Classification of orbit closures of 4–dimensional complex Lie algebras

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Let $\mathcal{L}_{\rm n}(\mathbb{C})$ be the variety of complex n–dimensional Lie algebras. The group $\rm GL_n(\mathbb{C})$ acts on it via change of basis. An orbit $\rm O(\mu)$ under this action consists of all structures isomorphic to μ . The aim of this paper is to give a complete classification of orbit closures of 4-dimensional Lie algebras, i.e., determining all $\mu \in \overline{O(\lambda)}$ where $\lambda \in \mathcal{L}_4(\mathbb{C})$. Starting with a classification of complex Lie algebras of dimension $\rm n \leq 4$, we study the behaviour of several Lie algebra invariants under degeneration, i.e. under transition to the orbit closure. As a corollary, we will show that all degenerations in $\mathcal{L}_3(\mathbb{C})$ can be realized via a one–parameter subgroup, but this is not the case in $\mathcal{L}_4(\mathbb{C})$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra of dimension n over a field K . Then $\mathfrak{g}$ determines a multiplication table relative to each basis $\{e_1,\dots,e_n\}$. If $[e_i,e_j]=\sum_{k=1}^n \gamma_{i,j}^k e_k$, then $(\gamma_{i,j}^k)\in K^{n^3}$ is called a structure for $\mathfrak{g}$ and the $\gamma_{i,j}^k$ the structure constants of $\mathfrak{g}$. The elements of $\mathcal{L}_n(K)$ are exactly the Lie algebra structures. They form an affine algebraic variety and the group $\mathbf{GL}_n(K)$ acts on $\mathcal{L}_n(K)$ by $(g*\mu)(x,y)=g(\mu(g^{-1}(x),g^{-1}(y)))$. The orbits under this action are the isomorphism classes. We say that λ *degenerates* to μ or μ is a *degeneration* of λ , if μ is in the Zariski closure of the orbit of λ . We denote this by $\mu\in\overline{O(\lambda)}$ or $\lambda \to_{\rm deg} \mu$. The degeneration is nontrivial if μ lies in the boundary of $O(\lambda)$. The classification of orbit closures of a given Lie algebra in general is not known. All the orbit closures of a given dimension have been determined only in low dimensions for nilpotent Lie algebras [GRH], [SEE]. Special kinds of degenerations, namely *contractions*, have been studied by physicists [LEV]. It is useful to study degenerations which can be realized via one–parameter subgroups. It has been asked whether every degeneration in $\mathcal{L}_n$ can be realized via a 1–PSG. This turns out to be true for $n\leq 3$ but does not hold for $n=4$ [STE]. In the case of nilpotent Lie algebras, however, it is true for all $n<7$. Nevertheless it does not hold in general: recently the first author discovered counterexamples for any dimension $n\geq 7$ [BUR].
In this paper we classify all possible degenerations of Lie algebra structures in $\mathcal{L}_4(\mathbb{C})$, i.e. we determine the Zariski closure of all Lie algebras $\lambda\in\mathcal{L}_4(\mathbb{C})$.

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2. Preliminaries

A point in $\mathcal{L}_n(K)$ is a Lie algebra structure which can be identified with the bilinear skew–symmetric mapping $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ defining the Lie bracket on $\mathfrak{g}$. Since the Jacobi identity and the antisymmetry are defined by polynomial conditions, i.e. by $(n^3 - n)/6$ algebraic equations, $\mathcal{L}_n(K)$ is an affine algebraic subvariety of $\mathrm{Hom}(\Lambda^2 V, V)$. $\mathbf{GL}_n(K)$ acts on $\mathcal{L}_n(K)$ via change of basis, i.e. by $(g * \mu)(x, y) = g(\mu(g^{-1}(x), g^{-1}(y)))$. An orbit $O(\mu)$ under this action consists of all structures in a single isomorphism class. We recall the following definitions:

Definition 1. A Lie algebra λ is said to *degenerate* to another Lie algebra μ , if μ is represented by a structure which lies in the Zariski closure of the $\mathbf{GL}_n(K)$–orbit of a structure which represents λ . In this case the entire orbit $O(\mu)$ lies in the closure of $O(\lambda)$. We denote this by $\lambda \to_{\mathrm{deg}} \mu$.

Definition 2. A degeneration $\lambda \to_{\mathrm{deg}} \mu$ is called a *one–parameter subgroup degeneration* (1–PSG) if it can be realized by a group homomorphism $g : K^* \to \mathbf{GL}_n(K),\ t \mapsto g_t$ such that $\mu = \lim_{t \to 0} g_t * \lambda$.

Example 1. Any n–dimensional Lie algebra λ degenerates to the abelian Lie algebra K^n : Let $g_t = t^{-1} E_n$, where E_n is the identity matrix. Then we have $(g_t * \lambda)(x, y) = t^{-1}\lambda(tx, ty) = t\lambda(x, y)$, hence $\lim_{t \to 0} g_t * \lambda = K^n$.

Remarks:

(1) The notion of a 1–PSG degeneration is independent of the choice of a basis.
(2) For $K = \mathbb{C}$ it is known that the usual analytic topology on $\mathbb{C}^{n^3}$ leads to the same degenerations as does the Zariski topology. Therefore the following condition will imply that $\lambda \to_{\mathrm{deg}} \mu :\quad \exists\, g_t \in \mathbf{GL}_n(\mathbb{C}(t))$ such that $\lim_{t \to 0} g_t * \lambda = \mu$. Here $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.
(3) The notion of degeneration is transitive: If $\lambda \to_{\mathrm{deg}} \mu$, $\mu \to_{\mathrm{deg}} \nu$, then $\lambda \to_{\mathrm{deg}} \nu$.
(4) Let $Z(\lambda)$ denote the center of the Lie algebra λ , $[\lambda, \lambda]$ the commutator subalgebra of λ, $\lambda^{(i)}$ the i^{th} derived commutator ideal, $\mathrm{ab}(\lambda)$ the dimension of a maximal abelian subalgebra of λ and $\mathrm{rank}(\lambda)$ the rank of λ.

The following Lemma is well known and will be used for determining the orbit closures in $\mathcal{L}_4(\mathbb{C})$.

Lemma 1. *Let $\lambda \to_{\mathrm{deg}} \mu$ be a nontrivial degeneration. Then the following inequalities hold:*

(1) $\dim O(\lambda) > \dim O(\mu)$
(2) $\dim \mathrm{Der}(\lambda) < \dim \mathrm{Der}(\mu)$
(3) $\dim [\lambda, \lambda] \geq \dim [\mu, \mu]$
(4) $\dim Z(\lambda) \leq \dim Z(\mu)$
(5) $\mathrm{ab}(\lambda) \leq \mathrm{ab}(\mu)$

(6) *If* λ *is solvable of step* k *, then* μ *is solvable of step* $\leq k$ *. The same holds for nilpotent Lie algebras. In that case,* $\dim \lambda^{(i)} \geq \dim \mu^{(i)}$ *where* $\lambda^{(1)} = \lambda, \lambda^{(i+1)} = [\lambda, \lambda^{(i)}]$ *.*

The proof (which may be found in [STE],[SEE],[GRH]) uses the following important fact: *Let* B *be a Borel subgroup of* $\mathbf{GL}_n(K)$ *and* G *be a complex reductive algebraic group acting rationally on some algebraic set* X *. Let* B *be a Borel subgroup of* G *. Then* $\overline{G * x} = G * \overline{(B * x)}$ *for all* $x \in X$ *.* [GRH]

The statements (1) and (2) are equivalent since we have $\dim O(\lambda) = (\dim \lambda)^2 - \dim \operatorname{Der}(\lambda)$. For the subvariety $\mathcal{N}_n(K)$ of $\mathcal{L}_n(K)$ consisting of nilpotent Lie algebras and $\lambda, \mu \in \mathcal{N}_n$ one can use the following fact: If $\lambda \to_{\deg} \mu$ and λ lies in a B–stable closed subset $\mathcal{R} \subset \mathcal{N}_n$, then μ must also be represented by a structure in $\mathcal{R}$.

Two results on 1–PSG's are the following Propositions:

Proposition 1. *If* $\lambda \to_{\deg} \mu$ *via a* 1*–PSG then* μ *is the associated* $\mathbb{Z}$*–graded Lie algebra given by the filtration on* λ *induced by* g_t *. Conversely, if* μ *is the associated graded Lie algebra given by some filtration on* λ *then* $\lambda \to_{\deg} \mu$ *via a* 1*–PSG.*

Proposition 2. *Every degeneration* $\lambda \to_{\deg} \mu$ *of nilpotent Lie algebras in* $\mathcal{N}_n(\mathbb{C})$ *can be obtained via a one–parameter subgroup for* $n < 7$ *, but not for* $n \geq 7$ *. Every degeneration in* $\mathcal{L}_3(\mathbb{C})$ *can be realized via a* 1*–PSG, but this is not the case in* $\mathcal{L}_4(\mathbb{C})$ *.*

Proposition 1 is proved in [GRH]; the result for $\mathcal{N}_n(\mathbb{C})$ in Proposition 2 is proved in [BUR]. The last part will follow from our classification results in section 4 .

3. Classification of complex Lie algebras up to dimension 4

In contrast to the semisimple case, the classification of solvable Lie algebras has not been achieved in general. Most results concern solvable Lie algebras of dimension $n < 7$. The classification for $n \leq 3$ is well known [JAC]; in dimension 4 there exists a classification over a perfect field K [PAZ]. Over the field of real numbers the classification has been obtained up to dimension 6 , see [TUR] and the references cited therein. The complex and real nilpotent Lie algebras of dimension $n = 7$ have been classified in [ROM]. For our purpose we need a list in dimension 4 over $\mathbb{C}$, such that every Lie algebra is isomorphic to exactly one algebra of the list. The lists for $K = \mathbb{C}$ we found in the literature unfortunately contain some errors. The list in [PAZ] is incomplete over $\mathbb{C}$ because of some surplus parameter restrictions. The list in [MUB] is over the field of real numbers and one has to find the isomorphisms over $\mathbb{C}$. The result of checking the details is the following:

Lemma 2. *Every complex* 3*–dimensional Lie algebra is isomorphic to one and only one Lie algebra of the following list:*

$\mathfrak{g}$	Lie brackets
$\mathbb{C}^3$	–
$\mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_1$
$\mathfrak{r}_3(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3$, $\lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1$
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$

We have $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$ iff $\mu = \lambda^{-1}$, or $\mu = \lambda$. Hence, for $|\lambda| = 1$, we have to parametrize $\lambda = e^{i\theta}$ with $\theta \in [0, \pi]$.

Lemma 3. *Every complex 4–dimensional Lie algebra is isomorphic to one and only one Lie algebra of the following list:*

$\mathfrak{g}$	Lie brackets
$\mathbb{C}^4$	–
$\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$	$[e_1, e_2] = e_1$
$\mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3$, $\lambda \in \mathbb{C}, 0 < \lvert\lambda\rvert \leq 1$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_1, [e_3, e_4] = e_3$
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$
$\mathfrak{n}_4(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$\mathfrak{g}_1(\alpha)$	$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = \alpha e_4$, $\alpha \in \mathbb{C}^*$
$\mathfrak{g}_2(\alpha, \beta)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha e_2 - \beta e_3 + e_4$, $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$ or $\alpha, \beta = 0$
$\mathfrak{g}_3(\alpha)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha(e_2 + e_3)$, $\alpha \in \mathbb{C}^*$
$\mathfrak{g}_4$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_2$
$\mathfrak{g}_5$	$[e_1, e_2] = \frac{1}{3}e_2 + e_3, [e_1, e_3] = \frac{1}{3}e_3, [e_1, e_4] = \frac{1}{3}e_4$
$\mathfrak{g}_6$	$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = 2e_4, [e_2, e_3] = e_4$
$\mathfrak{g}_7$	$[e_1, e_2] = e_3, [e_1, e_3] = e_2, [e_2, e_3] = e_4$
$\mathfrak{g}_8(\alpha)$	$[e_1, e_2] = e_3, [e_1, e_3] = -\alpha e_2 + e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4$, $\alpha \in \mathbb{C}$

Proof: The classification up to dimension 3 can be found in many books on Lie algebras, see for example [JAC]. Lemma 3 is proved in [STE] and we will only outline the main steps: First, up to isomorphism there is only one complex non-solvable Lie algebra in $\mathcal{L}_4$, namely $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$. Secondly, we take the list of all solvable Lie algebras in dimension 4 given in [PAZ], but drop the parameter restrictions given there. Over the complex numbers we check whether there are isomorphic algebras within each family of parameters or between all other families. That will give our above list. We test our result by comparing it with the list given in [MUB] over $\mathbb{R}$. Each Lie algebra given there is isomorphic over $\mathbb{C}$ to exactly one in our list. Comparing with the list (including parameter-restrictions) in [PAZ] we see that there are two Lie algebras for which the parameter restrictions given in [PAZ] are too strong.

4. Degenerations of Lie algebras in dimension 3 and 4

In this section we will determine the orbit closures of the Lie algebras of Lemma 2 and 3. In order to decide which Lie algebra structures are lying in the boundary of a given orbit $O(\lambda)$ one has to consider several isomorphism invariants which behave well under degeneration. Since the orbits are path-connected we know that $\mu \in \overline{O(\lambda)}$ implies $O(\mu) \subseteq \overline{O(\lambda)}$. We use these invariants to determine first the Lie algebra structures which do *not* belong to the orbit closure of a given orbit $O(\lambda)$: If μ has an invariant which does not satisfy a certain condition with the corresponding invariant of λ, then μ cannot belong to $\overline{O(\lambda)}$. A good example of such an invariant is the dimension of the commutator subalgebra, see Lemma 1. If $\dim[\lambda,\lambda] < \dim[\mu,\mu]$, then μ is not in the orbit closure of λ. The invariance arguments used here are given below. For the Lie algebra structures which cannot be excluded by these arguments we try to construct an appropiate matrix $g_t \in \mathbf{GL}_n(\mathbb{C}(t))$ which realizes the degeneration by $\lambda = \lim_{t\to 0} g_t * \mu$.

Invariance arguments:

(1) Lemma 1.
(2) Let $\mu \in \mathcal{L}_n(\mathbb{C})$ and $\{e_1,\dots,e_n\}$ a basis of μ. If $\operatorname{tr}(\operatorname{ad} e_i) = 0$ for all i, then $\operatorname{tr}(\operatorname{ad} x_i) = 0$ for all $\lambda \in \overline{O(\mu)}$ and each basis $\{x_1,\dots,x_n\}$ of λ.
(3) If $\lambda \in \mathcal{L}_n(\mathbb{C})$ is rigid, i.e., has an open orbit then no other $\mu \in \mathcal{L}_n(\mathbb{C})$ degenerates to λ, i.e. $\lambda \notin \overline{O(\mu)}$ for all $\mu \in \mathcal{L}_n(\mathbb{C})$ with $\mu \neq \lambda$.
(4) Let κ_λ denote the Killing form of $\lambda \in \mathcal{L}_n(\mathbb{C})$. If $\kappa_\lambda = (0)$, then $\kappa_\mu = (0)$ for all $\mu \in \overline{O(\lambda)}$.
(5) Let $\mu = (\gamma_i) \in \mathcal{L}_n(\mathbb{C})$ with structure constants $\gamma_1,\dots,\gamma_r$ and (i,j) be a pair of positive integers such that

$$c_{ij}(\mu) := \frac{\operatorname{tr}(\operatorname{ad} x)^i \cdot \operatorname{tr}(\operatorname{ad} y)^j}{\operatorname{tr}((\operatorname{ad} x)^i \circ (\operatorname{ad} y)^j)}$$

is *independent* of the Lie algebra elements x, y (i.e., the nominator polynomial and the denominator polynomial in the structure constants coincides). Then $c_{ij}(\mu) = c_{ji}(\mu)$

is a quotient of two polynomials in $\mathbb{C}[\gamma_1, \dots, \gamma_r]$. If neither of these polynomials is zero, we call $c_{i,j} \in \mathbb{C}(\gamma_1, \dots, \gamma_r)$ an (i,j) – *invariant* of μ . It is clearly an isomorphism invariant of μ . Suppose $\mu \in \mathcal{L}_n(\mathbb{C})$ has an (i,j) – invariant $c_{ij}(\mu)$. Then all $\lambda \in \overline{O(\mu)}$ have the same (i,j) – invariant.

(6) Let $\mu = (\gamma_i) \in \mathcal{L}_n(\mathbb{C})$ be as in (5) and assume that either $\operatorname{tr}(\operatorname{ad} x)^i \cdot \operatorname{tr}(\operatorname{ad} y)^j = 0$ or $\operatorname{tr}((\operatorname{ad} x)^i \circ (\operatorname{ad} y)^j) = 0 \quad \forall\, x, y \in \mu$ and some pair (i,j) . Then these equations hold for all $\lambda \in \overline{O(\mu)}$.

Example 2. If μ is not nilpotent, the invariant $c_{ij}(\mu)$ defined in (5) exists quite often, e.g., $c_{ij}(\mathfrak{r}_2 \oplus \mathbb{C}^2) = 1$, $c_{ij}(\mathfrak{r}_3 \oplus \mathbb{C}) = 2$, $c_{ij}(\mathfrak{g}_5) = 3$, $c_{ij}(\mathfrak{g}_6) = (2^i+2)(2^j+2)/(2^{i+j}+2)$ and

$$c_{ij}(\mathfrak{r}_{3,\lambda} \oplus \mathbb{C}) = 1 + \frac{\lambda^i + \lambda^j}{\lambda^{i+j}+1}, \quad c_{ij}(\mathfrak{g}_1(\alpha)) = \frac{(\alpha^i+2)(\alpha^j+2)}{\alpha^{i+j}+2}$$

for all i,j . There are also nice invariants for $\mathfrak{g}_2(\alpha,\beta)$, $\mathfrak{g}_3(\alpha)$, $\mathfrak{g}_8(\alpha)$. The algebra $\mathfrak{g}_4$ has an (i,j) – invariant (equal to 3) if and only if $i,j \equiv 0(3)$, and $c_{2i,2j}(\mathfrak{g}_7) = 2$. On the other hand, $\mathfrak{r}_2 \oplus \mathfrak{r}_2$ and $\mathfrak{sl}_2 \oplus \mathbb{C}$ have no (i,j) – invariant.

Proposition 4. *The orbit closures in dimension* 3 *are as follows:*

$\mathfrak{g}$	$\overline{O(\mathfrak{g})}$
$\mathbb{C}^3$	$\mathbb{C}^3$
$\mathfrak{n}_3(\mathbb{C})$	$\mathfrak{n}_3(\mathbb{C}),\ \mathbb{C}^3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C},\ \mathfrak{n}_3(\mathbb{C}),\ \mathbb{C}^3$
$\mathfrak{r}_3(\mathbb{C})$	$\mathfrak{r}_3(\mathbb{C}),\ \mathfrak{r}_{3,1}(\mathbb{C}),\ \mathfrak{n}_3(\mathbb{C}),\ \mathbb{C}^3$
$\mathfrak{r}_{3,\lambda\neq 1}(\mathbb{C})$	$\mathfrak{r}_{3,\lambda\neq 1}(\mathbb{C}),\ \mathfrak{n}_3(\mathbb{C}),\ \mathbb{C}^3$
$\mathfrak{r}_{3,1}(\mathbb{C})$	$\mathfrak{r}_{3,1}(\mathbb{C}),\ \mathbb{C}^3$
$\mathfrak{sl}_2(\mathbb{C})$	$\mathfrak{sl}_2(\mathbb{C}),\ \mathfrak{r}_{3,-1}(\mathbb{C}),\ \mathfrak{n}_3(\mathbb{C}),\ \mathbb{C}^3$

The possible degenerations are summarized in the following diagram:

$$\begin{tikzcd}
 & \mathfrak{r}_{3,\lambda^2\neq 1}(\mathbb{C}) \arrow[d] & \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C} \arrow[dl] \arrow[d] & \\
\mathfrak{sl}_2(\mathbb{C}) \arrow[r] \arrow[dr] & \mathfrak{n}_3(\mathbb{C}) \arrow[r] & \mathbb{C}^3 & \mathfrak{r}_3(\mathbb{C}) \arrow[l] \arrow[dl] \\
 & \mathfrak{r}_{3,-1}(\mathbb{C}) \arrow[u] & \mathfrak{r}_{3,1}(\mathbb{C}) \arrow[u] &
\end{tikzcd}$$

Recall that degeneration is transitive. If $A \to B$ and $B \to C$ is drawn in the diagram, then $A \to C$ can be be drawn also. However, to keep the diagram reasonably simple we have sometimes omitted the arrow $A \to C$.

Proof: We carry out the arguments for the Lie algebra $\mu = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$. All other cases are similar. First, the orbit closure of μ contains μ itself and the abelian Lie algebra $\mathbb{C}^3$, see Example 1 . Since μ has an one–dimensional commutator, all Lie algebras with higher–dimensional commutator cannot be contained in the orbit closure. That excludes $\mathfrak{r}_3(\mathbb{C}), \mathfrak{r}_{3,\lambda}(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})$. The only Lie algebra remaining is $\mathfrak{n}_3(\mathbb{C})$. It is not difficult to see that

$$\lim_{t\to 0} g_t * \mu \cong \mathfrak{n}_3(\mathbb{C}) \ \text{ with } \ g_t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & t^{-1} & 0 \\ t^{-1} & 0 & 1 \end{pmatrix}$$

Hence $\mathfrak{n}_3(\mathbb{C})$ is contained in the orbit closure of μ .

Proposition 5. *The orbit closures in dimension* 4 *are as follows:*

$\mathfrak{g}$	$\overline{O}(\mathfrak{g})$
$\mathbb{C}^4$	$\mathbb{C}^4$
$\mathfrak{n}_3 \oplus \mathbb{C}$	$\mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{n}_4$	$\mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{r}_2 \oplus \mathbb{C}^2$	$\mathfrak{r}_2 \oplus \mathbb{C}^2,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{r}_3 \oplus \mathbb{C}$	$\mathfrak{r}_3 \oplus \mathbb{C},\ \mathfrak{r}_{3,1} \oplus \mathbb{C},\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{r}_{3,\lambda^2\neq 1} \oplus \mathbb{C}$	$\mathfrak{r}_{3,\lambda^2\neq 1} \oplus \mathbb{C},\ \mathfrak{n}_4,\ \ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{r}_{3,1} \oplus \mathbb{C}$	$\mathfrak{r}_{3,1} \oplus \mathbb{C},\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{r}_{3,-1} \oplus \mathbb{C}$	$\mathfrak{r}_{3,-1} \oplus \mathbb{C},\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{r}_2 \oplus \mathfrak{r}_2$	$\mathfrak{r}_2 \oplus \mathfrak{r}_2,\ \mathfrak{g}_8(0),\ \mathfrak{g}_2(0,0),\ \mathfrak{r}_{3,\lambda\neq 0} \oplus \mathbb{C},\ \mathfrak{r}_3 \oplus \mathbb{C}$ $\mathfrak{r}_2 \oplus \mathbb{C}^2,\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_1(1)$	$\mathfrak{g}_1(1),\ \mathbb{C}^4$
$\mathfrak{g}_1(\alpha), \alpha \neq 1$	$\mathfrak{g}_1(\alpha), \alpha \neq 1,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_2(\alpha,\beta), \alpha \neq 0$	$\mathfrak{g}_2(\alpha,\beta), \alpha \neq 0,\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$ $(\alpha,\beta) = (1/27, 1/3) : \mathfrak{g}_5$ $(\alpha,\beta) = (\gamma/(\gamma+2)^3, (2\gamma+1)/(\gamma+2)^2) : \mathfrak{g}_1(\gamma)$
$\mathfrak{g}_2(0,0)$	$\mathfrak{g}_2(0,0),\ \mathfrak{r}_2 \oplus \mathbb{C}^2,\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_3(\alpha)$	$\mathfrak{g}_3(\alpha),\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$ $\alpha = 27/4 :\ \mathfrak{g}_1(-2)$
$\mathfrak{g}_4$	$\mathfrak{g}_4,\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_5$	$\mathfrak{g}_5,\ \mathfrak{g}_1(1),\ \ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_6$	$\mathfrak{g}_6,\ \mathfrak{g}_1(2),\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_7$	$\mathfrak{g}_7,\ \mathfrak{r}_{3,-1} \oplus \mathbb{C},\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_8(0)$	$\mathfrak{g}_8(0),\ \mathfrak{r}_3 \oplus \mathbb{C},\ \mathfrak{r}_{3,1} \oplus \mathbb{C},\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$
$\mathfrak{g}_8(\alpha), \alpha \neq 0$	$\mathfrak{g}_8(\alpha), \alpha \neq 0,\ \mathfrak{g}_2(\alpha/8, (1+\alpha)/4),\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$ $\alpha = 1/4 : \mathfrak{g}_6,\ \mathfrak{g}_1(2)$
$\mathfrak{sl}_2 \oplus \mathbb{C}$	$\mathfrak{sl}_2 \oplus \mathbb{C},\ \mathfrak{g}_7,\ \mathfrak{r}_{3,-1} \oplus \mathbb{C},\ \mathfrak{n}_4,\ \mathfrak{n}_3 \oplus \mathbb{C},\ \mathbb{C}^4$

Proof: The degenerations in the $4-$ dimensional case are very complicated because of the wealth of solvable Lie algebras with parameters. The table shows the result of the classification of orbit closures. For example, the orbit closure of $\mathfrak{g}_3(\alpha)$ contains for all α the algebras $\mathfrak{g}_3(\alpha), \mathfrak{n}_4, \mathfrak{n}_3 \oplus \mathbb{C}$ and $\mathbb{C}^4$. In the special case $\alpha = 27/4$ there is one more algebra contained in the orbit closure: $\mathfrak{g}_1(-2)$.
We will carry out the proof for $\mu = \mathfrak{r}_{3,\lambda} \oplus \mathbb{C}$ where $\lambda \neq 1, -1$ is fixed but arbitrary. The complete proof may be found in [STE]. As before, μ and $\mathbb{C}^4$ are contained in the orbit closure. Since μ has a $2-$dimensional commutator subalgebra we can apply Lemma 1 , part (3) to exclude the following algebras: $\mathfrak{g}_1(\alpha)$, $\mathfrak{g}_3(\alpha)$, $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$, $\mathfrak{g}_7$, $\mathfrak{sl}_2 \oplus \mathbb{C}$, $\mathfrak{g}_2(\alpha,\beta)$ with $(\alpha,\beta) \neq (0,0)$ and $\mathfrak{g}_8(\alpha)$ with $\alpha \neq 0$.
Next we compute that $\dim \mathrm{Der}(\mu) = 6$. By Lemma 1 , part (2) we know that algebras λ with $\dim \mathrm{Der}(\lambda) \leq 6$ except μ itself cannot be contained in the orbit closure of μ . We have $\dim \mathrm{Der}(\mathfrak{r}_2 \oplus \mathfrak{r}_2) = 4$, $\dim \mathrm{Der}(\mathfrak{g}_8(0)) = 5$ and $\dim \mathrm{Der}(\mathfrak{g}_2(0,0)) = \dim \mathrm{Der}(\mathfrak{r}_3 \oplus$

$\mathbb{C}) = 6$. Also $\dim \mathrm{Der}(\mathfrak{r}_{3,\lambda'}) = 6$ for $\lambda' \neq 1$. Hence all algebras $\mathfrak{r}_{3,\lambda'}$ except for $\lambda = \lambda'$ are not contained in the orbit closure of μ . If $\lambda \neq i, -i$, then μ has an $(1,1)$ – invariant

$$c_{11}(\mu) = \frac{(\lambda+1)^2}{\lambda^2+1}$$

The algebra $\mathfrak{r}_{3,1} \oplus \mathbb{C}$ has an $(1,1)$ – invariant $c'_{11} = 2$. By the invariance argument (5) we see that $\mathfrak{r}_{3,1} \oplus \mathbb{C}$ is not contained in the orbit closure of μ . Otherwise $c_{11}(\mu) = 2$, i.e., $\lambda = 1$. This is a contradiction. However, if $\lambda = i, -i$ the argument fails. But then the Killing form of μ is identical to zero in contrast to the Killing form of $\mathfrak{r}_{3,1} \oplus \mathbb{C}$. The algebra $\mathfrak{r}_2 \oplus \mathbb{C}^2$ has an $(1,1)$ – invariant $c'_{11} = 1$. Again this shows that the algebra is not contained in the orbit closure of μ since $c_{11} = 1$ means $\lambda = 0$, a contradiction. For $\lambda = i, -i$ the argument with the Killing form applies.
Finally there are only two algebras left: $\mathfrak{n}_4$ and $\mathfrak{n}_3 \oplus \mathbb{C}$. Both algebras belong to the orbit closure of μ . A computation shows

$$\lim_{t\to 0} \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{t(1-\lambda)} & \frac{1}{t(\lambda-1)} & 0 \\ 0 & \frac{1}{t^2(1-\lambda)} & \frac{1}{t^2\lambda(\lambda-1)} & \frac{-1}{t\lambda} \end{pmatrix} * \mu \cong \mathfrak{n}_4$$

and

$$\lim_{t\to 0} \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & \frac{1}{1-\lambda} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \mu \cong \mathfrak{n}_3 \oplus \mathbb{C}$$

This finishes the proof. □

Remark: It is very difficult to summarize the degenerations in dimension 4 in a diagram. However, degeneration trees starting from a particular algebra illustrate the classification result. Consider the rigid algebras in $\mathcal{L}_4(\mathbb{C})$, namely $\mathfrak{sl}_2 \oplus \mathbb{C}, \mathfrak{r}_2 \oplus \mathfrak{r}_2$. The degeneration tree of $\mathfrak{sl}_2 \oplus \mathbb{C}$ has no branches:

$$\mathfrak{sl}_2 \oplus \mathbb{C} \longrightarrow \mathfrak{g}_7 \longrightarrow \mathfrak{r}_{3,-1} \oplus \mathbb{C} \longrightarrow \mathfrak{n}_4 \longrightarrow \mathfrak{n}_3 \oplus \mathbb{C} \longrightarrow \mathbb{C}^4$$

The degeneration tree of $\mathfrak{r}_2 \oplus \mathfrak{r}_2$ is as follows:

$$\begin{tikzcd}
 & \mathfrak{g}_2(0,0) \arrow[dl] & \mathfrak{r}_2 \oplus \mathfrak{r}_2 \arrow[l] \arrow[r] \arrow[dl] \arrow[dr] & \mathfrak{g}_8(0) \arrow[dr] & \\
\mathfrak{r}_2 \oplus \mathbb{C}^2 \arrow[dr] & \mathfrak{r}_{3,-1} \oplus \mathbb{C} \arrow[dr] & & \mathfrak{r}_{3,\lambda^2\neq 1} \oplus \mathbb{C} \arrow[dl] & \mathfrak{r}_3 \oplus \mathbb{C} \arrow[dl] \\
 & \mathfrak{n}_3 \oplus \mathbb{C} \arrow[dr] & \mathfrak{n}_4 \arrow[l] & \mathfrak{r}_{3,1} \oplus \mathbb{C} \arrow[l] \arrow[dl] & \\
 & & \mathbb{C}^4 & &
\end{tikzcd}$$

Proposition 6. *In* $\mathcal{L}_4(\mathbb{C})$ *we have* $\mathfrak{r}_2 \oplus \mathfrak{r}_2 \to_{\deg} \mathfrak{n}_4$. *This degeneration cannot be realized via a* 1 *–PSG.*

Proof: Let $\mu = \mathfrak{r}_2 \oplus \mathfrak{r}_2$. By Proposition 5 we know that $\mathfrak{n}_4 \in \overline{O(\mu)}$. Indeed, one can easily check that

$$g_t := \begin{pmatrix} 1 & 0 & 0 & 0 \\ t^{-2} & -t^{-3} & 0 & 0 \\ -2t^{-1} & t^{-2} & 1 & 0 \\ -t^{-4} & \frac{1-t^2}{t^5} & -t^{-1} & t^{-3} \end{pmatrix}$$

is a matrix with: $\lim_{t\to 0} g_t * \mu = \mathfrak{n}_4$. Assume that the degeneration can be realized via a 1 –PSG. According to Proposition 2 , $\mathfrak{n}_4$ is the associated graded Lie algebra given by some filtration on μ . Such a filtration is a family of subspaces $(V_i)_{i\in\mathbb{Z}}$ such that $V_k \supset V_l$ for all $k \le l$ and $[V_j, V_k] \subseteq V_{j+k}$ for all $j, k \in \mathbb{Z}$. The associated graded Lie algebra is given by

$$\operatorname{gr}(\mu) = \bigoplus_j V_j/V_{j+1}$$

and Lie brackets $[x + V_{i+1}, y + V_{j+1}] = [xy] + V_{i+j+1}$ for $x \in V_i$ and $y \in V_j$. Since μ is 4 –dimensional, a filtration on μ can have $2, 3$ or 4 terms. First assume that the filtration has length four: $\mu = V_{i+4} \supset V_{i+3} \supset V_{i+2} \supset V_{i+1} \supset V_i = 0$. Let V_{i+j} be generated by $f_1, \dots, f_j$ and let e_1, e_2, e_3, e_4 be the standard basis of μ with $[e_1, e_2] = e_1$ and $[e_3, e_4] = e_3$. Then $f_j = \sum_{i=1}^4 \alpha_{ij} e_i$ with $\alpha_{ij} \in \mathbb{C}$. Let $A := (\alpha_{ij})_{i,j}$. Then $\det(A) \neq 0$ since $\{f_1, f_2, f_3, f_4\}$ is also a basis of μ . By assumption $\operatorname{gr}(\mu)$ is isomorphic to $\mathfrak{n}_4$. In fact, we may assume $\operatorname{gr}(\mu) = \mathfrak{n}_4$, i.e. $[\bar{f}_2, \bar{f}_4] = \bar{f}_1$ and $[\bar{f}_3, \bar{f}_4] = \bar{f}_2$. Rewriting these conditions with respect to the basis $\{e_1, e_2, e_3, e_4\}$ we obtain $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21} = \alpha_{11}\alpha_{41} - \alpha_{21}\alpha_{14} = \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} = \alpha_{11} - \alpha_{12}\alpha_{24} + \alpha_{22}\alpha_{14} = \alpha_{21} = \alpha_{31} = \alpha_{41} = 0$ and $(\alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14} - \alpha_{12})e_1 + \alpha_{22}e_2 + \alpha_{32}e_3 + \alpha_{42}e_4 \in < f_1 >$. This implies $\det(A) = 0$, a contradiction. The other two cases can be treated likewise. □

References

[BUR] D. BURDE, Degenerations of nilpotent Lie algebras, *J. Lie Theory* **9**, No. 1 (1999), 193–202.

[FIH] A. FIALOWSKI, J. O'HALLORAN, A comparison of deformations and orbit closure, *Comm. in Algebra* **18** (1990), 4121–4140.

[GRH] F. GRUNEWALD, J. O'HALLORAN, Varieties of nilpotent Lie algebras of dimension less than six, *J. Algebra* **112** (1988), 315–325.

[JAC] N. JACOBSON, *Lie algebras*, Interscience, New York, 1962.

[LEV] M. LEVY–NAHAS, Deformation and contraction of Lie algebras, *J. Math. Phys.* **8** (1967), 1211–1223.

[MUB] G. M. MUBARAKZJANOV, On solvable Lie algebras (Russian), *Izv. Vyssh. Uchebn. Zaved Mat.* **1** (1963), 114–123.

[PAZ] J. PATERA, H. ZASSENHAUS, Solvable Lie algebras of dimension ≤ 4 over perfect fields, *Linear Algebra and its applications* **142** (1990), 1–17.

[ROM] M. RHOMDANI, Classification of real and complex nilpotent Lie algebras of dimension 7, *Linear and Multilinear Algebra* **24** (1989), 167–189.

[SEE] C. SEELEY, Degenerations of 6–dimensional nilpotent Lie algebras over $\mathbb{C}$, *Comm. Algebra* **18** (1990), 3493–3505.

[STE] C. STEINHOFF, Klassifikation und Degeneration von Lie Algebren, *Diplomarbeit*, Düsseldorf (1997).

[TUR] P. TURKOWSKI, Solvable Lie algebras of dimension six, *J. Math. Phys.* **31** (1990), 1344–1350.