. If a Lie algebra admits a nonsingular derivation then there exists an affine structure by Corollary 3.1.2. Unfortunately for filform Lie algebras that is rarely the case. One should use additional methods to construct affine structures. Nevertheless it is interesting to determine the filiform Lie algebras possessing a nonsingular derivation. That holds in particular for algebras of dimension 10 and 11 where an affine structure does not exist in general. Hence the existence of a nonsingular derivation is a valuable information.

Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and $(e_1, \ldots, e_n)$ be an adapted basis. For $n \le 11$ the derivations of $\mathfrak{g}$ can be determined by a straightforward computation with a symbolic computer algebra package. In addition, if D is a nonsingular derivation of $\mathfrak{g}$ then the linear operators

$$L(e_i) = \begin{pmatrix} \operatorname{ad}(e_i) & D(e_i) \\ 0 & 0 \end{pmatrix}$$

define a faithful $\mathfrak{g}$–module of dimension $n + 1$, see Lemma 2.1.2. Whether such a faithful module exists can be also decided by a straightforward computation.

3.5.1. PROPOSITION. *Let $\lambda \in \mathcal{A}_{9,3}$. Then $\mathfrak{g}_\lambda$ does not possess a nonsingular derivation.*

PROOF. If $\lambda \in \mathcal{A}_9(K)$ and $D \in \operatorname{Der}(\mathfrak{g}_\lambda)$, then it is clear that the matrix of D with respect to the adapted basis $(e_1, \ldots, e_n)$ is lower-triangular. Hence its determinant is given by the product of the elements in the diagonal. If not all $\alpha_{k,s}$ are zero then $\det(D) = 0$ if and only if $D(e_9) = 0$. If not $(\alpha_{2,5}, \alpha_{3,7}, \alpha_{4,9}) = (0, 0, 0)$, the diagonal of D is of the form $(\xi, 2\xi, \ldots, 9\xi)$ for some $\xi \in K$. For $\lambda \in \mathcal{A}_{9,3}$ we obtain the equation $\xi(\alpha_{2,6} + \alpha_{3,8}) = 0$, hence $\xi = 0$ and $\det(D) = 0$. □

3.5.2. PROPOSITION. *If $\lambda \in \mathcal{A}^1_{10,1}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\begin{aligned}
\alpha_{2,6} &= 2\alpha^2_{2,5}/\alpha_{5,10} \\
\alpha_{3,8} &= -3\alpha_{2,6} \\
\alpha_{3,9} &= -3(10\alpha^3_{2,5} + \alpha_{2,7}\alpha^2_{5,10})/\alpha^2_{5,10} \\
\alpha_{3,10} &= (\alpha^4_{5,10}(\alpha_{2,5}\alpha_{2,9} + \alpha^2_{2,7}) - 21\alpha^3_{5,10}\alpha_{2,8}\alpha^2_{2,5} - 144\alpha^2_{5,10}\alpha_{2,7}\alpha^3_{2,5} \\
&\quad - 1152\alpha^6_{2,5})/(5\alpha^2_{2,5}\alpha^3_{5,10})
\end{aligned}$$

3.5.3. PROPOSITION. *If $\lambda \in \mathcal{A}^2_{10,1}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\begin{aligned}
\alpha_{2,6} &= 2\alpha_{2,5}^2/\alpha_{5,10}\\
\alpha_{3,8} &= 0\\
\alpha_{3,9} &= \alpha_{2,7}\\
\alpha_{3,10} &= (\alpha_{5,10}^2(\alpha_{2,7}^2 - \alpha_{2,5}\alpha_{2,9}) + \alpha_{5,10}\alpha_{2,8}\alpha_{2,5}^2 - 2\alpha_{2,7}\alpha_{2,5}^3)/(\alpha_{5,10}\alpha_{2,5}^2)
\end{aligned}$$

3.5.4. PROPOSITION. *If $\lambda \in \mathcal{A}^1_{10,4}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\begin{aligned}
\alpha_{3,8} &= -3\alpha_{2,6}\\
\alpha_{3,9} &= -(6\alpha_{2,5}\alpha_{2,7} + 15\alpha_{2,6}^2)/(2\alpha_{2,5})\\
\alpha_{3,10} &= -3(6\alpha_{2,5}^2\alpha_{2,7} + 20\alpha_{2,5}\alpha_{2,6}\alpha_{2,7} + 19\alpha_{2,6}^3)/(2\alpha_{2,5}^2)
\end{aligned}$$

3.5.5. PROPOSITION. *If $\lambda \in \mathcal{A}^2_{10,4}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\begin{aligned}
\alpha_{3,8} &= 0\\
\alpha_{3,9} &= \alpha_{2,7}\\
\alpha_{3,10} &= (\alpha_{2,5}\alpha_{2,8} - \alpha_{2,6}\alpha_{2,7})/\alpha_{2,5}\\
\alpha_{2,10} &= (\alpha_{2,5}(\alpha_{2,6}\alpha_{2,9} + \alpha_{2,7}\alpha_{2,7}) - \alpha_{2,6}\alpha_{2,7}^2)/(\alpha_{2,5}^2)
\end{aligned}$$

3.5.6. PROPOSITION. *If $\lambda \in \mathcal{A}^2_{11,1}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\begin{aligned}
\alpha_{3,i} &= 0,\ 8 \le i \le 11\\
\alpha_{2,7} &= 5\alpha_{2,6}^2/(4\alpha_{2,5})\\
\alpha_{2,8} &= 7\alpha_{2,6}^3/(4\alpha_{2,5}^2)\\
\alpha_{2,9} &= 21\alpha_{2,6}^4/(8\alpha_{2,5}^3)\\
\alpha_{2,10} &= 33\alpha_{2,6}^5/(8\alpha_{2,5}^4)\\
\alpha_{2,11} &= 429\alpha_{2,6}^6/(64\alpha_{2,5}^5)
\end{aligned}$$

3.5.7. PROPOSITION. *If $\lambda \in \mathcal{A}^1_{11,1}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\begin{aligned}
\alpha_{2,6} &= 0\\
\alpha_{3,9} &= (3\alpha_{2,7}\alpha_{3,7} + \alpha_{3,8}^2)/\alpha_{3,7}\\
\alpha_{3,10} &= (\alpha_{3,8}^3 - 5\alpha_{2,7}\alpha_{3,7}\alpha_{3,8} + 3\alpha_{2,8}\alpha_{3,7}^2)/(\alpha_{3,7}^2)\\
\alpha_{3,11} &= (2\alpha_{3,8}^4 - 3\alpha_{2,7}^2\alpha_{3,7}^2 + 25\alpha_{2,7}\alpha_{3,7}\alpha_{3,8}^3 - 13\alpha_{2,8}\alpha_{3,7}^2\alpha_{3,8} + 9\alpha_{2,9}\alpha_{3,7}^3)/(2\alpha_{3,7}^2)\\
\alpha_{2,11} &= (3\alpha_{2,8}^2\alpha_{3,7}^2 - 10\alpha_{2,7}^3\alpha_{3,7} - 2\alpha_{2,7}^2\alpha_{3,8}^2 - 10\alpha_{2,7}\alpha_{2,8}\alpha_{3,7}\alpha_{3,8} + 9\alpha_{2,7}\alpha_{2,9}\alpha_{3,7}^2\\
&\quad + 5\alpha_{2,9}\alpha_{3,7}\alpha_{3,8}^2 - 3\alpha_{2,10}\alpha_{3,7}^2\alpha_{3,8})/(2\alpha_{3,7}^3)
\end{aligned}$$

3.5.8. PROPOSITION. *If $\lambda \in \mathcal{A}^3_{11,1}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\alpha_{3,8} = 3\alpha_{2,6}\alpha_{3,7}(\alpha_{2,5} - \alpha_{3,7})/(2\alpha_{2,5}^2)$$

and $\alpha_{3,9}$, $\alpha_{3,10}$, $\alpha_{3,11}$, $\alpha_{2,11}$ are quotients of two certain polynomials with nonzero denominator. The polynomials depend on the condition whether $4\alpha_{2,5}$ equals $\alpha_{3,7}, 7\alpha_{3,7}$ or not.

3.5.9. PROPOSITION. *If $\lambda \in \mathcal{A}_{11,4}$ then $\mathfrak{g}_\lambda$ admits a nonsingular derivation iff*

$$\alpha_{2,6} = 0$$
$$\alpha_{2,8} = 0$$
$$\alpha_{3,9} = (\alpha_{4,9}\alpha_{4,11} - \alpha_{4,10}^2)/(6\alpha_{4,9})$$
$$\alpha_{3,11} = (36\alpha_{3,10}\alpha_{4,9}^2\alpha_{4,10} + 144\alpha_{2,9}\alpha_{4,9}^3 + \alpha_{4,9}^2\alpha_{4,11}^2 - 2\alpha_{4,9}\alpha_{4,10}^2\alpha_{4,11} + \alpha_{4,10}^4)/(36\alpha_{4,9}^3)$$

Note that the Jacobi identity implies $\alpha_{2,5} = \alpha_{3,7} = 0$, $\alpha_{3,8} = -\alpha_{2,6}$, $\alpha_{5,11} = 6\alpha_{4,9}$.

3.5.2. Affine structures of adjoint type. Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and $(e_1, \dots, e_n)$ be an adapted basis. Then $\mathfrak{g}$ is generated by e_1, e_2 and an affine structure on $\mathfrak{g}$ is given by a Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $[x, y] = L(x)y - L(y)x$ for all $x, y \in \mathfrak{g}$. In order to construct an affine structure we have to find linear operators $L(e_1), L(e_2)$ such that

$$[L(e_i), L(e_j)] = L([e_i, e_j])$$
$$[e_i, e_j] = L(e_i)e_j - L(e_j)e_i$$

where $L(e_{i+1}) = [L(e_1), L(e_i)]$ for $i \geq 2$. We will use the adjoint representation as follows. Let $L(e_2)$ be a strictly lower-triangular matrix and set

$$L(e_1) = \operatorname{ad}(e_1)$$

3.5.10. DEFINITION. Let $\lambda \in \mathcal{A}_n(K)$ with adapted basis $(e_1, \dots, e_n)$. An affine structure on $\mathfrak{g}_\lambda$ is called of *of adjoint type* with respect to $(e_1, \dots, e_n)$, if it is given by $L(e_1) = \operatorname{ad}(e_1)$ and $L(e_2)$ which is a strictly lower-triangular matrix.

For $n \leq 9$ adjoint structures always exist and moreover $L(e_2)$ is related to $\operatorname{ad}(e_2)$. We have the following result:

3.5.11. THEOREM. *Any filiform Lie algebra $\mathfrak{g}$ of dimension $n \leq 9$ admits an affine structure.*

PROOF. Any filiform Lie algebra of dimension $n \leq 6$ admits a nonsingular derivation. In fact, all nilpotent Lie algebras of dimension $n \leq 6$ can be graded by positive integers and hence possess a nonsingular derivation. Let $\lambda \in \mathcal{A}_n(K)$, $7 \leq n \leq 9$. Then the algebras $\mathfrak{g}_\lambda$ admit an affine cohomology class except for the cases $\lambda \in \mathcal{A}_{8,1}$, $\mathcal{A}_{8,3}$, $\mathcal{A}_{9,3}$. In that cases it turns out that we can find an appropriate $L(e_2)$ for the above construction. We will write down $L(e_2)$ only in the second case; for $\lambda \in \mathcal{A}_{8,1}$, $\mathcal{A}_{9,3}$ the construction is similar. Let $\lambda \in \mathcal{A}_{8,3}$ and define $L(e_2)$ by

$$L(e_2)e_1 = 0$$
$$L(e_2)e_2 = -\alpha_{2,5}e_4 + \alpha_{2,6}e_5 + \alpha_{2,7}e_6 + \alpha_{2,8}e_7$$
$$L(e_2)e_3 = \alpha_{2,6}e_6 + \alpha_{2,7}e_7 + \alpha_{2,8}e_8$$
$$L(e_2)e_4 = (13\alpha_{2,6} + \alpha_{3,8})e_7/7$$
$$L(e_2)e_5 = \alpha_{2,5}e_7 + 2(8\alpha_{2,6} - \alpha_{3,8})e_8/7$$
$$L(e_2)e_6 = 3\alpha_{2,5}e_8$$
$$L(e_2)e_7 = 0$$
$$L(e_2)e_8 = 0$$

Then it follows easily that L defines an affine structure of adjoint type on $\mathfrak{g}_\lambda$. □

For $n = 10$ we obtain affine structures as follows:

3.5.12. PROPOSITION. *If $\lambda \in \mathcal{A}_{10,5}$, $\mathcal{A}_{10,8}$, $\mathcal{A}_{10,2}$ then $\mathfrak{g}_\lambda$ always admits an affine structure of adjoint type. For $\lambda \in \mathcal{A}_{10,1}$, $\mathfrak{g}_\lambda$ admits an affine structure of adjoint type if and only if $\lambda \in \mathcal{A}^2_{10,1}$ with $2\alpha_{2,5}^2 = \alpha_{2,6}\alpha_{5,10}$. If $\lambda \in \mathcal{A}_{10,4}$ then $\mathfrak{g}_\lambda$ admits no affine structure of adjoint type.*

PROOF. The result follows by straightforward computation. An affine structure of adjoint type with respect to an adapted basis is completely described if $L(e_2)$ is given. Define the *first layer* of the lower-triangular matrix $L(e_2)$ to be the first lower diagonal, say $\{\lambda_1, \dots, \lambda_9\}$, the *second layer* the second lower diagonal $\{\lambda_{10}, \dots \lambda_{17}\}$ and so forth. It turns out that, knowing the first and second layer, the matrix $L(e_2)$ can be easily computed. For that reason we will describe the affine structures of adjoint type by giving the first, second and sometimes third layer of $L(e_2)$.

Case $\mathcal{A}_{10,5}$:

$$\begin{aligned}
1^{th} \text{ layer of } L(e_2) : &\quad \{0,0,0,0,0,0,0,0,0\} \\
2^{nd} \text{ layer of } L(e_2) : &\quad \{0,0,0,0,0,0,\frac{\alpha_{4,9}}{2},\alpha_{4,9}\} \\
3^{rd} \text{ layer of } L(e_2) : &\quad \{0,-3\alpha_{2,6},-\alpha_{2,6},-3\alpha_{2,6},-\frac{7}{2}\alpha_{2,6},0,\frac{11\alpha_{2,6}+\alpha_{4,10}}{2}\}
\end{aligned}$$

Case $\mathcal{A}_{10,8}$:

$$\begin{aligned}
1^{th} \text{ layer of } L(e_2) : &\quad \{0,0,0,0,0,0,0,0,0\} \\
2^{nd} \text{ layer of } L(e_2) : &\quad \{0,0,0,0,0,0,0,1\} \\
3^{rd} \text{ layer of } L(e_2) : &\quad \{0,\alpha_{2,6},\alpha_{2,6},\alpha_{2,6}-\alpha_{3,8},\alpha_{2,6}-2\alpha_{3,8},0,\frac{\alpha_{4,10}+7\alpha_{3,8}-3\alpha_{2,6}}{2}\}
\end{aligned}$$

Case $\mathcal{A}_{10,2}$:

$$\begin{aligned}
1^{th} \text{ layer of } L(e_2) : &\quad \{0,0,0,0,0,0,0,0,-\frac{\alpha_{5,10}}{2}\} \\
2^{nd} \text{ layer of } L(e_2) : &\quad \{0,0,0,0,0,0,0,0\}
\end{aligned}$$

Case $\mathcal{A}^2_{10,1}$ *with* $2\alpha_{2,5}^2 = \alpha_{2,6}\alpha_{5,10}$:

$$\begin{aligned} 1^{th} \text{ layer of } L(e_2): &\quad \{0,0,0,0,0,0,0,0,-\frac{\alpha_{5,10}}{2}\} \\ 2^{nd} \text{ layer of } L(e_2): &\quad \{0,-3\alpha_{2,5},-\alpha_{2,5},0,0,\alpha_{2,5},3\alpha_{2,5},\frac{15\alpha_{2,5}}{4}\} \end{aligned}$$

$\square$

For $n = 11$ the result is as follows:

3.5.13. Proposition. *If $\lambda \in \mathcal{A}^2_{11,1}$, $\mathcal{A}_{11,5}$ then $\mathfrak{g}_\lambda$ always admits an affine structure of adjoint type. If $\lambda \in \mathcal{A}_{11,4}$ then $\mathfrak{g}_\lambda$ admits an affine structure of adjoint type if and only if $\alpha_{2,6} = \alpha_{2,8} = 0$. If $\lambda \in \mathcal{A}^1_{11,1}$, $\mathcal{A}^3_{11,1}$ then $\mathfrak{g}_\lambda$ admits no affine structure of adjoint type.*

Proof. The affine structures are described as follows:

$$\textit{Case } \mathcal{A}^2_{11,1}:$$

$$\begin{aligned} 1^{th} \text{ layer of } L(e_2): &\quad \{0,0,0,0,0,0,0,0,0,0\} \\ 2^{nd} \text{ layer of } L(e_2): &\quad \{0,\alpha_{2,5},\alpha_{2,5},\alpha_{2,5},\alpha_{2,5},\alpha_{2,5},\alpha_{2,5}\alpha_{2,5},\alpha_{2,5}\} \end{aligned}$$

$$\textit{Case } \mathcal{A}_{11,5}:$$

$$\begin{aligned} 1^{th} \text{ layer of } L(e_2): &\quad \{0,0,0,0,0,0,0,0,0,0\} \\ 2^{nd} \text{ layer of } L(e_2): &\quad \{0,0,0,0,0,0,0,-\alpha_{5,11},-4\alpha_{5,11}\} \end{aligned}$$

$$\textit{Case } \mathcal{A}_{11,4} \quad \textit{with} \quad \alpha_{2,6} = \alpha_{2,8} = 0:$$

$$\begin{aligned} 1^{th} \text{ layer of } L(e_2): &\quad \{0,0,0,0,0,0,0,0,0,0\} \\ 2^{nd} \text{ layer of } L(e_2): &\quad \{0,0,0,0,0,-\frac{3\alpha_{4,9}}{5},-\alpha_{4,9},0,0\} \end{aligned}$$

$\square$

3.5.14. Remark. The construction of affine structures on filiform Lie algebras of dimension $n \leq 11$ is complete except for the cases $\lambda \in \mathcal{A}_{10,1}$, $\mathcal{A}_{10,4}$, $\mathcal{A}_{11,1}$, $\mathcal{A}_{11,4}$.

3.5.15. Remark. There exists another useful modification of the adjoint representation to construct affine structures. Let $\lambda \in \mathcal{A}_n(K)$ and $L(e_2)$ be lower-triangular. Instead of $L(e_1) = \mathrm{ad}(e_1)$ one takes

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & & \frac{n-1}{n} & 0 \end{pmatrix}$$

In some cases that construction is interesting, but we will not use it here.

CHAPTER 4

A refinement of Ado's theorem

Let $\mathfrak{g}$ be a Lie algebra of dimension n over a field K of characteristic zero. Ado's theorem states that there exists a *faithful* $\mathfrak{g}$*–module* of finite dimension. Hence $\mathfrak{g}$ may be embedded in the matrix algebra $\mathfrak{gl}_m(K)$ for some $m \in \mathbb{N}$. It arises the question about the size of m.

4.0.16. Definition. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic zero. Define an invariant of $\mathfrak{g}$ by

$$\mu(\mathfrak{g}, K) := \min\{\dim_K M \mid M \text{ is a faithful } \mathfrak{g}\text{–module}\}$$

We write $\mu(\mathfrak{g})$ if the field is fixed. By Ado's theorem, $\mu(\mathfrak{g})$ is finite. Following the details of the proof we see that $\mu(\mathfrak{g}) \le f(n)$ for a function f *only* depending on n. Interest for a refinement of Ado's theorem in this respect comes from the fact that the existence of left-invariant affine structures on a Lie group implies $\mu(\mathfrak{g}) \le n+1$ for its Lie algebra $\mathfrak{g}$. Then the question arises which Lie algebras satisfy this bound. Clearly all Lie algebras $\mathfrak{g}$ with trivial center satisfy $\mu(\mathfrak{g}) \le n$, since in that case the adjoint representation is a faithful representation of dimension n. The answer is not clear for nilpotent and solvable Lie algebras. As we know, not all nilpotent Lie algebras satisfy the bound $n+1$ for μ, although it is very difficult to find such algebras.

4.1. Elementary properties of μ

We state some general results on $\mu(\mathfrak{g})$. By Lemma 3.1.18 we have:

4.1.1. Lemma. *Let $\mathfrak{g}$ be a Lie algebra of dimension n. If $\mathfrak{g}$ has an affine structure then $\mu(\mathfrak{g}) \le n+1$. If $\mathfrak{g}$ has trivial center then $\mu(\mathfrak{g}) \le n$.*

In particular we obtain $\mu(\mathfrak{g}) \le n+1$ if $\mathfrak{g}$ is 2 or 3-step nilpotent, or admits a nonsingular derivation.

4.1.2. Lemma. *Let $\mathfrak{g}$, $\mathfrak{b}$ be Lie algebras with $\dim \mathfrak{g} = n$ and $\dim \mathfrak{b} = m$ such that there is an extension*

$$0 \to \mathfrak{b} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with $\mathfrak{z}(\mathfrak{h}) = \mathfrak{b}$, identifying $\iota(\mathfrak{b})$ with $\mathfrak{b}$. Then $\mu(\mathfrak{g}) \le n+m$.

Proof. By assumption, $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{b}$. We obtain an $\mathfrak{h}/\mathfrak{b}$–module structure on $\mathfrak{h}$ by:

$$\mathfrak{h}/\mathfrak{b} \times \mathfrak{h} \to \mathfrak{h},\ ([x], v) \mapsto x \bullet v = [x, v]_{\mathfrak{h}}$$

Then $\mathfrak{h}$ together with this action $x \bullet v$ is a faithful $\mathfrak{g}$–module of dimension $n+m$: Suppose $[x] \bullet v = 0$ for all $v \in \mathfrak{h}$. Then $[x, v] = 0$ for all $v \in \mathfrak{h}$, hence $x \in \mathfrak{z}(\mathfrak{h}) \subseteq \mathfrak{b}$. This implies $[x] = [0]$ in $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{b}$. Hence the module is faithful. $\square$

For the following result see [**10**]:

4.1.3. Proposition. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$. Then $\mathfrak{g}$ satisfies $\mu(\mathfrak{g}) \geq n$.*

Proof. We may assume that $n \geq 3$ and K is algebraically closed. Let M be a faithful $\mathfrak{g}$–module. There is a unique decomposition of M into submodules indexed by the characters of $\mathfrak{g}$,

$$M = \bigoplus_c M_c$$

with the following property: the module tensor product $M_c \otimes K_{-c}$ is nilpotent, where K_c denotes the 1–dimensional $\mathfrak{g}$–module given by the character c. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. It is 1–dimensional. Since M is faithful it acts non-trivially on one factor $M_c \otimes K_{-c}$. We may assume that M is nilpotent by replacing M by $M_c \otimes K_{-c}$. We have a natural descending filtration on M by $M^1 = M$, $M^{i+1} = \mathfrak{g} \bullet M^i$ for $i \geq 1$. If $M^i = M^{i+1}$ for some i then $M^i = 0$. Since M is faithful and $\dim \mathfrak{g} = n$ we have $M^n \supseteq \mathfrak{g}^{n-1} \bullet M^1 = \mathfrak{z} \bullet M \neq 0$. This implies $\dim M^i/M^{i+1} \geq 1$ for all $i = 1, \dots, n$ and hence $\dim M \geq n$. $\square$

4.1.4. Proposition. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and suppose that there exists an extension $0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$. Then $\mu(\mathfrak{g}) = n$.*

Proof. We proceed as in Proposition 3.1.8 and reduce the question to the case that $\dim \mathfrak{z}(\mathfrak{h}) = 1$ and $\mathfrak{h} \in \mathfrak{F}_{n+1}(K)$. Let $(f_1, \dots, f_{n+1})$ be an adapted basis for $\mathfrak{h}$. The adjoint representation of $\mathfrak{h}$ then induces a $\mathfrak{g}$–module structure on $M = \operatorname{span}\{f_1, f_3, f_4, \dots f_{n+1}\}$. That is a faithful $\mathfrak{g}$–module of dimension n since the center of $\mathfrak{g}$ acts nontrivially. This implies $\mu(\mathfrak{g}) \leq n$ and hence $\mu(\mathfrak{g}) = n$ by Proposition 4.1.3. $\square$

4.2. Explicit formulas for μ

In some cases we can determine $\mu(\mathfrak{g})$ by an explicit formula in the dimension of $\mathfrak{g}$. The first case is that $\mathfrak{g}$ is *abelian*. Then $\mathfrak{g}$ is a vector space and any faithful representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a d–dimensional vector space, turns $\varphi(\mathfrak{g})$ into an n–dimensional commutative subalgebra of the matrix algebra $M_d(K)$. There is an upper bound of n in terms of d. Since φ is a monomorphism, $n \leq d^2$. A sharp bound was proved by Schur [**73**] over $\mathbb{C}$ and by Jacobson [**51**] over any field K:

4.2.1. Proposition. *Let M be a commutative subalgebra of $M_d(K)$ over an arbitrary field K. Then $\dim M \leq [d^2/4] + 1$, where $[x]$ denotes the integral part of x. This bound is sharp.*

Denote by $\lceil x \rceil$ the ceiling of x, i.e., the least integer greater or equal than x.

4.2.2. Proposition. *Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n over an arbitrary field K. Then $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1} \rceil$.*

Proof. By Proposition 4.2.1, a faithful $\mathfrak{g}$–module has dimension d with $n \leq [d^2/4]+1$. This implies $d \geq \lceil 2\sqrt{n-1} \rceil$. It is easy to construct commutative subalgebras of $M_d(K)$ of dimension exactly equal to $[d^2/4] + 1$. Hence $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1} \rceil$. $\square$

4.2.3. Definition. Let $\mathfrak{h}_m(K)$ be a $(2m+1)$–dimensional vector space over K with basis $(x_1, \dots, x_m, y_1, \dots, y_m, z)$. Denote by $\mathfrak{h}_m(K)$ the 2–step nilpotent Lie algebra defined by $[x_i, y_i] = z$ for $i = 1, \dots, m$. It is called *Heisenberg Lie algebra of dimension* $2m + 1$.

4.2.4. LEMMA. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with 1–dimensional center $\mathfrak{z}(\mathfrak{g})$ spanned by $z \in \mathfrak{g}$. Then a representation $L : \mathfrak{g} \to \mathfrak{gl}(V)$ is faithful iff $\mathfrak{z}(\mathfrak{g})$ acts nontrivially. In that case $L(z)v \neq 0$ for some nonzero $v \in V$.*

PROOF. If L is not faithful then $\ker L$ is a nonzero ideal in $\mathfrak{g}$ intersecting $\mathfrak{z}(\mathfrak{g})$ nontrivially since $\mathfrak{g}$ is nilpotent. Hence $z \in \ker L$ and $L(z) = 0$. Conversely if $L(z) \neq 0$ then $\ker L = 0$ and L is faithful. □

4.2.5. PROPOSITION. *The Heisenberg Lie algebras satisfy $\mu(\mathfrak{h}_m(K)) = m + 2$.*

PROOF. It is well known that $\mathfrak{h}_m(K)$ has a faithful representation of dimension $m+2$ as follows. Let $\gamma z + \sum_{j=1}^m (\alpha_j x_j + \beta_j y_j)$ correspond to the $(m+2) \times (m+2)$ matrix

$$\begin{pmatrix} 0 & \alpha_1 & \dots & \alpha_m & \gamma \\ & 0 & \dots & 0 & \beta_1 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & 0 & \beta_m \\ 0 & & \dots & & 0 \end{pmatrix}$$

There are no faithful representations of *smaller* dimension for $\mathfrak{h}_m(K)$. It seems that this fact is not mentioned in the standard literature. Assume that $L : \mathfrak{h}_m(K) \to \mathfrak{gl}(V)$ is a faithful representation of minimal degree. By the lemma we may fix a $v \in V$ with $L(z)v \neq 0$. We have to show $\dim V \geq m + 2$. The *evaluation map* is defined by

$$e_v : \mathfrak{h}_m(K) \to V,\ x \mapsto L(x)v$$

It is clear that $\ker e_v$ is a subalgebra of $\mathfrak{h}_m(K)$ not containing z. In fact, $\ker e_v$ is an *abelian* Lie algebra: Let $x, y \in \ker e_v$. Then $[x, y] \in \ker e_v$, hence $L([x, y])v = 0$. On the other hand, $[x, y] \in \mathfrak{z}(\mathfrak{g})$ so that $[x, y]$ is a multiple of z. Since $L(z)v \neq 0$ it follows $[x, y] = 0$. We have

$$\dim V \geq \dim \operatorname{im} e_v = \dim \mathfrak{h}_m(K) - \dim \ker e_v$$

The number on the right hand becomes minimal if the dimension of $\ker e_v$ becomes maximal. This happens if $\ker e_v$ is a maximal subalgebra not containing z. Any such abelian subalgebra has dimension m, hence $\dim \operatorname{im} e_v \geq m + 1$. If we show $v \notin \operatorname{im} e_v$ then it follows $\dim V \geq \dim \operatorname{im} e_v \geq m + 2$ and the proof is finished.

Assume $v \in \operatorname{im} e_v$. Let K be algebraically closed. By Lie's theorem we may assume that $L(x_i), L(y_i)$ are upper triangular endomorphisms and hence that the commutator $L(z) = [L(x_i), L(y_i)]$ is nilpotent. In particular $L(z)v = v$ is impossible. But since $v \in \operatorname{im} e_v$ there must be an $x \in \mathfrak{g}$ with $L(x)v = v$. That x is not contained in the center of $\mathfrak{g}$ and not in $\ker e_v$. There exists some $y \in \ker e_v$ such that $[x, y] = z$. Otherwise x would commute with $\ker e_v$ and $\operatorname{span}\{\ker e_v, x\} = \{\ker e_v\}$ because $\ker e_v$ is maximal abelian. In that case $x \in \ker e_v$ and $v = L(x)v = 0$, which is a contradiction. It follows

$$L(z)v = [L(x), L(y)]v = L(x)L(y)v - L(y)L(x)v = 0$$

by using $L(y)v = 0$ and $L(x)v = v$. This is a contradiction. □

4.2.6. PROPOSITION. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra of dimension n with 1–dimensional center. Then n is odd and $\mu(\mathfrak{g}) = (n + 3)/2$.*

Proof. The commutator subalgebra $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{z}(\mathfrak{g})$ is 1–dimensional. Hence the Lie algebra structure on $\mathfrak{g}$ is defined by a skew-symmetric bilinear form $V \wedge V \to K$ where V is the subspace of $\mathfrak{g}$ complementary to $K = [\mathfrak{g},\mathfrak{g}]$. It follows from the classification of such forms that $\mathfrak{g}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_m(K)$ with $n = 2m+1$. It follows $\mu(\mathfrak{g}) = m+2 = (n+3)/2$. □

4.2.7. Proposition. *Let $\mathfrak{g}$ be a filiform Lie algebra of dimension n with abelian commutator algebra. Then $\mu(\mathfrak{g}) = n$.*

Proof. First $\mathfrak{g}$ has a faithful $\mathfrak{g}$–module M of dimension $n+1$, since $\mathfrak{g}$ admits an affine structure, see Proposition 3.1.16. It is easy to see that M has a faithful submodule of dimension n. Hence $\mu(\mathfrak{g}) = n$ by Proposition 4.1.3. □

In the same way we obtain:

4.2.8. Proposition. *Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ with $n < 10$. Then $\mu(\mathfrak{g}) = n$.*

4.3. A general bound for μ

In 1937 Birkhoff [**13**] proved a special case of Ado's theorem. He showed

$$\mu(\mathfrak{g}) \le 1 + n + n^2 + \cdots + n^{k+1}$$

for a nilpotent Lie algebra $\mathfrak{g}$ of dimension n and class k. His construction used the universal enveloping algebra of $\mathfrak{g}$. In 1969 this method was slightly improved by Reed [**70**]:

4.3.1. Proposition. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n and nilpotency class k. Then $\mu(\mathfrak{g}) \le 1 + n^k$.*

For a solvable Lie algebra, Reed gives the following bound:

4.3.2. Proposition. *Let $\mathfrak{g}$ be a solvable Lie algebra of dimension n over an algebraically closed field of characteristic zero. Then $\mu(\mathfrak{g}) < 1 + n + n^n$.*

4.3.3. Remark. The idea of the proof is as follows, see [**70**]. By embedding $\mathfrak{g}$ in a splittable solvable Lie algebra of dimension $\dim \mathfrak{g} + \dim \mathfrak{g}/\mathfrak{n}$, where $\mathfrak{n}$ denotes the nilradical of $\mathfrak{g}$, the situation is in principle reduced to the case that $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$. If $\mathfrak{g}$ is a semidirect product of a Lie algebra $\mathfrak{h}$ and a nilpotent ideal $\mathfrak{n}$ containing the center of $\mathfrak{g}$, then a faithful representation of $\mathfrak{g}$ is constructed extending the faithful representation of $\mathfrak{n}$ from Proposition 4.3.1: let ρ' be the representation of $\mathfrak{g}$ of degree $1+n^k$ extending the faithful representation of $\mathfrak{n}$ and ρ be the direct sum of ρ' and ad, the adjoint representation of $\mathfrak{g}$. Since $\ker(\mathrm{ad})$ is the center of $\mathfrak{g}$ and since ρ' is faithful on $\mathfrak{n}$ and hence on the center, it follows that ρ is faithful on $\mathfrak{g}$. It is clear that the degree of ρ is not greater than the degree of ρ' plus $\dim \mathfrak{g}$. It would be interesting to know whether that construction also works if we start with a faithful representation of $\mathfrak{n}$ of minimal degree. In other words, is it true that

$$\mu(\mathfrak{g}) \le \mu(\mathfrak{n}) + \dim \mathfrak{g}$$

where $\mathfrak{g}$ is a solvable Lie algebra and $\mathfrak{n}$ its nilradical. An answer does not appear to be known.

It is interesting to note that the estimate $\mu(\mathfrak{g}) \le \mu(\mathfrak{n}) + \dim(\mathfrak{g}/\mathfrak{n})$ is not true in general. The following example is due to H. Abels. Let $\mathfrak{n}$ be a k–dimensional abelian Lie algebra over $\mathbb{C}$ with basis $(v_1, \ldots, v_k)$. Define $D_n : \mathfrak{n} \to \mathfrak{n}$ by $D_n v_j = v_{j+1}$ for $1 \le j \le k$ and $D_n v_k = 0$, and form the semidirect product $\mathfrak{g} = \mathbb{C} \ltimes_D \mathfrak{n}$ with the derivation $D := \mathrm{Id}_n + D_n$. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation of $\mathfrak{g}$ and $\rho(D) = \rho(D)_s + \rho(D)_n$ the additive Jordan decomposition then $\mathbb{C}\rho(D)_n + \rho(\mathfrak{n})$ is isomorphic to the standard graded filiform of dimension $k+1$: $\rho(D)_n$ acts by taking commutators in $\mathfrak{gl}(V)$ on the isomorphic copy $\rho(\mathfrak{n})$ of $\mathfrak{n}$ in the corresponding way as D_n acts on $\mathfrak{n}$. In particular $\dim V \ge k+1$ by Proposition 4.1.3 and hence $\mu(\mathfrak{g}) \ge k+1$. In fact $\mu(\mathfrak{g}) = k+1$, whereas $\mu(\mathfrak{n}) = \lceil 2\sqrt{k-1}\,\rceil$.

The general bounds for μ are not very good if one keeps in mind that it is quite difficult to find Lie algebras $\mathfrak{g}$ with $\mu(\mathfrak{g}) \ge \dim \mathfrak{g} + 2$. In particular for $\mathfrak{g} \in \mathfrak{F}_n(K)$ the bound $\mu(\mathfrak{g}) < 1 + n^{n-1}$ is very rough. We can improve this bound:

4.3.4. THEOREM. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n and nilpotency class k. Denote by $p(j)$ the number of partitions of j and let*

$$\nu(n,k) = \sum_{j=0}^{k} \binom{n-j}{k-j} p(j)$$

Then $\mu(\mathfrak{g}) \le \nu(n,k)$.

PROOF. Construct a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, such that $\rho(X)$ is nilpotent for all $X \in \mathfrak{g}$ as follows: Let $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$. Since $\mathfrak{g}$ is k-step nilpotent, $\mathfrak{g}^{k+1} = 0$. Choose a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$ such that the first n_1 elements span $\mathfrak{g}^k$, the first n_2 elements span $\mathfrak{g}^{k-1}$ and so on. We will take V as a quotient of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem the ordered monomials

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n},\ \alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{Z}_+^n$$

form a basis for $U(\mathfrak{g})$. Let $t = \sum_\alpha c_\alpha x^\alpha$ be an element of $U(\mathfrak{g})$ where only finitely many c_α are nonzero. Define an order function as follows:

$$\mathrm{ord}(x_j) = \max\{m :\ x_j \ \in \ \mathfrak{g}^m\}$$
$$\mathrm{ord}(x^\alpha) = \sum_{j=1}^{n} \alpha_j \,\mathrm{ord}(x_j)$$
$$\mathrm{ord}(t) = \min\{\mathrm{ord}(x^\alpha) \ :\ c_\alpha \ne 0\}$$

and $\mathrm{ord}(\mathbf{1}_{U(\mathfrak{g})}) = 0$, $\mathrm{ord}(0) = \infty$. Let $U^m(\mathfrak{g}) = \{t \in U(\mathfrak{g}) \ :\ \mathrm{ord}(t) \ge m\}$. One can show that $U^m(\mathfrak{g})$ is an ideal of $U(\mathfrak{g})$ having finite codimension. Define

$$V = U(\mathfrak{g})/U^m(\mathfrak{g})$$

Choose a basis $\{t_1, \ldots, t_\ell\}$ of V such that $t_1, \ldots, t_{\ell_1}$ span $U^{m-1}(\mathfrak{g})/U^m(\mathfrak{g})$, $t_1, \ldots, t_{\ell_2}$ span $U^{m-2}(\mathfrak{g})/U^m(\mathfrak{g})$ and so on. Then it is easy to check that the desired representation of $\mathfrak{g}$ is obtained by setting

$$\rho(X)(t_j) = Xt_j \mod U^m(\mathfrak{g})$$

If $m > k$ then $\rho(X) \cdot \mathbf{1}_{U(\mathfrak{g})} = X \ne 0$ for all $X \in \mathfrak{g}$, so that ρ is faithful.
Now we will construct a bound for $\dim V$: Choose m minimal, i.e., $m = k+1$. Let

$\mathcal{B} = \{x^\alpha \mid \operatorname{ord}(x^\alpha) \le k\}$ be a basis for V as above. Then $x_1, \dots, x_{n_1}$ have order k, $x_{n_1+1}, \dots, x_{n_2}$ have order $k-1$ and so on. Hence

$$\#\mathcal{B} = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid \sum_{j=1}^k (k-j+1)(\alpha_{n_{j-1}+1} + \dots + \alpha_{n_j}) \le k\}$$

with $n_0 = 0$. On the other hand, $\dim \mathfrak{g}^k \ge 1$, $\dim \mathfrak{g}^{k-1} \ge 2$ and so on. We can choose the x_i such that $\operatorname{ord}(x_1) = k,\ \operatorname{ord}(x_2) \ge k-1,\ \operatorname{ord}(x_3) \ge k-2, \dots, \operatorname{ord}(x_k) = \dots = \operatorname{ord}(x_n) \ge 1$. If actually $\operatorname{ord}(x_i) = k+1-i$ for $i = 1, \dots, k$ and $\operatorname{ord}(x_{k+1}) = \dots = \operatorname{ord}(x_n) = 1$, then $\#\mathcal{B}$ will be maximal, i.e., $\#\mathcal{B} \le p(n,k)$, where

$$p(n,k) = \#\big\{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid \left(\sum_{j=1}^k (k-j+1)\alpha_j\right) + \alpha_{k+1} + \dots + \alpha_n \le k\big\}.$$

Using the generating function $(1/(1-x))^{r+1} = \sum_{k \ge 0} \binom{r+k}{k} x^k$ for $|x| < 1$ we obtain

$$\begin{aligned}\#\{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid \sum_{j=1}^n \alpha_j \le k\} &= \#\{(\alpha_0, \dots, \alpha_n) \in \mathbb{Z}_+^{n+1} \mid \sum_{j=0}^n \alpha_j = k\} \\ &= \binom{n+k}{k}.\end{aligned}$$

Since $p(k) = \#\{(\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+^k \mid k\alpha_1 + (k-1)\alpha_2 + \dots + \alpha_k = k\}$ we have

$$p(n,k) = \nu(n,k) = \sum_{j=0}^k \binom{n-j}{k-j} p(j).$$

□

We give two examples:

4.3.5. Example. Let $\mathfrak{g} = \operatorname{span}\{x_1, \dots, x_6\}$ and define Lie brackets by

$$[x_2, x_6] = -x_1,\ [x_3, x_6] = -x_2,\ [x_4, x_5] = -x_1,\ [x_5, x_6] = -x_3$$

We obtain a 4–step nilpotent Lie algebra of dimension 6 with

$$\operatorname{ord}(x_1) = 4,\ \operatorname{ord}(x_2) = 3,\ \operatorname{ord}(x_3) = 2,\ \operatorname{ord}(x_i) = 1$$

for $i = 4, 5, 6$. By the above theorem there is a faithful $\mathfrak{g}$–module of dimension $\nu(6,4) = 51$. The bound of Proposition 4.3.1 is $1 + n^k = 1297$.

4.3.6. Example. Let $\mathfrak{g} = \operatorname{span}\{x_1, \dots, x_6\}$ and define Lie brackets by

$$[x_6, x_i] = x_{i-1} \text{ for } 2 \le i \le 5$$

We obtain a filiform nilpotent Lie algebra of dimension 6. There is a faithful $\mathfrak{g}$–module of dimension $\nu(6,5) = 45$. Here $1 + n^k = 7777$. The bounds are not very good. Because of Proposition 4.2.8 we know $\mu(\mathfrak{g}) = 6$.

In the following we study some properties of $\nu(n,k)$. Let $\nu(n,0) := 1$.

4.3.7. Lemma. *$\nu(n+1,k) = \nu(n,k) + \nu(n,k-1)$ for $1 \le k \le n$.*

PROOF.

$$\begin{aligned}
\nu(n,k)+\nu(n,k-1) &= \sum_{j=0}^{k}\binom{n-j}{k-j}p(j)+\sum_{j=0}^{k-1}\binom{n-j}{k-j-1}p(j) \\
&= \sum_{j=0}^{k-1}\left[\binom{n-j}{k-j}+\binom{n-j}{k-j-1}\right]p(j)+\binom{n-k}{0}p(k) \\
&= \sum_{j=0}^{k-1}\binom{n+1-j}{k-j}p(j)+p(k) \\
&= \nu(n+1,k)
\end{aligned}$$

□

4.3.8. LEMMA. *Let $\eta(q)=\prod_{j=1}^{\infty}(1-q^j)^{-1}$. For $2\le k\le n-1$ it holds*

$$\nu(n,k)<\binom{n}{k}\eta(\tfrac{k}{n})^{-1} \tag{39}$$

PROOF. Denote by $p_k(j)$ the number of those partitions of j in which each term in the partition does not exceed k. We have

$$\sum_{j=0}^{k}p(j)q^j<\sum_{j=0}^{\infty}p_k(j)q^j=\prod_{j=1}^{k}\frac{1}{1-q^j}$$

for $|q|<1$. Hence

$$\nu(n,k)=\sum_{j=0}^{k}\binom{n-j}{k-j}p(j)<\sum_{j=0}^{k}\binom{n}{k}q^jp(j)<\binom{n}{k}\prod_{j=1}^{k}\frac{1}{1-q^j}$$

with $q=k/n$. □

4.3.9. LEMMA. *We have $\nu(n,k-1)<\nu(n,k)$ for all n,k with $2\le k\le[\frac{n+3}{2}]$ and $\nu(n,k-1)>\nu(n,k)$ for all n,k with $[\frac{n+3}{2}]+1\le k<n$. In particular for $k(n)=[(n+3)/2]$:*

$$\nu(n,k)\le\nu(n,k(n))\quad \textit{for } 1\le k\le n. \tag{40}$$

PROOF. Let n be even. We first show $\nu(n,k-1)<\nu(n,k)$ for all k with $2\le k\le(n+2)/2$. By Lemma 4.3.7 that means $\nu(n+1,k)-\nu(n,k)=\nu(n,k-1)<\nu(n,k)$, hence $2\nu(n,k)-\nu(n+1,k)>0$. Since

$$\sum_{j=0}^{k}\left(2\binom{n-j}{k-j}-\binom{n+1-j}{k-j}\right)p(j)=\sum_{j=0}^{k}\frac{(n-2k+j+1)(n-j)!}{(k-j)!(n-k+1)!}p(j)$$

the inequality is equivalent to

$$\sum_{j=0}^{k}(n-2k+j+1)\binom{n-j}{k-j}p(j)>0 \tag{41}$$

It is certainly true for $k<(n+2)/2$ since then all coefficients in the sum are positive. For $k=k(n)=(n+2)/2$ however the first term is negative. Nevertheless (41) holds since already the sum of the first four terms is positive for $n\ge4$ and $k=k(n)$.

For the second part we must show that the sum in (41) is negative for $n \geq k \geq k(n)+1$. For $k = k(n)+1 = (n+4)/2$ that means

$$\sum_{j=0}^{k} (j-3)\binom{n-j}{k-j}p(j) < 0, \quad \text{or} \quad \sum_{j=0}^{k} j\binom{n-j}{k-j}p(j) < 3\nu(n,k) \tag{42}$$

Estimating carefully we obtain

$$\sum_{j=0}^{k} p(j)jq^j < \sum_{j=0}^{\infty} jp_k(j)q^j = \sum_{j=1}^{k} \frac{jq^j}{1-q^j} \prod_{j=1}^{k} \frac{1}{1-q^j} < 10$$

for $q = k/n$, $k = k(n)+1$ and $n \geq 355$. Hence

$$\sum_{j=0}^{k} j\binom{n-j}{k-j}p(j) < \binom{n}{k}\sum_{j=0}^{k} p(j)jq^j < 10\binom{n}{k} < 3\nu(n,k)$$

The last inequality follows from summing up the first terms of $\nu(n,k)$ for $k = k(n)+1$:

$$\nu(n,k) > \frac{1745}{512}\binom{n}{k}$$

For $n \leq 355$ the inequality (42) is also true. We have verified it with the computer algebra system Pari. If $k \geq k(n)+1$, then the sum in (41) is negative. That follows from the fact that $\nu(n,k)$ is unimodal for fixed n. A function $F(n,k)$ with $0 \leq k \leq n$ is called unimodal if there exists a sequence K with $K(n) \leq K(n+1) \leq K(n)+1$ such that for all $n \geq 0$

$$\begin{gathered} F(n,0) < F(n,1) < F(n,2) < \cdots < F(n,K(n)-1) \leq F(n,K(n)), \\ F(n,K(n)) > F(n,K(n)+1) > \cdots > F(n,n-1) > F(n,n) > F(n,n+1) = 0. \end{gathered}$$

The unimodality of $\nu(n,k)$ can be proved by induction. Finally the proof for n odd is done likewise. $\square$

4.3.10. Proposition. *There is the following estimate for $\nu(n,k)$:*

$$\nu(n,k) \leq \frac{3}{\sqrt{n}}\,2^n \quad \text{for fixed } n \geq 1 \text{ and all } 1 \leq k \leq n \tag{43}$$

Proof. Using the two preceding lemmas and Stirling's formula for the binomial coefficient we obtain

$$\nu(n,k) \leq \nu(n,k(n)) < \binom{n}{k(n)}\frac{1}{\eta\left(\frac{k(n)}{n}\right)} < \frac{2^n}{\sqrt{\pi n/2}} \cdot \frac{1}{\eta\left(\frac{k(n)}{n}\right)} < \frac{2.81}{\sqrt{n}}2^n$$

for $n \geq 355$. For $n \leq 355$ the proposition is true also. $\square$

4.3.11. Corollary. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. Then*

$$\mu(\mathfrak{g}) < \frac{3}{\sqrt{n}}\,2^n$$

4.3.12. Remark. If $k, n \to \infty$ with $\frac{k}{n} \le \delta$ for some fixed $\delta > 0$ then one has asymptotically $\nu(n,k) \sim \binom{n}{k}\left(\eta(\frac{k}{n})\right)^{-1}$ and $\eta\left(\frac{k(n)}{n}\right)^{-1} \sim 3.4627466$. The proposition shows that the bound $\mu(\mathfrak{g}) \le \nu(n,k)$ is much better than $1+n^k$, especially if k is not small in comparison to n. For small k we can give better bounds. Note that $k = 1$ corresponds to the abelian case.

4.3.13. Lemma. *Let $\alpha = \sqrt{\frac{2}{3}}\pi$. Then for $n \ge 3$ we have*

$$\frac{\sqrt{n}}{\sqrt{n+1}-1} < 1 + \frac{\pi}{\sqrt{6n}} < e^{\alpha\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-1\right)}$$

Proof. Using the inequality

$$1 + \frac{1}{2n} - \frac{1}{8n^2} < \sqrt{1+\frac{1}{n}}$$

and $\exp(x) > 1 + x + x^2/2$ for $x > 0$ we obtain

$$\begin{aligned} e^{\alpha\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-1\right)} &> \exp\left(\alpha\sqrt{n}\left(\frac{1}{2n} - \frac{1}{8n^2}\right)\right) = \exp\left(\frac{\pi}{\sqrt{6n}}\left(1 - \frac{1}{4n}\right)\right) \\ &> 1 + \frac{\pi}{\sqrt{6n}} - \frac{\pi}{4n\sqrt{6n}} + \frac{\pi^2}{12n} - \frac{\pi^2}{24n^2} + \frac{\pi^2}{192n^3} \\ &> 1 + \frac{\pi}{\sqrt{6n}} \end{aligned}$$

for $n \ge 1$. On the other hand we have for $n \ge 17$

$$\begin{aligned} \frac{1}{1+\frac{\pi}{\sqrt{6n}}} &< 1 - \frac{\pi}{\sqrt{6n}} + \frac{\pi^2}{6n} < 1 + \frac{1}{2n} - \frac{1}{8n^2} - \frac{1}{\sqrt{n}} \\ &< \sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n+1}-1}{\sqrt{n}} \end{aligned}$$

Taking reciprocal values yields the second part of the lemma. For $3 \le n \le 16$ one verifies the lemma directly. □

4.3.14. Lemma. *Let $\alpha = \sqrt{\frac{2}{3}}\pi$. Then*

$$\nu(n-1, n-1) < e^{\alpha\sqrt{n}} \quad \textit{for all } \; n \ge 1.$$

Proof. Let $\alpha = \sqrt{\frac{2}{3}}\pi$. In [**2**], section 14.7 formula (11), the following upper bound for $p(n)$ is proved:

$$p(n) < \frac{\pi}{\sqrt{6n}} e^{\alpha\sqrt{n}} \quad \text{for all } \; n \ge 1$$

We want to prove the proposition by induction on n. By Lemma 4.3.13 we have

$$1 + \frac{\pi}{\sqrt{6n}} < e^{\alpha\sqrt{n+1}-\alpha\sqrt{n}}$$

which holds for all $n \geq 1$. Assuming the claim for $n-1$ it follows for n:

$$\begin{aligned}\nu(n,n) = \nu(n-1,n-1) + \nu(n) &< e^{\alpha\sqrt{n}} + \frac{\pi}{\sqrt{6n}} e^{\alpha\sqrt{n}} \\ &= \left(1 + \frac{\pi}{\sqrt{6n}}\right) e^{\alpha\sqrt{n}} < e^{\alpha\sqrt{n+1}}\end{aligned}$$

$\square$

4.3.15. Lemma. *Let $\alpha = \sqrt{\frac{2}{3}}\pi$. Then*

$$\nu(n,n-1) < \sqrt{n} e^{\alpha\sqrt{n}} \quad \textit{for all } n \geq 1.$$

Proof. It follows from Lemma 4.3.13 that

$$\sqrt{n} e^{\alpha\sqrt{n}} < \left(\sqrt{n+1} - 1\right) e^{\alpha\sqrt{n+1}}$$

By induction on n and Lemma 4.3.14 we have:

$$\begin{aligned}\nu(n+1,n) = \nu(n,n) + \nu(n,n-1) &< e^{\alpha\sqrt{n+1}} + \sqrt{n} e^{\alpha\sqrt{n}} \\ &< \sqrt{n+1} e^{\alpha\sqrt{n+1}}\end{aligned}$$

$\square$

We improve the bound of Proposition 4.3.10 for filiform Lie algebras:

4.3.16. Proposition. *Let $\mathfrak{g}$ be a filiform Lie algebra of dimension n and $\alpha = \sqrt{\frac{2}{3}}\pi$. Then*

$$\mu(\mathfrak{g}) < 1 + e^{\alpha\sqrt{n-1}}$$

Proof. Using the construction of Theorem 4.3.4 with $(x_1, \dots, x_n) = (e_n, \dots, e_1)$ we obtain a faithful module V with basis

$$\mathcal{B} = \left\{ e_n^{\alpha_n} \cdots e_1^{\alpha_1} \mid \sum_{j=2}^{n} (j-1)\alpha_j + \alpha_1 \leq n-1 \right\}$$

for $\mathfrak{g} = \operatorname{span}\{e_1, \dots, e_n\}$ and $\dim V = \nu(n, n-1)$. Here $\operatorname{ord}(e_i) = i-1,\ i = 2, \dots, n$ and $\operatorname{ord}(e_1) = 1$. The elements e_i of $\mathfrak{g}$ act on V by

$$e_i e_j = [e_i, e_j] + e_j e_i \text{ for } i < j$$

where $e_j e_i$ is an element of V for $j \geq i$. Let U be the submodule of V generated by e_1. It has a basis of all monomials $e_n^{\alpha_n} \cdots e_1^{\alpha_1}$ with $\alpha_1 \neq 0$, hence $\dim U = \nu(n-1, n-2)$. The factor module V/U is a faithful $\mathfrak{g}$– module of dimension

$$\nu(n, n-1) - \nu(n-1, n-2) = \nu(n-1, n-1)$$

Its basis $\widetilde{\mathcal{B}}$ contains the monomials $e_n^{\alpha_n} \cdots e_2^{\alpha_2}$ of maximal order, i.e., with

$$\sum_{j=2}^{n} (j-1)\alpha_j = n-1$$

These are $p(n-1)$ monomials, the number of partitions of $n-1$. We may omit these monomials from $\widetilde{\mathcal{B}}$, except for e_n in order to preserve faithfulness. Then we obtain a faithful module of dimension

$$\nu(n-1,n-1) - p(n-1) + 1 = 1 + \nu(n-2,n-2)$$

The claim follows by Lemma 4.3.14. $\square$

4.4. Faithful modules of dimension $n+1$

In this section we will study faithful $\mathfrak{g}$–modules of dimension $n+1$ for all Lie algebras $\mathfrak{g} \in \mathfrak{F}_n(K)$ with $n \le 11$. In case that there exists such a module for $\mathfrak{g}$ we have

$$n \le \mu(\mathfrak{g}) \le n+1$$

Because of the results of section 3.5 we only have to consider Lie algebra laws λ from

$$\mathcal{A}_{10,4},\ \mathcal{A}^1_{10,1},\ \mathcal{A}^2_{10,1},\ \mathcal{A}_{11,4},\ \mathcal{A}^1_{11,1},\ \mathcal{A}^3_{11,3}$$

In all other cases $\mathfrak{g}_\lambda$ admits an affine structure and hence possesses a faithful module of dimension $n+1$. Let $\lambda \in \mathcal{A}_n(K)$ and $(e_1, \dots, e_n)$ be an adapted basis. The Lie algebra $\mathfrak{g}_\lambda$ is generated by e_1, e_2. A $\mathfrak{g}_\lambda$–module M is given by a linear representation $L : \mathfrak{g}_\lambda \to \mathfrak{gl}(M)$. In order to construct such a module it is sufficient to find linear operators $L(e_1), L(e_2)$ such that

$$[L(e_i), L(e_j)] = L([e_i, e_j])$$

where $L(e_{i+1}) = [L(e_1), L(e_i)]$.

4.4.1. Definition. Let $\mathfrak{g} \in \mathfrak{F}_n(K)$ and M be a $\mathfrak{g}$–module given by $L(e_1), L(e_2)$ where $(e_1, \dots, e_n)$ denotes an adapted basis of $\mathfrak{g}$. Then M is called a *Δ–module* if $L(e_1)$ and $L(e_2)$ are simultaneously strictly upper triangular matrices with respect to some basis of M such that each entry of $L(e_1)$ is equal to 0 or 1 and in each row and in each column of $L(e_1)$ there is *at most one nonzero entry*, and $\dim M = n+1$.

Note that such a Δ–module, if it exists, is nilpotent, i.e., all $L(e_i)$ are nilpotent linear transformations. The center of a Δ–module is given by $\ker L(e_1) \cap \ker L(e_2)$. We will introduce a combinatorical type for Δ–modules as follows:

4.4.2. Definition. Define the *first layer* of the matrix $L(e_1)$ to be the first upper diagonal, say $\{\lambda_1, \dots, \lambda_n\}$, the *second layer* of $L(e_1)$ the second upper diagonal $\{\lambda_{n+1}, \dots, \lambda_{2n-1}\}$ and so on. Let N_1 denote the set of indices i such that $\lambda_i = 0$ in the first layer of $L(e_1)$, and N_j the set of indices i such that $\lambda_i = 1$ in the j–th layer of $L(e_1)$ for $j = 2, \dots, n$. Define the *combinatorical type* of a Δ–module M with respect to the adapted basis of $\mathfrak{g}$ to be

$$\operatorname{type}(M) = \{N_1 \mid N_2 \mid \cdots \mid N_n\}$$

The combinatorical type of M more precisely is the type of $L(e_1)$. Empty sets N_i are omitted in this notation. If $L(e_1)$ is of full Jordan block form, then $N_j = \emptyset$ for all j. In that case we set $\operatorname{type}(M) = \emptyset$. Note that not all Δ–modules are faithful.

4.4.3. Example. *Let $\lambda \in \mathcal{A}_5(K)$. The Lie brackets of $\mathfrak{g}_\lambda$ with respect to an adapted basis $\{e_1, \dots, e_5\}$ are given by $[e_1, e_i] = e_{i+1}$, $[e_2, e_3] = \alpha e_5$. A faithful Δ–module M for $\mathfrak{g}_\lambda$ of type $\{4, 5 \mid 9\}$ is given as follows:*

$$L(e_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; L(e_2) = \begin{pmatrix} 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; L(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Denote by $\{f_1, \dots, f_6\}$ the basis of M. The center of M is 1–dimensional. It is generated by f_1.

Note that $L(e_2)$ is described by the first two layers. The module M is faithful since $L(e_5)$ is nonzero. In the following we will describe for each filiform Lie algebra of the above classes the Δ–modules we have found. Of course, the construction requires a systematic study of Δ–modules. It is very helpful to know which types can never yield faithful modules. Recall that a module of a filiform Lie algebra $\mathfrak{g}_\lambda$ with $\lambda \in \mathcal{A}_n(K)$ is faithful if and only if the center $\mathfrak{z}(\mathfrak{g}_\lambda)$ of $\mathfrak{g}_\lambda$ acts nontrivially, i.e., if $L(e_n)$ is not the zero transformation, where e_n generates $\mathfrak{z}(\mathfrak{g}_\lambda)$. After that it will be useful to reduce the number of possible types for Δ–modules. This will be done in Chapter 5. Here we will only state the result of the construction of Δ–modules.

4.4.1. Δ–modules for $n = 10$.

4.4.4. Theorem. *If $\lambda \in \mathcal{A}^2_{10,4}$, $\mathcal{A}^2_{10,1}$ then $\mathfrak{g}_\lambda$ always has a faithful Δ–module. If $\lambda \in \mathcal{A}^1_{10,4}$, $\mathcal{A}^1_{10,1}$ satisfies*

$$3\alpha_{2,6} + \alpha_{3,8} = 0$$

then $\mathfrak{g}_\lambda$ possesses a faithful Δ–module. Hence $\mu(\mathfrak{g}_\lambda) \le 11$ for these algebras.

4.4.5. Remark. The theorem implies together with the preceding results that all Lie algebras $\mathfrak{g}_\lambda$ with $\lambda \in \mathcal{A}_{10}(K)$ have a faithful module of dimension 11 with the possible exception of the cases where λ satisfies $2\alpha_{2,5} + \alpha_{3,7} \neq 0$, $\alpha_{3,7} = -\alpha_{2,5}$, $3\alpha_{2,6} + \alpha_{3,8} \neq 0$.

Proof. A Δ–module of a given type is computed as follows: Denote the first layer of $L(e_2)$ by $\{x_1, \dots, x_{10}\}$, the second layer by $\{x_{11}, \dots, x_{19}\}$ and so on. The two matrices $L(e_1), L(e_2)$ generate a Δ–module if and only if certain polynomial equations in the variables $x_1, \dots, x_{55}$ holds. These equations can be solved successively. First one has to solve a subsystem of equations which has only the variables $\{x_1, \dots, x_{10}\}$, then a subsystem of equations in the variables $\{x_{11}, \dots, x_{19}\}$ can be solved. As it turns out, once we have solved the equations involving the first two layers of $L(e_2)$, the remaining equations can be easily solved by substitutions of certain x_i which appear as linear monomials. More or less only the first two layers involve non-trivial computations. For that reason we will describe the Δ–modules here by specifying the *type* of $L(e_1)$ and the *first and second layer* of $L(e_1)$. A complete solution may be found in [**21**] and the references cited therein. For the computations we have used Reduce. The constructed faithful modules are as follows:

$$\textit{Case}\; \mathcal{A}^2_{10,1} \quad \textit{with} \quad 33\alpha_{2,6} - 20\alpha_{2,8} = 0 :$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9\} \\
1^{th} \text{ layer of } L(e_2) &: \quad \{-\frac{10\alpha_{5,10}}{11}, 0,0,0,0,0,0,0,0,0\} \\
2^{nd} \text{ layer of } L(e_2) &: \quad \{-\frac{23\alpha_{2,6}\alpha_{5,10}}{22\alpha_{2,5}}, -\alpha_{2,5}, -\alpha_{2,5}, 0, \alpha_{2,5}, \alpha_{2,5}, 0, 1, -2\}
\end{aligned}$$

$$\textit{Case}\, \mathcal{A}^2_{10,1} \quad \textit{with} \quad \gamma = 33\alpha_{2,6} - 20\alpha_{2,8} \neq 0 \textit{ and } 726\alpha^2_{2,5} - \gamma\alpha_{5,10} = 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{1, 9, 10 \mid 19\} \\
1^{th} \text{ layer of } L(e_2) &: \quad \{1, 0,0,0,0,0,0, \frac{\alpha_{5,10}}{11}, 0, \frac{1}{\alpha^2_{2,5}}\} \\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, -\alpha_{2,5}, -\alpha_{2,5}, 0, \alpha_{2,5}, \alpha_{2,5}, 0, \frac{2\alpha^2_{2,5}\alpha^2_{5,10}}{121}, -\frac{2\alpha_{5,10}}{11}\}
\end{aligned}$$

$$\textit{Case}\, \mathcal{A}^2_{10,1} \quad \textit{with} \quad \gamma = 33\alpha_{2,6} - 20\alpha_{2,8} \neq 0 \textit{ and } 726\alpha^2_{2,5} - \gamma\alpha_{5,10} \neq 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 19\} \\
1^{th} \text{ layer of } L(e_2) &: \quad \{-\frac{660\alpha^2_{2,5}\alpha_{5,10}}{726\alpha^2_{2,5} - \gamma\alpha_{5,10}}, 0,0,0,0,0,0, \frac{66\alpha_{2,5}}{\gamma}, 0, \frac{8712\alpha^2_{2,5}}{\gamma^2}\} \\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, -\alpha_{2,5}, -\alpha_{2,5}, 0, \alpha_{2,5}, \alpha_{2,5}, 0, 1, -\frac{132\alpha^2_{2,5}}{\gamma}\}
\end{aligned}$$

$$\textit{Case}\, \mathcal{A}^2_{10,4} \quad \textit{with} \quad 33\alpha_{2,6} - 20\alpha_{2,8} = 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9\} \\
1^{th} \text{ layer of } L(e_2) &: \quad \{0,0,0,0,0,0,0,0,0,0\} \\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, -\alpha_{2,5}, -\alpha_{2,5}, 0, \alpha_{2,5}, \alpha_{2,5}, 0, 1, -2\}
\end{aligned}$$

$$\textit{Case}\, \mathcal{A}^2_{10,4} \quad \textit{with} \quad \gamma = 33\alpha_{2,6} - 20\alpha_{2,8} \neq 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 19\} \\
1^{th} \text{ layer of } L(e_2) &: \quad \{0,0,0,0,0,0,0, \frac{66\alpha^2_{2,5}}{\gamma}, 0, \frac{8712\alpha^4_{2,5}}{\gamma^2}\} \\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, -\alpha_{2,5}, -\alpha_{2,5}, 0, \alpha_{2,5}, \alpha_{2,5}, 0, 1, -\frac{132\alpha^2_{2,5}}{\gamma}\}
\end{aligned}$$

$$\textit{Case}\, \mathcal{A}^1_{10,1} \quad \textit{with} \quad 3\alpha_{2,6} + \alpha_{2,8} = 0, \textit{ and } \alpha_{2,6} = 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9\}\\
1^{th} \text{ layer of } L(e_2) &: \quad \{-\frac{10\alpha_{2,5}}{11}, 0, 0, 0, 0, 0, 0, \alpha_{5,10}, 0, 2\alpha_{5,10}\}\\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, 7\alpha_{2,5}, 3\alpha_{2,5}, 2\alpha_{2,5}, \alpha_{2,5}, \alpha_{2,5}, -\frac{11\alpha_{2,5}}{5}, \alpha_{5,10}, -2\alpha_{5,10}\}
\end{aligned}$$

$$\mathit{Case}\, \mathcal{A}^1_{10,1} \quad \mathit{with} \quad 3\alpha_{2,6} + \alpha_{2,8} = 0,\ 22\alpha_{2,5}^2 - \alpha_{2,6}\alpha_{5,10} = 0,\ \mathit{and}\ \alpha_{2,6} \neq 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{1, 9, 10 \mid 19\}\\
1^{th} \text{ layer of } L(e_2) &: \quad \{1, 0, 0, 0, 0, 0, 0, \frac{\alpha_{5,10}}{11}, 0, \frac{1}{\alpha_{2,5}^2}\}\\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, 7\alpha_{2,5}, 3\alpha_{2,5}, 2\alpha_{2,5}, \alpha_{2,5}, \alpha_{2,5}, 0, \frac{2\alpha_{2,5}^2\alpha_{5,10}^2}{121}, -\frac{2\alpha_{5,10}}{11}\}
\end{aligned}$$

$$\mathit{Case}\, \mathcal{A}^1_{10,1} \quad \mathit{with} \quad 3\alpha_{2,6} + \alpha_{2,8} = 0,\ \gamma = 22\alpha_{2,5}^2 - \alpha_{2,6}\alpha_{5,10} \neq 0,\ \mathit{and}\ \alpha_{2,6} \neq 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 19\}\\
1^{th} \text{ layer of } L(e_2) &: \quad \{-\frac{20\alpha_{2,5}^2\alpha_{5,10}}{\gamma}, 0, 0, 0, 0, 0, 0, \frac{2\alpha_{2,5}}{\alpha_{2,6}}, 0, \frac{8\alpha_{2,5}^4}{\alpha_{2,6}^2}\}\\
2^{nd} \text{ layer of } L(e_2) &: \quad \{0, 7\alpha_{2,5}, 3\alpha_{2,5}, 2\alpha_{2,5}, \alpha_{2,5}, \alpha_{2,5}, -\frac{7\gamma\alpha_{2,5}}{5\alpha_{2,6}\alpha_{5,10}}, 1, -\frac{4\alpha_{2,5}^2}{\alpha_{2,6}}\}
\end{aligned}$$

$$\mathit{Case}\, \mathcal{A}^1_{10,4} \quad \mathit{with} \quad 3\alpha_{2,6} + \alpha_{2,8} = 0,\ \mathit{and}\ \alpha_{2,6} \neq 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 19\}\\
1^{th} \text{ layer of } L(e_2) &: \quad \{0, 0, 0, 0, 0, 0, 0, \frac{2\alpha_{2,5}^2}{\alpha_{2,6}}, 0, \frac{8\alpha_{2,5}^4}{\alpha_{2,6}^2}\}\\
2^{nd} \text{ layer of } L(e_2) &: \quad \{14\alpha_{2,5}, 7\alpha_{2,5}, 3\alpha_{2,5}, 2\alpha_{2,5}, \alpha_{2,5}, \alpha_{2,5}, 0, 1, -\frac{4\alpha_{2,5}^2}{\alpha_{2,6}}\}
\end{aligned}$$

$$\mathit{Case}\, \mathcal{A}^1_{10,4} \quad \mathit{with} \quad 3\alpha_{2,6} + \alpha_{2,8} = 0,\ \mathit{and}\ \alpha_{2,6} = 0:$$

$$\begin{aligned}
L(e_1) \text{ is of type } &: \quad \{9\}\\
1^{th} \text{ layer of } L(e_2) &: \quad \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\\
2^{nd} \text{ layer of } L(e_2) &: \quad \{14\alpha_{2,5}, 7\alpha_{2,5}, 3\alpha_{2,5}, 2\alpha_{2,5}, \alpha_{2,5}, \alpha_{2,5}, 0, 1, -2\}
\end{aligned}$$

We have $L(e_{10}) \neq 0$ in all cases, i.e., the constructed Δ–modules are faithful. $\square$

4.4.2. Δ–modules for $n = 11$. To shorten the formulas we have set $\alpha_{3,7} = 1$ for $\lambda \in \mathcal{A}^1_{11,1}$ in the result, but not in the calculation. Similarily we have set $\alpha_{2,5} = 1$ for $\lambda \in \mathcal{A}^3_{11,1}$. On the other hand, we could have assumed this by changing the base so that it stays adapted and satisfies $\alpha_{3,7} = 1$ respectively $\alpha_{2,5} = 1$.

4.4.6. Theorem. *If $\lambda \in \mathcal{A}_{11,4}$ then $\mathfrak{g}_\lambda$ possesses a faithful Δ–module. If $\lambda \in \mathcal{A}^1_{11,1}$ satisfies $\alpha_{2,6} = 0$, $\alpha_{3,9} = 3\alpha_{2,7} + \alpha^2_{3,8}$ then $\mathfrak{g}_\lambda$ possesses a faithful Δ–module. If $\lambda \in \mathcal{A}^3_{11,1}$ satisfies $\alpha_{2,6} = \alpha_{3,8}/(3\alpha_{3,7}(1 - \alpha_{3,7}))$ and*

$$\alpha_{3,9} = \frac{12\alpha_{2,7}\alpha_{3,7} + \alpha^2_{2,6}(\alpha_{3,7} - 1)(3\alpha^2_{3,7} + 7\alpha_{3,7} - 1)}{4(2 + \alpha_{3,7})} \tag{44}$$

then $\mathfrak{g}_\lambda$ possesses a faithful Δ–module. Hence $\mu(\mathfrak{g}_\lambda) \le 12$ for these algebras.

Proof. The constructed faithful Δ–modules of dimension 12 are as follows:

$$\textit{Case}\ \mathcal{A}_{11,4} \quad \textit{with} \quad \gamma = 2\alpha_{2,6} - \alpha_{4,10} \neq 0 :$$

$$\begin{aligned} L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 20\} \\ 1^{th} \text{ layer of } L(e_2) &: \quad \{\frac{\alpha^2_{4,9}}{\gamma}, 0, 0, 0, 0, 0, 0, 0, 0, 1, \frac{7\alpha^2_{4,9}}{4\gamma}\} \\ 2^{nd} \text{ layer of } L(e_2) &: \quad \{0, 0, 0, 0, 0, -\alpha_{4,9}, -3\alpha_{4,9}, 0, 0, 0\} \end{aligned}$$

$$\textit{Case}\ \mathcal{A}_{11,4} \quad \textit{with} \quad 2\alpha_{2,6} - \alpha_{4,10} = 0 :$$

$$\begin{aligned} L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 20\} \\ 1^{th} \text{ layer of } L(e_2) &: \quad \{1, 0, 0, 0, 0, 0, 0, 0, 0, 1, \frac{7}{3}\} \\ 2^{nd} \text{ layer of } L(e_2) &: \quad \{0, 0, 0, 0, 0, -\alpha_{4,9}, -3\alpha_{4,9}, 0, 0, 0\} \end{aligned}$$

$$\textit{Case}\ \mathcal{A}^1_{11,1} \quad \textit{with} \quad \alpha_{2,6} = 0,\ \alpha_{3,9} = 3\alpha_{2,7} + \alpha^2_{3,8},\ \alpha_{3,8} \neq 0 :$$

$$\begin{aligned} L(e_1) \text{ is of type } &: \quad \{9, 10 \mid 20\} \\ 1^{th} \text{ layer of } L(e_2) &: \quad \{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \frac{6}{\alpha_{3,8}}\} \\ 2^{nd} \text{ layer of } L(e_2) &: \quad \{\frac{5}{2}, 5, 0, -2, -1, 0, 0, 0, -\frac{1}{\alpha_{3,8}}, \frac{18}{\alpha^2_{3,8}}\} \end{aligned}$$

$$\textit{Case}\ \mathcal{A}^1_{11,1} \quad \textit{with} \quad \alpha_{2,6} = 0,\ \alpha_{3,9} = 3\alpha_{2,7} + \alpha^2_{3,8},\ \alpha_{3,8} = 0 :$$

$$\begin{aligned} L(e_1) \text{ is of type } &: \quad \{10\} \\ 1^{th} \text{ layer of } L(e_2) &: \quad \{0,0,0,0,0,0,0,0,1,0,2\} \\ 2^{nd} \text{ layer of } L(e_2) &: \quad \{\frac{5}{2},5,0,-2,-1,0,0,0,1,-2\} \end{aligned}$$

$$\textit{Case}\, \mathcal{A}^3_{11,1} \quad \textit{with}\, (44),\ \alpha_{3,8} \neq 0 :$$

$$\begin{aligned} L(e_1) \text{ is of type } &: \quad \{9,10 \mid 20\} \\ 1^{th} \text{ layer of } L(e_2) &: \quad \{0,0,0,0,0,0,0,0,1,0,-\frac{4}{\alpha_{2,6}}\} \\ 2^{nd} \text{ layer of } L(e_2) &: \quad \{\gamma_1,\gamma_2,\gamma_3,1-2\alpha_{3,7},1-\alpha_{3,7},1,1,0,\frac{2}{\alpha_{2,6}},\frac{8}{\alpha_{2,6}^2}\} \end{aligned}$$

$$\textit{Case}\, \mathcal{A}^3_{11,1} \quad \textit{with}\, (44),\ \alpha_{3,8} = 0 :$$

$$\begin{aligned} L(e_1) \text{ is of type } &: \quad \{10\} \\ 1^{th} \text{ layer of } L(e_2) &: \quad \{0,0,0,0,0,0,0,0,1,0,2\} \\ 2^{nd} \text{ layer of } L(e_2) &: \quad \{\gamma_1,\gamma_2,\gamma_3,1-2\alpha_{3,7},1-\alpha_{3,7},1,1,0,-2\}, \end{aligned}$$

$$\gamma_1 = \frac{2-10\alpha_{3,7}+16\alpha_{3,7}^2-5\alpha_{3,7}^3}{2(1-\alpha_{3,7}^2)}$$

$$\gamma_2 = \frac{(2-5\alpha_{3,7})(1-\alpha_{3,7})}{2+\alpha_{3,7}}$$

$$\gamma_3 = \frac{2-5\alpha_{3,7}}{2+\alpha_{3,7}}$$

$\square$

Note that the Jacobi identity implies that $1-\alpha_{3,7}^2 \neq 0$. The constructed modules are faithful, since $L(e_{11})$, which depends only on the first two layers of $L(e_2)$, is nonzero. To give an example, consider the first Δ–module constructed for $\lambda \in \mathcal{A}^3_{11,1}$. Here $L(e_{11})$ is the zero matrix except for the right upper corner element which is $-22/\alpha_{2,6}$, hence nonzero.

4.4.7. Remark. The theorem is consistent with the computations from section 3.5.1: The Lie algebras admitting a nonsingular derivation also admit a faithful Δ–module.

CHAPTER 5

Counterexamples to the Milnor conjecture

5.1. An open problem

In this chapter we will give counterexamples to the Milnor conjecture. The conjecture may be formulated as follows:

5.1.1. Milnor Conjecture. *Every solvable Lie algebra admits an affine structure.*

To obtain the counterexamples we construct Lie algebras $\mathfrak{g}$ of dimension n which satisfy $\mu(\mathfrak{g}) \geq n+2$. Such Lie algebras do not possess a faithful module of dimension $n+1$ and hence admit no affine structure. In general it is not clear how to find such algebras.

5.1.2. Problem. *Find the Lie algebras* $\mathfrak{g}$ *which satisfy* $\mu(\mathfrak{g}) \geq n+2$.

We will solve this problem for filiform Lie algebras of dimension $n = 10$ and $n = 11$. We determine all Lie algebras $\mathfrak{g} \in \mathfrak{F}_n(\mathbb{C})$, $n = 10, 11$ with $\mu(\mathfrak{g}) \geq n+2$.

As a corollary we obtain that there are nilpotent Lie groups which do not admit any left-invariant structure. We will always assume here that $K = \mathbb{C}$. The proof is based on the study of combinatorical types of faithful Δ–modules and the explicit computation of Δ–modules of a given type. The first step is the following:

5.1.3. Lemma. *Let* $\lambda \in \mathcal{A}_n(\mathbb{C})$. *If* $\mathfrak{g}_\lambda$ *has a faithful module* M *of dimension* $n+1$ *then there also exists a faithful* Δ*–module for* $\mathfrak{g}_\lambda$.

Proof. The proof of Proposition 4.1.3 implies that M can be replaced by a faithful module of dimension $n+1$ which is nilpotent. Hence we may assume that M is nilpotent. Then by Lie's theorem there exists a basis $f_1, \dots, f_{n+1}$ such that $L(e_1), L(e_2)$ are simultaneously strictly upper triangular matrices. Applying suitable base changes which keep the upper triangular form of both operators we may assume that the entries of $L(e_1)$ are 0 or 1 and that in each row and in each column of $L(e_1)$ there is at most one nonzero entry. The suitable base changes are of the form $f_i \mapsto \alpha_{1i} f_1 + \cdots + \alpha_{ii} f_i$. By definition we obtain a faithful Δ–module. $\square$

Let $\mathfrak{g} \in \mathfrak{F}_n(K)$. Choose an adapted basis $(e_1, \dots, e_n)$ so that the corresponding law λ belongs to $\mathcal{A}_n(K)$. If there exists no faithful Δ–module then $\mu(\mathfrak{g}) \geq n+2$. Hence we have reduced the above problem to the study of Δ–modules. Unfortunately we have to classify, in a certain sense, the faithful Δ–modules because of the lack of a better invariant for the problem. Clearly this method is not suitable to study the problem in more generality. Nevertheless it indicates that there should be counterexamples to the Milnor conjecture in any dimension $n \geq 13$. The candidates are certain filiform algebras $\mathfrak{g}$ where $H^2(\mathfrak{g}, K)$ does not contain an affine cohomology class. Recall that we write $\mathfrak{g} \in \mathfrak{A}_n^2(K)$ for filiform

Lie algebras which satisfy property (b) and (c), but not (d): $\mathfrak{g}$ does not contain a one-codimensional subspace $U \subseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$, $\mathfrak{g}^{\frac{n-4}{2}}$ is abelian for n even, and $[\mathfrak{g}^1, \mathfrak{g}^1]$ is not contained in $\mathfrak{g}^6$. We propose the following problem:

5.1.4. Open problem. *Does a Lie algebra $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$ satisfy $\mu(\mathfrak{g}) \geq n+2$ if and only if there is no affine $[\omega] \in H^2(\mathfrak{g}, K)$?*

For $n = 13$ we have studied the Δ–modules and we believe that the statement is true. For higher dimensions, however, a classification of Δ–modules is a hopeless approach.

5.1.5. Remark. The condition that $\mathfrak{g}$ admits an affine structure is not necessarily equivalent to the fact that $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$.

5.2. Filiform Lie algebras of dimension 10

Any filiform Lie algebra of dimension 10 over $\mathbb{C}$ has an adapted basis $(e_1, \ldots, e_{10})$ such that the Lie brackets are given by:

$$
\begin{aligned}
[e_1, e_i] &= e_{i+1}, \ 2 \leq i \leq 9 \\
[e_2, e_3] &= \alpha_{2,5}e_5 + \alpha_{2,6}e_6 + \alpha_{2,7}e_7 + \alpha_{2,8}e_8 + \alpha_{2,9}e_9 + \alpha_{2,10}e_{10} \\
[e_2, e_4] &= \alpha_{2,5}e_6 + \alpha_{2,6}e_7 + \alpha_{2,7}e_8 + \alpha_{2,8}e_9 + \alpha_{2,9}e_{10} \\
[e_2, e_5] &= (\alpha_{2,5} - \alpha_{3,7})e_7 + (\alpha_{2,6} - \alpha_{3,8})e_8 + (\alpha_{2,7} - \alpha_{3,9})e_9 + (\alpha_{2,8} - \alpha_{3,10})e_{10} \\
[e_2, e_6] &= (\alpha_{2,5} - 2\alpha_{3,7})e_8 + (\alpha_{2,6} - 2\alpha_{3,8})e_9 + (\alpha_{2,7} - 2\alpha_{3,9})e_{10} \\
[e_2, e_7] &= (\alpha_{2,5} - 3\alpha_{3,7} + \alpha_{4,9})e_9 + (\alpha_{2,6} - 3\alpha_{3,8} + \alpha_{4,10})e_{10} \\
[e_2, e_8] &= (\alpha_{2,5} - 4\alpha_{3,7} + 3\alpha_{4,9})e_{10} \\
[e_2, e_9] &= -\alpha_{5,10}e_{10} \\
[e_3, e_4] &= \alpha_{3,7}e_7 + \alpha_{3,8}e_8 + \alpha_{3,9}e_9 + \alpha_{3,10}e_{10} \\
[e_3, e_5] &= \alpha_{3,7}e_8 + \alpha_{3,8}e_9 + \alpha_{3,9}e_{10} \\
[e_3, e_6] &= (\alpha_{3,7} - \alpha_{4,9})e_9 + (\alpha_{3,8} - \alpha_{4,10})e_{10} \\
[e_3, e_7] &= (\alpha_{3,7} - 2\alpha_{4,9})e_{10} \\
[e_3, e_8] &= \alpha_{5,10}e_{10} \\
[e_4, e_5] &= \alpha_{4,9}e_9 + \alpha_{4,10}e_{10} \\
[e_4, e_6] &= \alpha_{4,9}e_{10} \\
[e_4, e_7] &= -\alpha_{5,10}e_{10} \\
[e_5, e_6] &= \alpha_{5,10}e_{10}
\end{aligned}
$$

The Jacobi identity holds if and only if the parameters $\{\alpha_{k,s} \mid (k,s) \in \mathcal{I}_{10}\}$ satisfy the following equations:

$$
\begin{aligned}
0 &= \alpha_{5,10}(2\alpha_{2,5} - \alpha_{3,7} - \alpha_{4,9}) \\
0 &= \alpha_{4,9}(2\alpha_{2,5} + \alpha_{3,7}) - 3\alpha_{3,7}^2 \\
0 &= \alpha_{5,10}(2\alpha_{2,7} + \alpha_{3,9}) - \alpha_{4,10}(2\alpha_{2,5} + \alpha_{3,7}) - 3\alpha_{4,9}(\alpha_{2,6} + \alpha_{3,8}) + 7\alpha_{3,7}\alpha_{3,8}
\end{aligned}
$$

To determine the algebras $\mathfrak{g} \in \mathfrak{F}_{10}(\mathbb{C})$ with $\mu(\mathfrak{g}) \geq n+2 = 12$ it suffices to consider the laws $\lambda \in \mathcal{A}_{10,10}$, see Remark 4.4.5, defined by:

$$\mathcal{A}_{10,10} \quad \text{if } \ \alpha_{3,7} = -\alpha_{2,5} \neq 0$$

Note that $\mathcal{A}_{10,10}$ is the union of $\mathcal{A}^1_{10,1}$ and $\mathcal{A}^1_{10,4}$. The Jacobi identity holds if and only if

$$\begin{aligned} \alpha_{4,9} &= 3\alpha_{2,5} \\ \alpha_{4,10} &= (\alpha_{5,10}(2\alpha_{2,7} + \alpha_{3,9}) - \alpha_{2,5}(16\alpha_{3,8} + 9\alpha_{2,6}))/\alpha_{2,5} \end{aligned}$$

The free parameters are then $\alpha_{2,6}, \alpha_{2,7}, \alpha_{2,8}, \alpha_{2,9}, \alpha_{2,10}, \alpha_{3,8}, \alpha_{3,9}, \alpha_{3,10}, \alpha_{5,10}$.

5.2.1. Remark. There is a classification of all algebras $\mathfrak{g} \in \mathfrak{F}_{10}(\mathbb{C})$, see [**16**]. So we could compute $\mu(\mathfrak{g})$ for each isomorphism class. However, it is easier to compute the invariant $\mu(\mathfrak{g}_\lambda)$ just for all $\lambda \in \mathcal{A}_{10,10}$.

We need the following lemma:

5.2.2. Lemma. *Let $\lambda \in \mathcal{A}_{10}(\mathbb{C})$ and M be a faithful Δ–module for $\mathfrak{g}_\lambda$. Then we may assume that the combinatorical type of M is one of the following:*

(1)	$\emptyset$		
(2)	$\{i\}$	$i = 5, \dots, 10$	
(3)	$\{i, i+1\}$	$i = 5, \dots, 9$	
(4)	$\{i, i+1 \mid 10+i\}$	$i = 5, \dots, 9$	
(5)	$\{i, i+1, j \mid 10+i\}$	$i = 5, 6, 7$	$j > i+2$
(6)	$\{i, j, j+1 \mid 10+j\}$	$j = 6, 7, 8, 9$	$i < j-1$

Proof. The module M is given by the linear operators $L(e_1)$ and $L(e_2)$ which are of the following form:

$$L(e_1) := \begin{pmatrix} 0 & \lambda_1 & \lambda_{11} & \dots & \lambda_{53} & \lambda_{55} \\ 0 & 0 & \lambda_2 & \dots & \lambda_{51} & \lambda_{54} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_9 & \lambda_{19} \\ 0 & 0 & 0 & \dots & 0 & \lambda_{10} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad L(e_2) := \begin{pmatrix} 0 & x_1 & x_{11} & \dots & x_{53} & x_{55} \\ 0 & 0 & x_2 & \dots & x_{51} & x_{54} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_9 & x_{19} \\ 0 & 0 & 0 & \dots & 0 & x_{10} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where $\lambda_i \in \{0, 1\}$ such that in each row and each column of $L(e_1)$ there is at most one nonzero entry (equal to 1). The module M is faithful iff $L(e_{10})$ is nonzero. Let $n = 10$ and

$$\lambda_{i,j} := \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \lambda_k, \quad \lambda_{i,j,k} := \prod_{\substack{\ell=1 \\ \ell \neq i,j,k}}^{n} \lambda_\ell$$

$$r_i := (-1)^{n-2} \binom{n-2}{i-1}, \ 1 \leq i \leq n-1$$

Then $L(e_{10}) = (a_{i,j})_{1\le i,j\le 11}$ is given as follows: $a_{i,j} = 0$ except for

$$a_{1,n} = \sum_{i=1}^{n-1} r_i\lambda_{i,n}x_i, \quad a_{2,n+1} = \sum_{i=1}^{n-1} r_i\lambda_{1,i+1}x_{i+1}$$

$$a_{1,n+1} = \sum_{i=1}^{n-1} r_i\lambda_{i,i+1}x_{i+n} + \sum_{i=1}^{n-1}\sum_{j=1}^{i-1} r_i\lambda_{i+1,j+1,j}\,\lambda_{j+n}x_{i+1} + \sum_{i=1}^{n-1}\sum_{j=1+1}^{n-1} r_i\lambda_{j+1,j,i}\,\lambda_{j+n}x_i$$

Using this formulas we can determine the combinatorical types

$$\mathrm{type}(M) = \{N_1 \mid N_2 \mid \cdots \mid N_n\}$$

of M which can yield a faithful module. For the definition of the sets N_i see Definition 4.4.2. If N_1 contains more than 3 elements, then $L(e_{10}) = 0$ and M is not faithful. If $|N_1| = 3$ then M is not faithful if not $\{i, i+1\} \subseteq N_1$. It is straightforward to check that the types which can be faithful are given as follows:

(1)	$\emptyset$		
(2)	$\{i\}$,		$i = 1, \dots, n+1$
(3)	$\{i, i+1 \mid N_2\}$,	$\lvert N_2\rvert \le 1$,	$i = 1, \dots, n$
(4)	$\{1, i \mid N_i\}$,	$\lvert N_i\rvert \le 1$,	$i = 3, \dots, n+1$
(5)	$\{i, n+1 \mid N_{n-i}\}$,	$\lvert N_{n-1}\rvert \le 1$,	$i = 2, \dots, n-1$
(6)	$\{i, i+1, j \mid i+n \mid N_k\}$,	$\lvert N_k\rvert \le 1$,	$i = 1, \dots, n-3,\ j > i+2$
(7)	$\{i, j, j+1 \mid j+n \mid N_k\}$,	$\lvert N_k\rvert \le 1$,	$i = 3, \dots, n,\ i < j-1$
(8)	$\{i, i+1, i+2 \mid N_2\}$,	$1 \le \lvert N_2\rvert \le 2$,	$i = 1, \dots, n-1$

where N_2 in case (8) is $\{i+n\}$ or $\{i+n+1\}$ or $\{i+n, i+n+1\}$, and N_2 in case (3) is $\emptyset$ or $\{i+n\}$. The notation $|N_i| \le 1$ means that N_i is either $\emptyset$ or consists of an element which is uniquely determined by the rule that in each row and column of $L(e_1)$ there is at most one nonzero entry. It is not difficult to see that this list can be reduced by various arguments, such as adding trivial 1–dimensional modules and possibly going to the dual module M^* of M. The type of M^* results from reflecting the matrix $L(e_1)$, which determines the type of M, on its antidiagonal. For more details see [**18**],[**21**]. We obtain the list of types given above. □

Now we are ready to determine the Lie algebras $\mathfrak{g} \in \mathfrak{F}_{10}(\mathbb{C})$ with $\mu(\mathfrak{g}) \ge 12$:

5.2.3. Theorem. *If $\lambda \in \mathcal{A}_{10,10}$ satisfies $3\alpha_{2,6} + \alpha_{3,8} \ne 0$ then $\mathfrak{g}_\lambda$ has no faithful module of dimension 11 and hence admits no affine structure. In fact, it holds $12 \le \mu(\mathfrak{g}_\lambda) \le 22$. For all other choices of $\lambda \in \mathcal{A}_{10}(\mathbb{C})$ it holds $\mu(\mathfrak{g}_\lambda) \le 11$.*

Proof. Note that $\alpha_{2,6}$ and $\alpha_{3,8}$ are free parameters for $\lambda \in \mathcal{A}_{10,10}$. The construction of faithful Δ–modules in section 4.4 shows the last part of the theorem. Assume that $\lambda \in \mathcal{A}_{10,10}$, and that $\mathfrak{g}_\lambda$ has a faithful module M of dimension 11. By Lemma 5.1.3 we may assume that M is a faithful Δ–module. By Lemma 5.2.2 we may assume that the combinatorical type of M is one of the list given there. The proof consists of solving the module equations, which are polynomial equations in the variables x_i of $L(e_2)$ with coefficients $\alpha_{i,j}$ of the Lie algebra, for each type of the list. This means a lot of computations. However, the computation for each type is straightforward and the polynomial equations involved can be solved using a certain algorithm: one has to solve the equations involving the first layer to obtain a typical subsystem of equations with variables from the second layer of $L(e_2)$. After that in general the equations can be solved by applying a few substitutions. Many types very soon yield a contradiction. As an example for a more complicated type, let M be of type $\{10\}$. If $x_{19} \neq 0$, then the equations involving the variables $\{x_1, \dots, x_{10}\}$ immediately yield $x_1 = x_2 = \cdots = x_8 = 0$ and $x_{10} = 0$ so that we obtain the following subsystem in the variables $x_{11}, \dots, x_{14}$:

$$
\begin{aligned}
0 &= -x_{14}x_{12} + 2x_{14}x_{11} - x_{14}\alpha_{2,5} - x_{13}x_{11} + 3x_{13}\alpha_{2,5} - 3x_{12}\alpha_{2,5} + x_{11}\alpha_{2,5}\\
0 &= -4x_{14}x_{13} + 7x_{14}x_{12} - x_{14}\alpha_{2,5} + 5x_{13}^2 - 12x_{13}x_{12} + 5x_{13}\alpha_{2,5} + 4x_{12}^2 - 7x_{12}\alpha_{2,5} + 3\alpha_{2,5}^2\\
0 &= -10x_{14}^2 + 31x_{14}x_{13} - 6x_{14}x_{12} + 7x_{14}\alpha_{2,5} - 25x_{13}^2 + 10x_{13}x_{12} - 12x_{13}\alpha_{2,5} - \alpha_{2,5}^2\\
0 &= -50x_{14}^2 + 175x_{14}x_{13} - 70x_{14}x_{12} + 101x_{14}\alpha_{2,5} - 150x_{13}^2 + 120x_{13}x_{12} - 175x_{13}\alpha_{2,5}\\
&\quad - 24x_{12}^2 + 70x_{12}\alpha_{2,5} - 49\alpha_{2,5}^2\\
0 &= -10x_{14}^2 + 35x_{14}x_{13} - 13x_{14}x_{12} + 8x_{14}\alpha_{2,5} - 30x_{13}^2 + 22x_{13}x_{12} - 17x_{13}\alpha_{2,5}\\
&\quad - 4x_{12}^2 + 7x_{12}\alpha_{2,5} - 4\alpha_{2,5}^2\\
0 &= -40x_{14}^2 + 144x_{14}x_{13} - 64x_{14}x_{12} + 94x_{14}\alpha_{2,5} - 125x_{13}^2 + 110x_{13}x_{12} - 163x_{13}\alpha_{2,5}\\
&\quad - 24x_{12}^2 + 70x_{12}\alpha_{2,5} - 48\alpha_{2,5}^2\\
0 &= -10x_{14}^2 + 45x_{14}x_{13} - 36x_{14}x_{12} + 11x_{14}x_{11} + 8x_{14}\alpha_{2,5} - 45x_{13}^2 + 63x_{13}x_{12} - 16x_{13}x_{11}\\
&\quad - 24x_{13}\alpha_{2,5} - 18x_{12}^2 + 6x_{12}x_{11} + 22x_{12}\alpha_{2,5} - 5x_{11}\alpha_{2,5} - 14\alpha_{2,5}^2
\end{aligned}
$$

A computation by hand or by a computer algebra system using Groebner bases shows that these equations imply $\alpha_{2,5} = 0$. That is a contradiction. The case $x_{19} = 0$ is similar. The following types however lead to faithful Δ–modules

$$\{9\},\ \{1, 9, 10 \mid 19\},\ \{1, 8, 9 \mid 18\},\ \{9, 10 \mid 19\},\ \{8, 9 \mid 18\}$$

But in all these cases it follows $3\alpha_{2,6} + \alpha_{3,8} = 0$.
Let now $\lambda \in \mathcal{A}_{10,10}$ with $3\alpha_{2,6} + \alpha_{3,8} \neq 0$. It remains to prove that $\mathfrak{g} = \mathfrak{g}_\lambda$ satisfies $\mu(\mathfrak{g}) \leq 22$. We will construct a faithful $\mathfrak{g}$–module as in Theorem 4.3.4. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ together with a basis of ordered monomials $e^\alpha = e_{10}^{\alpha_{10}} \dots e_1^{\alpha_1}$ and an order function. Consider

$$U^m(\mathfrak{g}) = \{t \in U(\mathfrak{g}) \mid \operatorname{ord}(t) \geq m\}$$

It follows that $U^m(\mathfrak{g})$ is an ideal of $U(\mathfrak{g})$ of finite codimension. Let V be the quotient module $U(\mathfrak{g})/U^m(\mathfrak{g})$. It is a faithful $\mathfrak{g}$–module if m is greater than the nilpotency class of $\mathfrak{g}$. If we choose $m = 10$ then V has vector space basis

$$\{e_{10}^{\alpha_{10}} \cdots e_1^{\alpha_1} \mid 9\alpha_{10} + \cdots + 2\alpha_3 + \alpha_2 + \alpha_1 \ \leq\ 9\}$$

The elements e_i of $\mathfrak{g}$ act on V by $e_ie_j = [e_i, e_j] + e_je_i$ for $i < j$. Consider the following quotient module $\widehat{V}$ of V with vector space basis, ordered by weight:

$$
\begin{gathered}
\{e_{10},\ e_9,\ e_5^2,\ e_8,\ e_5e_4,\ e_4e_3^2,\ e_7,\ e_5e_3,\ e_5e_2^2,\ e_4^2,\ e_4e_3e_2,\ e_4e_2^3,\ e_3^3,\ e_3^2e_2^2,\ e_6,\\
e_5e_2,\ e_4e_3,\ e_4e_2^2,\ e_3^2e_2,\ e_3e_2^3,\ e_2^5,\ e_5,\ e_4e_2,\ e_3^2,\ e_3e_2^2,\ e_2^4,\ e_4,\ e_3e_2,\ e_2^3,\ e_3,\ e_2^2,\ e_2,\ 1\}
\end{gathered}
$$

The module $\widehat{V}$ is a faithful $\mathfrak{g}$–module of dimension 33 with a center Z containing e_{10}. Taking the quotient module of $\widehat{V}$ by a maximal subspace of Z not containing e_{10} we obtain a faithful $\mathfrak{g}$–module of dimension 27. Repeating this procedure finally we obtain a faithful $\mathfrak{g}$–module of dimension 22. $\square$

5.3. Filiform Lie algebras of dimension 11

Any filiform Lie algebra of dimension 11 over $\mathbb{C}$ has an adapted basis $(e_1, \dots, e_{11})$ such that the Lie brackets are given by:

$$
\begin{aligned}
[e_1, e_i] &= e_{i+1},\ 2 \le i \le 10\\
[e_2, e_3] &= \alpha_{2,5}e_5 + \alpha_{2,6}e_6 + \alpha_{2,7}e_7 + \alpha_{2,8}e_8 + \alpha_{2,9}e_9 + \alpha_{2,10}e_{10} + \alpha_{2,11}e_{11}\\
[e_2, e_4] &= \alpha_{2,5}e_6 + \alpha_{2,6}e_7 + \alpha_{2,7}e_8 + \alpha_{2,8}e_9 + \alpha_{2,9}e_{10} + \alpha_{2,10}e_{11}\\
[e_2, e_5] &= (\alpha_{2,5} - \alpha_{3,7})e_7 + (\alpha_{2,6} - \alpha_{3,8})e_8 + (\alpha_{2,7} - \alpha_{3,9})e_9\\
&\quad + (\alpha_{2,8} - \alpha_{3,10})e_{10} + (\alpha_{2,9} - \alpha_{3,11})e_{11}\\
[e_2, e_6] &= (\alpha_{2,5} - 2\alpha_{3,7})e_8 + (\alpha_{2,6} - 2\alpha_{3,8})e_9 + (\alpha_{2,7} - 2\alpha_{3,9})e_{10} + (\alpha_{2,8} - 2\alpha_{3,10})e_{11}\\
[e_2, e_7] &= (\alpha_{2,5} - 3\alpha_{3,7} + \alpha_{4,9})e_9 + (\alpha_{2,6} - 3\alpha_{3,8} + \alpha_{4,10})e_{10}\\
&\quad + (\alpha_{2,7} - 3\alpha_{3,9} + \alpha_{4,11})e_{11}\\
[e_2, e_8] &= (\alpha_{2,5} - 4\alpha_{3,7} + 3\alpha_{4,9})e_{10} + (\alpha_{2,6} - 4\alpha_{3,8} + 3\alpha_{4,10})e_{11}\\
[e_2, e_9] &= (\alpha_{2,5} - 5\alpha_{3,7} + 6\alpha_{4,9} - \alpha_{5,11})e_{11}\\
[e_3, e_4] &= \alpha_{3,7}e_7 + \alpha_{3,8}e_8 + \alpha_{3,9}e_9 + \alpha_{3,10}e_{10} + \alpha_{3,11}e_{11}\\
[e_3, e_5] &= \alpha_{3,7}e_8 + \alpha_{3,8}e_9 + \alpha_{3,9}e_{10} + \alpha_{3,10}e_{11}\\
[e_3, e_6] &= (\alpha_{3,7} - \alpha_{4,9})e_9 + (\alpha_{3,8} - \alpha_{4,10})e_{10} + (\alpha_{3,9} - \alpha_{4,11})e_{11}\\
[e_3, e_7] &= (\alpha_{3,7} - 2\alpha_{4,9})e_{10} + (\alpha_{3,8} - 2\alpha_{4,10})e_{11}\\
[e_3, e_8] &= (\alpha_{3,7} - 3\alpha_{4,9} + \alpha_{5,11})e_{11}\\
[e_4, e_5] &= \alpha_{4,9}e_9 + \alpha_{4,10}e_{10} + \alpha_{4,11}e_{11}\\
[e_4, e_6] &= \alpha_{4,9}e_{10} + \alpha_{4,10}e_{11}\\
[e_4, e_7] &= (\alpha_{4,9} - \alpha_{5,11})e_{11}\\
[e_5, e_6] &= \alpha_{5,11}e_{11}
\end{aligned}
$$

The parameters $\{\alpha_{k,s} \mid (k,s) \in \mathcal{I}_{11}\}$ satisfy the Jacobi identity if and only if the following equations hold:

$$
\begin{aligned}
0 &= \alpha_{4,9}(2\alpha_{2,5} + \alpha_{3,7}) - 3\alpha_{3,7}^2\\
0 &= \alpha_{4,10}(2\alpha_{2,5} + \alpha_{3,7}) + 3\alpha_{4,9}(\alpha_{2,6} + \alpha_{3,8}) - 7\alpha_{3,7}\alpha_{3,8}\\
0 &= \alpha_{5,11}(2\alpha_{2,5} - \alpha_{3,7} - \alpha_{4,9}) + \alpha_{4,9}(6\alpha_{4,9} - 4\alpha_{3,7})\\
0 &= -\alpha_{5,11}(2\alpha_{2,7} + \alpha_{3,9}) + \alpha_{4,11}(2\alpha_{2,5} + \alpha_{3,7}) + 3\alpha_{4,10}(\alpha_{2,6} + \alpha_{3,8}) - 4\alpha_{3,8}^2\\
&\quad + 2\alpha_{4,9}(2\alpha_{2,7} + 3\alpha_{3,9}) - 8\alpha_{3,7}\alpha_{3,9}
\end{aligned}
$$

We want to determine the Lie algebras $\mathfrak{g} \in \mathfrak{F}_{11}(\mathbb{C})$ with $\mu(\mathfrak{g}) \geq 13$. Because of Theorem 4.4.6 it is sufficient to consider laws $\lambda \in \mathcal{A}^1_{11,1}$, $\mathcal{A}^3_{11,1}$, i.e., filiform laws $\lambda \in \mathcal{A}_{11}(\mathbb{C})$ with

$$\begin{gathered}2\alpha_{2,5} + \alpha_{3,7} \neq 0,\ \alpha_{3,7} \neq 0,\ 10\alpha_{3,7} - \alpha_{2,5} \neq 0,\\ (2\alpha_{2,5}^2 - 5\alpha_{3,7}^2)(4\alpha_{2,5}^2 - 4\alpha_{2,5}\alpha_{3,7} + 3\alpha_{2,7}^2) \neq 0\end{gathered}$$

The equations then imply $\alpha_{3,7}^2 \neq \alpha_{2,5}^2$. The Jacobi identity is satisfied if and only if

$$\begin{aligned}
\alpha_{4,9} &= 3\alpha_{3,7}^2/(2\alpha_{2,5} + \alpha_{3,7})\\
\alpha_{4,10} &= (7\alpha_{3,8}\alpha_{3,7} - 3\alpha_{4,9}(\alpha_{3,8} + \alpha_{2,6}))/(2\alpha_{2,5} + \alpha_{3,7})\\
\alpha_{4,11} &= (\alpha_{3,9}(8\alpha_{3,7} + \alpha_{5,11} - 6\alpha_{4,9}) + \alpha_{2,7}(2\alpha_{5,11} - 4\alpha_{4,9}) - 3\alpha_{4,10}(\alpha_{2,6} + \alpha_{3,8})\\
&\quad + 4\alpha_{3,8}^2)/(2\alpha_{2,5} + \alpha_{3,7})\\
\alpha_{5,11} &= 3\alpha_{3,7}^3(4\alpha_{2,5} - 7\alpha_{3,7})/(2(2\alpha_{2,5} + \alpha_{3,7})(\alpha_{2,5}^2 - \alpha_{3,7}^2))
\end{aligned}$$

Recall that we may assume $\alpha_{3,7} = 1$ for $\lambda \in \mathcal{A}^1_{11,1}$ and $\alpha_{2,5} = 1$ for $\lambda \in \mathcal{A}^3_{1,11}$. We obtain the following result:

5.3.1. Theorem. *For $\lambda \in \mathcal{A}^1_{11,1}$ it holds $\mu(\mathfrak{g}_\lambda) \leq 12$ iff λ satisfies $\alpha_{2,6} = 0,\ \alpha_{3,9} = 3\alpha_{2,7} + \alpha_{3,8}^2$. For $\lambda \in \mathcal{A}^3_{1,11}$ it holds $\mu(\mathfrak{g}_\lambda) \leq 12$ iff λ satisfies*

$$\begin{aligned}
\alpha_{2,6} &= 2\alpha_{3,8}/(3\alpha_{3,7}(1 - \alpha_{3,7}))\\
\alpha_{3,9} &= (12\alpha_{2,7}\alpha_{3,7} + \alpha_{2,6}^2(\alpha_{3,7} - 1)(3\alpha_{3,7}^2 + 7\alpha_{3,7} - 1))/(4(2 + \alpha_{3,7}))
\end{aligned}$$

For $\lambda \in \mathcal{A}_{11}(\mathbb{C})$ we have $11 \leq \mu(\mathfrak{g}_\lambda) \leq 12$ except for cases described above where we have $13 \leq \mu(\mathfrak{g}_\lambda) \leq 22$.

Proof. The construction of faithful Δ–modules in Theorem 4.4.6 shows that $11 \leq \mu(\mathfrak{g}_\lambda) \leq 12$ for all $\lambda \in \mathcal{A}_{11}(\mathbb{C})$ except for cases described above. The other statements can be proved in the same way as in Theorem 5.2.3. However, the combinatorical types for Δ–modules which have to be studied, are different for $n = 11$. Here we have the following result:

5.3.2. Lemma. *Let $\lambda \in \mathcal{A}_{11}(\mathbb{C})$ and M be a faithful Δ–module for $\mathfrak{g}_\lambda$. Then we may assume that the combinatorical type of M is one of the following:*

(1)	$\emptyset$		
(2)	$\{i\}$	$i = 6, \dots, 11$	
(3)	$\{i, i+1\}$	$i = 6, \dots, 10$	
(4)	$\{i, i+1 \mid 11 + i\}$	$i = 6, \dots, 10$	
(5)	$\{i, i+1, j \mid 11 + i\}$	$i = 6, 7, 8$	$j > i + 2$
(6)	$\{i, j, j+1 \mid 11 + j\}$	$j = 6, \dots, 10$	$i < j - 1$

For each type we have to do the analogous computations as for $n = 10$, only more, to see if there exists a faithful Δ–module of one of the above types. To construct a faithful module of dimension 22 take the following quotient module $\widehat{V}$ of V with vector space basis:

$$\{e_{11}, e_{10},\ e_9,\ e_5^2,\ e_8,\ e_5e_4,\ e_4e_3^2,\ e_7,\ e_5e_3,\ e_5e_2^2,\ e_4^2,\ e_4e_3e_2,\ e_4e_2^3,\ e_3^3,\ e_3^2e_2^2,\ e_6,$$
$$e_5e_2,\ e_4e_3,\ e_4e_2^2,\ e_3^2e_2,\ e_3e_2^3,\ e_2^5,\ e_5,\ e_4e_2,\ e_3^2,\ e_3e_2^2,\ e_2^4,\ e_4,\ e_3e_2,\ e_2^3,\ e_3,\ e_2^2,\ e_2,\ 1\}$$

The module $\widehat{V}$ is a faithful $\mathfrak{g}$–module of dimension 34 with a center Z containing e_{10}. Taking the quotient module of $\widehat{V}$ by a maximal subspace of Z not containing e_{10} we obtain a faithful $\mathfrak{g}$–module of dimension 28. Repeating this procedure finally we obtain a faithful $\mathfrak{g}$–module of dimension 22. $\square$

5.3.3. Remark. The Lie algebras $\mathfrak{a}(r,s,t)$ which were discussed in [**10**],[**18**] are special cases of algebras $\mathfrak{g}_\lambda$ with $\lambda \in \mathcal{A}^3_{1,11}$ and

$$\begin{aligned}\alpha_{2,5} &= 1\\ \alpha_{3,7} &= 1-r\\ \alpha_{3,8} &= -s\\ \alpha_{3,9} &= -t\end{aligned}$$

where the other $\alpha_{k,s}$ are zero, except for $\alpha_{4,9},\dots,\alpha_{5,11}$ which are polynomials in r,s,t given by the Jacobi identity.

5.4. Filiform Lie algebras of dimension 12

If $\lambda \in \mathcal{A}_{12}(\mathbb{C})$ satisfies $2\alpha_{2,5}+\alpha_{3,7}\neq 0$ then $\mathfrak{g}_\lambda$ belongs to one of the three classes $\mathfrak{A}^1_{12},\mathfrak{A}^2_{12},\mathfrak{A}^3_{12}$, see section 3.2. According to Theorems 3.3.1 and 3.3.7 all Lie algebras from the first and second class admit canonical affine structures given by central extensions. Hence they satisfy $\mu(\mathfrak{g})\le n+1=13$. The situation for $n=12$ is different from the cases $n\ge 13$, where algebras from the second class in general do not admit such extensions. It is therefore interesting to ask whether the Milnor conjecture holds for all $\mathfrak{g}\in\mathfrak{F}_{12}(\mathbb{C})$. This is not the case and there exist again counterexamples. Although we will not give a proof here, we formulate the following result:

5.4.1. Theorem. *Let $n=12$ and suppose that $\lambda\in\mathcal{A}^3_n(\mathbb{C})$ satisfies*

$$2\alpha_{2,5}^2+\alpha_{2,6}\alpha_{6,12}\neq 0$$

Then $\mu(\mathfrak{g}_\lambda)\ge n+2$ and $\mathfrak{g}_\lambda$ does not admit an affine structure.

The proof works as before in dimension 10 and 11. One has to reduce the combinatorical types for possible faithful Δ–modules and then solve the polynomial equations. This requires really a lot of computations, so that we will omit these here. If $2\alpha_{2,5}^2+\alpha_{2,6}\alpha_{6,12}=0$ and another condition holds, then we have constructed faithful Δ–modules of type $\{11,12\mid 23\}$.

CHAPTER 6

Deutsche Zusammenfassung

Die vorliegende Arbeit ist affinen Strukturen auf Lie Algebren und Darstellungen nilpotenter Lie Algebren gewidmet. Der Ursprung affiner Strukturen liegt in den linksinvarianten affinen Strukturen auf Lie Gruppen. Diese Strukturen spielen eine besondere Rolle für das Studium von Fundamentalgruppen affiner Mannigfaltigkeiten und von affinen kristallographischen Gruppen. Sie ordnen sich ein in die Theorie kompakter Mannigfaltigkeiten mit geometrischer Struktur. Beispiele solcher "geometrischer Mannigfaltigkeiten" sind euklidische, hyperbolische, projektive und nicht zuletzt affine Mannigfaltigkeiten. Die Fundamentalgruppe einer kompakten vollständigen affinen Mannigfaltigkeit ist eine affine kristallographische Gruppe, kurz ACG genannt. Sie ist eine natürliche Verallgemeinerung einer euklidischen kristallographischen Gruppe (ECG), welche eine diskrete Untergruppe der euklidischen Bewegungsgruppe mit kompaktem Quotienten ist. Historisch gesehen haben zuerst Bieberbach und Schönflies um 1911 wichtige Resultate über solche Gruppen bewiesen. Bieberbach bewies, daß jede ECG eine abelsche Untergruppe von endlichem Index besitzt, die aus Paralleltranslationen besteht. Außerdem zeigte er, daß es in jeder Dimension bis auf Isomorphie nur endlich viele ECG's gibt. Das Studium affiner Mannigfaltigkeiten und ACG's wurde insbesondere durch die Arbeiten von Auslander [**3**] und Milnor [**63**] begründet. Ein natürliches Problem ist die Verallgemeinerung der Sätze von Bieberbach. Im allgemeinen bleiben diese Sätze nicht mehr richtig, aber man kann analoge Aussagen formulieren und auch, bis auf einige Ausnahmen, beweisen. Eine prominente Ausnahme allerdings stellt die sogenannte *Auslander-Vermutung* dar, die besagt, daß jede ACG virtuell polyzyklisch ist, also eine polyzyklische Untergruppe von endlichem Index besitzt. Diese Vermutung hat auch Milnor beschäftigt, der gezeigt hatte, daß jede virtuell polyzyklische Gruppe als Fundamentalgruppe einer vollständigen affinen Mannigfaltigkeit vorkommt. Er fragte, ob man die Mannigfaltigkeit kompakt wählen kann, und ob die Fundamentalgruppe einer nicht notwendig kompakten vollständigen affinen Mannigfaltigkeit auch virtuell polyzyklisch sein muß. Die letztere Aussage wäre eine Verallgemeinerung der Auslander-Vermutung. Sie gilt aber nicht, wie Margulis 1983 gezeigt hat. Die Auslander-Vermutung bleibt hingegen weiterhin offen. Milnor betrachtete in diesem Zusammenhang linksinvariante affine Strukturen auf Lie Gruppen. Diese liefern zum einen wichtige Beispiele affiner Mannigfaltigkeiten, denn der Quotient einer solchen Lie Gruppe nach einer diskreten Untergruppe wird zu einer vollständigen Mannigfaltigkeit, zum anderen ist auch eine virtuell polyzyklische Gruppe virtuell in einer zusammenhängenden Lie Gruppe enthalten. Milnor stellte in seiner Arbeit [**64**] von 1977 die folgende Frage:

Läßt jede auflösbare Lie Gruppe eine vollständige affine Struktur zu, oder anders gefragt, läßt jede einfach zusammenhängende auflösbare Lie Gruppe eine einfach transitive Operation zu durch affine Transformationen auf einem $\mathbb{R}^n$?

Zu dieser Zeit waren nur Spezialfälle bekannt, in denen die Antwort positiv ist. Auslander hatte bewiesen, daß umgekehrt jede Lie Gruppe mit vollständiger linksinvarianter affiner Struktur auflösbar sein muß. Viele Mathematiker glaubten, daß Milnors Frage zu bejahen sei. Das Problem wurde als die *Milnor-Vermutung* bekannt. Sie läßt sich auch rein algebraisch formulieren. Dann ist die Frage, ob jede auflösbare Lie Algebra eine gewisse algebraische Struktur, die wir affin nennen werden, zuläßt. In der Folgezeit erschienen viele Artikel, die eine positive Antwort zu beweisen suchten. Tatsächlich wurden mehrere sogenannte Beweise veröffentlicht. Seit 1993 sind allerdings Gegenbeispiele in Dimension 11 von Benoist, von Grunewald und dem Autor selbst bekannt. In dieser Arbeit geben wir in systematischer Weise neue Gegenbeispiele auf dem Lie Algebra Niveau mit kürzeren Beweisen:

THEOREM. *Es gibt filiform nilpotente Lie Algebren der Dimension* $10 \leq n \leq 12$*, die keine affine Struktur zulassen. Andererseits besitzen alle filiformen Lie Algebren der Dimension* $n \leq 9$ *eine affine Struktur.*

Eine Lie Algebra $\mathfrak{g}$ der Dimension n über einem Körper K heißt filiform nilpotent, wenn sie nilpotent der Stufe $p = n - 1$ ist, also $\mathfrak{g}^p = 0$ und $\mathfrak{g}^{p-1} \neq 0$ gilt. Hierbei ist $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ für $k \geq 1$. Das Prinzip der Gegenbeispiele beruht darauf, daß wir filiforme Algebren der Dimension n bestimmen, die keinen treuen Modul der Dimension $n+1$ besitzen. Da jede Lie Algebra einer Lie Gruppe mit linksinvarianter affiner Struktur einen solchen Modul aber besitzen muß, erhalten wir Gegenbeispiele zur Milnorschen Vermutung. Es ist aber keineswegs klar, wie man solche Lie Algebren finden kann.

Die Arbeit ist wie folgt aufgebaut. In Kapitel 1 geben wir einen Überblick über den Hintergrund der Milnor-Vermutung. Wir werden erklären wie die Milnor-Vermutung rein algebraisch formuliert werden kann. Dann leiten wir Folgerungen aus unseren Gegenbeispielen für die Darstellungen von Lie Algebren und für endlich erzeugte nilpotente Gruppen ab.

In Kapitel 2 tragen wir alle nötigen algebraischen Voraussetzungen zusammen, die zum Studium der Milnor-Vermutung gebraucht werden. Wir behandeln Lie Algebra Kohomologie und Deformationstheorie von Lie Algebren, die wir anwenden, um die Existenz gewisser adaptierter Basen für filiforme Lie Algebren zu zeigen.

In Kapitel 3 beweisen wir notwendige und hinreichende Kriterien für die Existenz affiner Strukturen auf Lie Gruppen und Lie Algebren. Unter anderem beweisen wir folgende Kriterien:

THEOREM. *Sei* $\mathfrak{g}$ *eine filiforme Lie Algebra, die eine Erweiterung*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

besitzt mit einer Lie Algebra $\mathfrak{h}$ *und deren Zentrum* $\mathfrak{z}(\mathfrak{h})$*. Dann läßt* $\mathfrak{g}$ *eine affine Struktur zu.*

THEOREM. *Sei* $\mathfrak{g}$ *eine filiforme Lie Algebra über einem Körper* K*, so daß es eine affine Kohomologieklasse* $[\omega] \in H^2(\mathfrak{g}, K)$ *gibt. Dann läßt* $\mathfrak{g}$ *eine affine Struktur zu.*

Hierbei heißt ein 2–Kozykel $\omega : \mathfrak{g} \wedge \mathfrak{g} \to K$ *affin*, wenn er auf $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}$ nicht verschwindet. In diesem Fall haben alle Elemente der Klasse $[\omega] \in H^2(\mathfrak{g}, K)$ diese Eigenschaft und die Klasse heißt dann affin. Die Umkehrung der obigen Theoreme gilt im allgemeinen allerdings nicht. Um diese Kriterien anwenden zu können, bestimmen wir für alle filiformen

Lie Algebren der Dimension $n \leq 11$ explizit die Kohomologiegruppen $H^2(\mathfrak{g}, K)$. Alle expliziten Rechnungen sind hier, und in der ganzen Arbeit, mit dem Computeralgebra System Reduce ausgeführt und überprüft worden.

Diese Daten sind auch für das Studium der Betti Zahlen nilpotenter Lie Algebren nützlich. Im Studium affiner Strukturen auf Lie Algebren treten neue Phänomene in höheren Dimensionen auf, nämlich für $n \geq 12$. Wir studieren die filiformen Lie Algebren $\mathfrak{g}$ der Dimension $n \geq 12$ mit den folgenden Eigenschaften: $\mathfrak{g}$ enthält keinen 1–kodimensionalen Teilraum $U \supseteq \mathfrak{g}^1$ mit $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$, und $\mathfrak{g}^{(n-4)/2}$ ist abelsch, sofern n gerade ist. Dabei ist $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ und $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$. Diese Algebren zerfallen in natürlicher Weise in zwei verschiedene Klassen, nämlich je nach dem, ob $[\mathfrak{g}^1, \mathfrak{g}^1] \subseteq \mathfrak{g}^6$ gilt oder nicht. Wir bezeichen diese beiden Klassen mit $\mathfrak{A}_n^1(K)$ and $\mathfrak{A}_n^2(K)$. Für die erste Klasse gilt folgende Erweiterungseingenschaft:

Theorem. *Sei* $\mathfrak{g} \in \mathfrak{A}_n^1(K)$, $n \geq 12$. *Dann hat* $\mathfrak{g}$ *eine Erweiterung*

$$0 \to \mathfrak{z}(\mathfrak{h}) \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

mit einer Lie Algebra $\mathfrak{h} \in \mathfrak{A}_{n+1}^1(K)$. *Damit besitzt* $\mathfrak{g}$ *eine affine Structur.*

Das Theorem gilt auch für $\mathfrak{g} \in \mathfrak{A}_{12}^2(K)$, aber im allgemeinen nicht mehr für $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$. Hier hat $H^2(\mathfrak{g}, K)$ entweder die Dimension 2 oder 3. Wenn die Dimension gleich 2 ist, kann es keine affine Kohomologieklasse geben. Komplementär zu diesen Ergebnissen, also für $n \leq 11$, studieren wir die Existenz affiner Strukturen auf allen filiformen Lie Algebren $\mathfrak{g} \in \mathfrak{F}_n(K)$. Dazu wenden wir verschiedenste Konstruktionsprinzipien für affine Strukturen an.

In Kapitel 4 studieren wir ein sehr interessantes Problem über den *Satz von Ado*, welches direkt im Zusammenhang mit der Milnor Vermutung auftaucht. Für eine endlichdimensionale Lie Algebra $\mathfrak{g}$ sei $\mu(\mathfrak{g})$ die *minimale* Dimension eines treuen $\mathfrak{g}$–Moduls. Das ist eine Invariante von $\mathfrak{g}$, die endlich ist nach dem Satz von Ado, und die man nicht so leicht bestimmen kann, besonders nicht für auflösbare und nilpotente Lie Algebren. Man möchte obere Schranken für $\mathfrak{g}$ finden, insbesondere eine lineare Schranke in der Dimension von $\mathfrak{g}$. Falls $\mathfrak{g}$ ein triviales Zentrum hat, gilt $\mu(\mathfrak{g}) \leq \dim \mathfrak{g}$. Wenn $\mathfrak{g}$ eine affine Struktur zuläßt, dann folgt $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$. Im allgemeinen ist nicht bekannt, ob $\mu(\mathfrak{g})$ polynomial in der Dimension von $\mathfrak{g}$ wächst oder nicht. Nachdem Birkhoff schon 1937 eine Schranke für die minimale Dimension treuer Moduln nilpotenter Lie Algebren angegeben hatte, verbesserte Reed diese Schranke 1969. Er zeigte $\mu(\mathfrak{g}) < 1 + n^n$ für nilpotente Lie Algebren der Dimension n. Dabei benutzte auch er die universelle Einhüllende von $\mathfrak{g}$ wie Birkhoff. Wir können diese Schranke verbessern:

Theorem. *Es gilt* $\mu(\mathfrak{g}) < \frac{3}{\sqrt{n}} 2^n$ *für eine nilpotente Lie Algebra der Dimension* n.

Die für affine Strukturen relevante Schranke $\mu(\mathfrak{g}) \leq n + 1$ ist allerdings wesentlich schärfer. Wir bestimmen später alle filiformen Lie Algebren der Dimension $n \leq 11$ über $\mathbb{C}$, die diese Schranke erfüllen. Für spezielle Klassen von Lie Algebren können wir $\mu(\mathfrak{g})$ auch explizit ausrechnen. Die Ergebnisse sind selbst in einfachen Fällen nicht offensichtlich. Betrachtet man etwa abelsche Lie Algebren, so erhalten wir:

Theorem. *Sei* $\mathfrak{g}$ *eine abelsche Lie Algebra der Dimension* n *über einem beliebigen Körper* K. *Dann gilt* $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1} \rceil$.

Hierbei ist $\lceil x \rceil$ die kleinste ganze Zahl, die größer oder gleich x ist.

In Kapitel 5 kommen wir dann zu den Gegenbeispielen für $n \leq 11$. Wir stellen auch neue Gegenbeispiele in Dimension 12 vor, nämlich gewisse filiforme Lie Algebren $\mathfrak{g}$ mit den folgenden Eigenschaften: $\mathfrak{g}$ enthält keinen 1–kodimensionalen Teilraum $U \supseteq \mathfrak{g}^1$ mit $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$, und $\mathfrak{g}^4$ is nicht abelsch.

Der Beweis beruht auf der expliziten Klassifikation aller treuen Δ–Moduln. Diese Methode ist allerdings ungeeignet, das Problem in größerer Allgemeinheit zu studieren. Es muß offen bleiben, wie man die Lie Algebren mit $\mu(\mathfrak{g}) \geq \dim \mathfrak{g} + 2$ bestimmen kann. Nach den Ergebnissen meiner Arbeit denke ich, daß die folgende Frage, die immerhin für $n = 13$ wahr ist, interessant ist:

Offenes Problem. *Sei $\mathfrak{g} \in \mathfrak{A}_n^2(K)$, $n \geq 13$. Gilt $\mu(\mathfrak{g}) \geq n + 2$ dann und nur dann, wenn es keine affine Kohomologieklasse $[\omega]$ in $H^2(\mathfrak{g}, K)$ gibt ?*

Bibliography

[1] H. Abels, G. A. Margulis, G. A. Soifer: *Properly discontinuous groups of affine transformations with orthogonal linear part.* C. R. Acad. Sci. Paris **324** (1997), 253–258.

[2] T. M. Apostol: *Introduction to Analytic Number Theory* (1998), 5th Printing, Springer-Verlag, New York, Heidelberg, Berlin.

[3] L. Auslander: *The structure of complete locally affine manifolds.* Topology **3** (1964), 131–139.

[4] L. Auslander: *Simply transitive groups of affine motions.* Am. J. of Math. **99** (1977), 809–826.

[5] L. Auslander, F. E. A. Johnson: *On a conjecture of C. T. C. Wall.* J. London Math. Soc. **14** (1976), 331–332.

[6] G. F. Armstrong, G. Cairns, B. Jessup: *Explicit Betti numbers for a family of nilpotent Lie algebras.* Proc. Amer. Math. Soc. **125** (1997), 381–385.

[7] G. F. Armstrong, S. Sigg. *On the cohomology of a class of nilpotent Lie algebras.* Bull. Austral. Math. Soc. **54** (1996), 517–527.

[8] O. Baues: *Left-symmetric algebras for* $\mathfrak{gl}(n)$. Trans. Amer. Math. Soc. (1998), to appear.

[9] Y. Benoist: *Nilvariétiés Projectives.* Comment. Math. Helvetici **69** (1994), 447–473.

[10] Y. Benoist: *Une nilvariété non affine.* J. Diff. Geom. **41** (1995), 21–52.

[11] J. P. Benzécri: *Variétés localement affines.* Thèse, Princeton Univ. , Princeton, N. J. (1955).

[12] S. Berman, Y. Krylyuk: *Universal central extensions of twisted and untwisted Lie algebras extended over commutative rings.* J. Algebra **173**, 301–347.

[13] G. Birkhoff: *Representability of Lie algebras and Lie groups by matrices.* Ann. Math. **38** (1937), 562–532.

[14] N. Bourbaki: *Elements of Mathematics. Lie Groups and Lie algebras.* Chapter 1–3, Springer-Verlag (1989).

[15] N. Boyom: *Sur les structures affines homotopes à zéro des groupes de Lie.* J. Diff. Geom. **31** (1990), 859–911.

[16] L. Boza, F. J. Echarte, J. Núnez: *Classification of complex filiform Lie algebras of dimension* 10. Algebras, Groups and Geometries **11** (1994), 253–276.

[17] F. Bratzlavsky: *Classification des algèbres de Lie nilpotentes de dimension* n*, de classe* $n-1$*, dont l'idéal dérivé est commutatif.* Bull. de la classe des sciences Bruxelles **60** (1974), 858–865.

[18] D. Burde, F. Grunewald: *Modules for certain Lie algebras of maximal class.* J. Pure Appl. Algebra **99** (1995), 239-254.

[19] D. Burde: *Left-symmetric structures on simple modular Lie algebras.* J. Algebra **169** (1994), 112–138.

[20] D. Burde: *Left-invariant affine structures on reductive Lie groups.* J. Algebra **181** (1996), 884–902.

[21] D. Burde: *Affine structures on nilmanifolds.* Int. J. of Math. **7** (1996), 599-616.

[22] D. Burde: *Simple left-symmetric algebras with solvable Lie algebra.* Manuscripta math. **95** (1998), 397-411.

[23] D. Burde: *A refinement of Ado's Theorem.* Archiv Math. **70** (1998), 118-127.

[24] G. Cairns, B. Jessup, J. Pitkethly: *On the Betti numbers of nilpotent Lie algebras of small dimension.* Birkhäuser, Prog. Math. **145**, (1997), 19–31.

[25] G. Cairns, B. Jessup: *New bounds on the Betti numbers of nilpotent Lie algebras.* Comm. Algebra **25** (1997), 415–430.

[26] R. Carles: *Sur certaines classes d'algèbres de Lie rigides.* Math. Ann. **272** (1985), 477-488.

[27] R. Carles, Y. Diakité: *Sur les variétés d'algèbres de Lie de dimension* ≤ 7. J. Algebra **91** (1984), 53–63.

[28] Y. Carriére: *Autour de la conjecture de L. Markus sur les variétés affines.* Invent. Math. **95** (1989), 615–627.

[29] Y. Carriére, F. Dal'bo, G. Meigniez: *Inexistence de structures affines sur les fibres de Seifert.* Math. Ann. **296** (1993), 743–753.

[30] C. Deninger, W. Singhof: *On the cohomology of nilpotent Lie algebras.* Bull. Soc. Math. France **116** (1988), 3–14.

[31] K. Dekimpe, M. Hartl: *Affine structures on* 4*–step nilpotent Lie algebras.* J. Pure Appl. Math. **129** (1998), 123–134.

[32] K. Dekimpe, W. Malfait: *Affine structures on a class of virtually nilpotent groups.* Topology Appl. **73** (1996), 97–119.

[33] J. Dixmier: *Cohomologie des algèbres de Lie nilpotentes.* Acta Sci. Math. Szeged **16** (1955), 246–250.

[34] J. Dixmier, W. G. Lister: *Derivations of nilpotent Lie algebras.* Proc. Amer. Math. Soc. **8** (1957), 155–158.

[35] A. Fialowski, D. Fuchs: *Construction of miniversal deformations of Lie algebras.* To appear in J. Funct. Anal. (1999).

[36] D. Fried, W. Goldman: *Three-dimensional affine crystallographic groups.* Adv. Math. **47** (1983), 1–49.

[37] D. Fried, W. Goldmann, M. Hirsch: *Affine manifolds with nilpotent holonomy.* Comm. Math. Helv. **56** (1981), 487–523.

[38] M. Gerstenhaber: *On the deformation of rings and algebras.* Ann. of Math. **79** (1964), 59–104.

[39] W. Goldman, M. Hirsch: *Affine manifolds and orbits of algebraic groups.* Trans. Amer. Math. Soc. **295** (1986), 175–198.

[40] J. Gómez, A. Jimenéz-Merchán, Y. Khakimdjanov: *Low-dimensional filiform Lie algebras.* J. Pure Appl. Algebra **130** (1998), 133–158.

[41] W. Goldman, Y. Kamishima: *The fundamental group of a compact flat Lorentz space form is virtually polycyclic.* J. Diff. Geom. **19** (1984), 233–240.

[42] M. Goze, Y. B. Khakimdjanov: *Sur les algèbres de Lie nilpotentes admettant un tore de dérivations.* Manuscripta Math. **84** (1994), 115–224.

[43] F. Grunewald, G. Margulis: *Transitive and quasitransitive actions of affine groups preserving a generalized Lorentz-structure.* J. Geom. Phys. **5**, No.4 (1988), 493–531.

[44] F. Grunewald, D. Segal: *On affine crystallographic groups.* J. Diff. Geom. **40** (1994), 563–594.

[45] Y. B. Khakimdjanov: *Characteristically nilpotent Lie algebras.* Math. USSR **70** (1991), 65–78.

[46] Y. B. Khakimdjanov: *Variètè des lois d'algebres de Lie nilpotentes.* Geom. Dedicata **40** (1991), 269–295.

[47] S. Halperin: *Le complexe de Koszul en algèbre et topologie.* Ann. l'Inst. Fourier **37** (1987), 77–97.

[48] J. Helmstetter: *Radical d'une algèbre symétrique a gauche.* Ann. Inst. Fourier **29** (1979), 17–35.

[49] M. W. Hirsch, W. P. Thurston: *Foliated bundles, invariant measures and flat manifolds.* Ann. of Math. **101** (1975), 369–390.

[50] N. Jacobson: *A note on automorphisms and derivations of Lie algebras.* Proc. Amer. Math. Soc. **6** (1955), 281–283.

[51] N. Jacobson: *Schur's theorem on commutative matrices.* Bull. Amer. Math. Soc. **50** (1944), 431–436.

[52] H. Kim, H. Lee: *The Euler characteristic of a certain class of projectively flat manifolds.* Topology and its Appl. **40** (1991), 195–201.

[53] H. Kim: *Complete left-invariant affine structures on nilpotent Lie groups.* J. Diff. Geom. **24** (1986), 373–394.

[54] A. W. Knapp: *Lie groups, Lie algebras, and cohomology.* Math. Notes **34** (1988), Princeton University Press.

[55] B. Kostant, D. Sullivan: *The Euler characteristic of an affine space form is zero.* Bull. Amer. Math. Soc. **81** (1975), 937–938.

[56] A. A. Kirillov, Y. A. Neretin: *The variety A_n of n–dimensional Lie algebra structures.* Amer. Math. Soc. Transl. **137** (1987), 21–30.

[57] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry.* Vols. I and II, Wiley-Interscience Publishers, New York and London (1963, 1969).

[58] H. Koch: *Generator and relation ranks for finite- dimensional nilpotent Lie algebras.* Algebra and Logic **16** (1978), 246–253.
[59] N. H. Kuiper: *Sur les surfaces localement affines.* Colloque de Géometrie différentielle, Strasbourg (1953), 79–86.
[60] L. Markus: *Cosmological models in differential geometry.* Notes of the University of Minnesota (1962).
[61] G. A. Margulis: *Complete affine locally flat manifolds with a free fundamental group.* J. Soviet Math. **36** (1987), 129–139.
[62] A. Medina: *Sur quelques algèbres symétriques à gauche dont l'algèbre de Lie sous-jacente est résoluble.* C. R. Acad. Sc. **286** (1978), 173–176.
[63] J. Milnor: *On the existence of a connection with curvature zero.* Comment. Math. Helvetici **32** (1958), 215–223.
[64] J. Milnor: *On fundamental groups of complete affinely flat manifolds.* Advances in Math. **25** (1977), 178–187.
[65] A. Mizuhara: *On the radical of a left-symmetric algebra.* Tensor N. S. **36** (1982), 300-302.
[66] T. Nagano, K. Yagi: *The affine structures on the real two torus.* Osaka J. Math. **11** (1974), 181–210.
[67] A. Nijenhuis, R. W. Richardson: *Deformations of Lie algebra structures.* J. Math. Mech. **17** (1967), 89–105.
[68] M. Nisse: *Structure affine des infranilvariétés et infrasolvariétés.* C. R. Acad. Sci. Paris **310** (1990), 667–670.
[69] G. Rauch: *Effacement et deformation*, Ann. Inst. Fourier **22** (1972), 239–269.
[70] B. E. Reed: *Representations of solvable Lie algebras.* Michigan Math. J. **16** (1969), 227-233.
[71] L. J. Santharoubane: *Cohomology of Heisenberg Lie algebras.* Proc. Amer. Math. Soc. **87** (1983), 23–28.
[72] J. Scheuneman: *Affine structures on three-step nilpotent Lie algebras.* Proc. Amer. Math. Soc. **46** (1974), 451–454.
[73] I. Schur: *Zur Theorie vertauschbarer Matrizen.* J. Reine Angew. Mathematik **130** (1905), 66–76.
[74] C. Seeley: 7*–dimensional nilpotent Lie algebras.* Trans. Amer. Math. Soc. **335** (1993), 479–496.
[75] D. Segal: *The structure of complete left-symmetric algebras.* Math. Ann. **293** (1992), 569–578.
[76] J. Smillie: *An obstruction to the existence of affine structures.* Invent. Math. **64** (1981), 411–415.
[77] T. Springer: *Aktionen reduktiver Gruppen auf Varietäten.* Algebraic Transformation Groups and Invariant Theory, Birkhäuser Verlag (1989), 4–39.
[78] W. P. Thurston: *Three-dimensional Geometry and Topology Vol. 1.* Princeton Mathematical Series **35**, Princeton University Press (1997).
[79] M. Vergne: *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes.* Bull. Soc. Math. France **98** (1970), 81-116.
[80] E. B. Vinberg: *Convex homogeneous cones.* Transl. Moscow Math. Soc. **12** (1963), 340–403.
[81] J. A. Wolf: *Spaces of constant curvature.* McGraw-Hill Book company (1967).