Simple left–symmetric algebras with solvable Lie algebra

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Left–symmetric algebras (LSAs) are Lie admissible algebras arising from geometry. The leftinvariant affine structures on a Lie group G correspond bijectively to LSA–structures on its Lie algebra. Moreover if a Lie group acts simply transitively as affine transformations on a vector space, then its Lie algebra admits a complete LSA–structure. In this paper we study *simple* LSAs having only trivial two–sided ideals. Some natural examples and deformations are presented. We classify simple LSAs in low dimensions and prove results about the Lie algebra of simple LSAs using a canonical root space decomposition. A special class of complete LSAs is studied.

0. Introduction

Left–symmetric algebras (LSAs) first have been studied in the theory of affine manifolds, affine structures on Lie groups and convex homogeneous cones, see [AUS], [MIL], [VIN]. Let G be a Lie group with a left–invariant affine structure. Then this structure induces a flat torsionfree left–invariant affine connection ∇ on G, that is, a connection in the tangent bundle with zero torsion and zero curvature:

$$\begin{aligned} \nabla_X Y - \nabla_Y X - [X,Y] &= 0 \\ \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z &= 0 \end{aligned}$$

for all left–invariant vector fields X,Y,Z. By defining $x.y = \nabla_X Y$ we obtain a nonassociative algebra structure on the Lie algebra $\mathfrak{g}$ of G satisfying

$$(x,y,z) = (y,x,z) \tag{0.1}$$

where $(x,y,z) = x.(y.z) - (x.y).z$ is the associator of x,y,z.

A nonassociative algebra A with product $(x,y) \mapsto x.y$ over an arbitrary field K is called *left–symmetric algebra*, or LSA, if it satisfies the identity (0.1) for all $x,y,z \in A$. The commutator

$$[x,y] = x.y - y.x \tag{0.2}$$

defines a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$. Thus an LSA is a Lie admissible algebra. Note that not every Lie algebra admits an LSA–structure. In particular, there exist nilpotent Lie algebras without any LSA–structure (counterexamples to the Milnor conjecture, see [BU3]). There is a large literature on LSAs. Special classes of LSAs have also been investigated. An LSA with commuting right multiplications is called *Novikov algebra*. It satisfies in addition to (0.1) the identity $(x.y).z = (x.z).y$, i.e., $R(y)R(z) = R(z)R(y)$. Novikov algebras do arise in the study of local translationally invariant Lie algebras and Poisson brackets of hydrodynamic type, see [CHE], [BAN], [OSB]. S. P. Novikov asked whether there exist *simple* Novikov algebras. Zelmanov proved that a simple Novikov algebra over a field of characteristic zero is 1–dimensional [ZEL] : This is far from being true for LSAs in general: There are *a lot* of simple LSAs. An interesting subclass is given by the *complete* LSAs, where all operators $Id + R(x)$ are bijective. This condition arises naturally in the context of affine transformations as follows: If a Lie group G acts simply transitively as affine transformations on a vector space, then its Lie algebra is solvable and admits a complete LSA–structure [AUS]. As it turns out there are infinitely many nonisomorphic simple complete LSAs for dimension $n \geq 5$.
In this note we classify simple LSAs in low dimensions and prove algebraic properties of the Lie algebra in general. It turns out that the Lie algebra of a simple LSA cannot be nilpotent. We study complete simple LSAs with a special root space decomposition.

1. Definitions, lemmas and examples

Let $(A, .)$ be an LSA over a field K . We will always assume that A is finite–dimensional and K has characteristic zero. For infinite–dimensional LSAs and for LSAs over fields of prime characteristic see [BU1], [OSB], [XXU]. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Lie algebra of A defined by (0.2) . The left– and right multiplications $R(x), L(x)$ are defined by $L(x)y = x.y = R(y)x$. Substituting (0.2) in (0.1) we obtain $[x,y].z = x.(y.z) - y.(x.z)$. This implies that the map $L : \mathfrak{g} \to \mathfrak{gl}(A)$ with $x \mapsto L(x)$ is a Lie algebra representation, i.e., $L([x,y]) = [L(x), L(y)]$. Using L and R we may characterize left–symmetry also by the following identity: $[L(x), R(y)] = R(x.y) - R(y)R(x)$.

Definition 1.1. Let A be an LSA and $T(A) = \{x \in A \mid \operatorname{tr} R(x) = 0\}$. The largest left ideal of A which is contained in $T(A)$ is called the *radical* of A , shortly $\operatorname{rad}(A)$. An LSA is called *complete* if $A = \operatorname{rad}(A)$.

Lemma 1.1. *Let A be an LSA over K . Then the following conditions are equivalent:*

(a) A *is complete.*
(b) A *is right nil, i.e.,* $R(x)$ *is a nilpotent linear transformation, for all* $x \in A$.
(c) $R(x)$ *has no eigenvalue in* $K \setminus \{0\}$ *, for all* $x \in A$.
(d) $\operatorname{tr}(R(x)) = 0$ *for all* $x \in A$.
(e) $Id + R(x)$ *is bijective for all* $x \in A$.

For a proof see [SEG].

In the following we will often assume $K = \mathbb{C}$ for simplicity, although almost everything can be done for $K = \mathbb{R}$ as well.

Definition 1.2. Let $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$, $\lambda \in \mathfrak{g}^*$ and $\mathfrak{g}^\lambda(V) = \{v \in V \mid \exists m \in \mathbb{N} \text{ with } (\varrho(x) - \lambda(x)Id)^m v = 0 \; \forall x \in \mathfrak{g}\}$ the root space of ϱ corresponding to the weight λ. Let A be a complete LSA with Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We consider two representations of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$, namely ad and L. Denote by $\mathfrak{h}^\lambda, \mathfrak{g}^\lambda$ the root space of the representation ad and L respectively. We have

$$\mathfrak{g} = \bigoplus_\lambda \mathfrak{h}^\lambda = \bigoplus_\lambda \mathfrak{g}^\lambda$$

If $\mathfrak{g}^\lambda = \mathfrak{h}^\lambda$ for all λ, then the above decomposition of $\mathfrak{g}$ is called the *canonical decomposition* of A. Denote by Λ the set of λ with $\mathfrak{g}^\lambda \neq 0$.

Lemma 1.2. *Every complex complete LSA has a a unique canonical decomposition.*

For a proof see [GIV]. Let $\mathfrak{h}$ be the canonical Cartan subalgebra, i.e., the unique Cartan subalgebra of the canonical decomposition. The semisimple parts of $\mathrm{ad}(x)$ and $L(x)$ coincide for all $x \in \mathfrak{h}$. For all $\lambda, \mu \in \mathfrak{g}^*$ we have $L(\mathfrak{g}^\lambda)(\mathfrak{g}^\mu) \subseteq \mathfrak{g}^{\lambda+\mu}$.

Definition 1.3. A complex complete LSA is called *special*, if $\dim \mathfrak{g}^\lambda = 1$ for all $\lambda \in \Lambda$ in the canonical decomposition. In particular, $\mathrm{rank}(\mathfrak{g}) = \dim \mathfrak{h} = \dim \mathfrak{g}^0 = 1$. For such an LSA define a *graph* Γ, where the set of $\lambda \in \Lambda$ corresponds to the set of vertices of Γ and two vertices, λ and μ, are joined by an directed edge from λ to μ, if $\mathfrak{g}^\lambda$ is mapped to $\mathfrak{g}^\mu$ by a left multiplication, see [GIV]. For simplicity we will omit the trivial edges from λ to λ, i.e., the loops.

Let A be a LSA. In general, $\mathrm{rad}(A)$ is not an ideal in A. In [MIZ] the following is proved:

Lemma 1.3. *Let A be a complex LSA with nilpotent Lie algebra. Then $rad(A)$ is an ideal of A.*

Definition 1.4. Let A be an LSA and define

$$C_i(A) = \bigcap_{x_1,\ldots,x_i \in A} \ker(R(x_1)\cdots R(x_i)), \quad D_i(A) = \bigoplus_{x_1,\ldots,x_i \in A} \mathrm{im}(R(x_1)\cdots R(x_i))$$

In [HEL] it is proved:

Lemma 1.4. *The sets $C_i(A), D_i(A)$ are bilateral ideals in A with $C_j(A) \subseteq C_{j+1}(A)$ and $D_j(A) \supseteq D_{j+1}(A)$ for all $j \in \mathbb{N}$.*

Note that in particular $C_1(A) = \ker(L)$ is a two–sided ideal. If A is a nontrivial simple LSA then $C_j(A) = 0$ and $D_j(A) = A$ for all j. In [MED] so called *SGAs* are defined to be LSAs where all $L(x)$ are derivations of the Lie algebra $\mathfrak{g}$. For any SGA however, $C_1(A)$ is nontrivial. Hence there are no nontrivial simple SGAs.

Lemma 1.5. *Let A be an LSA with reductive Lie algebra of 1–dimensional center. Then A is simple.*

Proof: Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ be the Lie algebra with center $\mathfrak{z} \simeq K$. Suppose I is a proper two–sided ideal in A . Then it is also a proper Lie ideal in $\mathfrak{g}$ and both I and $\mathfrak{g}/I$ inherit an LSA–structure from A . Since a semisimple Lie algebra does not admit LSA–structures [BU1], we conclude that I must be equal to $\mathfrak{s}_1 \oplus K$, where $\mathfrak{s}_1$ is a semisimple ideal of $\mathfrak{s}$. Hence $\mathfrak{g}/I$ is semisimple and admits an LSA–structure. This is a contradiction. □

The lemma allows us to find nontrivial examples of simple LSAs, i.e., examples other than simple associative algebras: There exist infinitely many nonisomorphic simple LSA–structures on $\mathfrak{gl}_n(K)$. They are classified in [BU2]. Any such structure arises by a deformation of the associative matrix algebra structure.

It is somewhat surprising that there are also simple LSAs with *solvable* Lie algebra. Consider the following construction:

Example 1.1. Let $\mathfrak{t}_n(K)$ be the solvable Lie algebra of upper–triangular matrices and $\tau : \mathfrak{gl}_n(K) \to \mathfrak{t}_n(K)$ the linear map defined by $\tau(e_{ij}) = e_{ij},\ \tau(e_{ji}) = 0$ for $i < j$ and $\tau(e_{ii}) = \frac{1}{2}e_{ii}$ where e_{ij} is the canonical basis of $\mathfrak{gl}_n(K)$. Define an LSA–structure on $\mathfrak{t}_n(K)$ by

$$x.y = xy + \tau(xy^t + yx^t) \tag{1.2}$$

Since $\tau(xy^t + yx^t)$ is symmetric in x and y , it follows (0.2) . To prove (0.1) note that $e_{ij}.e_{kl} = \delta_{jk}e_{il} + \delta_{jl} \cdot \tau(e_{ik} + e_{ki})$ for $i < j$, $k < l$. In case $i \leq k \leq m$ a straightforward calculation shows $(e_{ij}, e_{kl}, e_{mn}) = (\delta_{lm}\delta_{jn} + \delta_{ln}\delta_{jm})e_{ik} - \delta_{jl}\delta_{km}e_{in} - \delta_{jl}\delta_{kn}e_{im} = (e_{kl}, e_{ij}, e_{mn})$ The other cases can be treated likewise. Hence formula (1.2) defines an LSA of dimension $\frac{1}{2}n(n+1)$. It is not simple since $I_n =< e_{1j} \mid j = 1, \dots, n >$ is a two–sided ideal of dimension n . However, the algebra I_n is indeed a *simple* LSA for all $n \in \mathbb{N}$.

The following table shows n–dimensional examples of simple LSAs, complete and incomplete, with solvable Lie algebra:

algebra	basis	nonzero products
I_n	$e_1, \dots, e_n$	$e_1.e_1 = 2e_1,\ e_1.e_j = e_j,\ e_j.e_j = e_1$ for $j = 2, \dots n$
A_n	$e_1, \dots, e_n$	$e_1.e_i = \alpha_i e_i,\ e_j.e_{n+2-j} = e_1$ with $\alpha_{n+2-j} = \alpha_1 - \alpha_j$, $\alpha_j \neq 0$ for $i = 1, \dots, n,\ j = 2, \dots, n$

The algebra I_n is a subalgebra of the LSA $\mathfrak{t}_n(K)$ with $e_i = e_{1i}$. Hence it is left–symmetric. Let J be an ideal in I_n . If $J \neq 0$ then it follows by suitable multiplication $e_1 \in J$. But then $J = I_n$. Hence I_n is a simple algebra. It is not complete since $\operatorname{tr} R(e_1) = 2$. This implies $\operatorname{rad}(I_n) = 0$. I_n is not associative for $n > 1$ since $(e_1, e_1, e_2) = -e_2$. Its Lie algebra is 2 –step solvable, given by $\mathfrak{g} =< e_i,\ i = 1, \dots, n \mid [e_1, e_j] = e_j$ for $j = 2, \dots, n >$. The algebras A_n with parameters $\alpha_1, \dots, \alpha_n$ are left–symmetric if and only if $\alpha_{n+2-i} = \alpha_1 - \alpha_i$ for all $i = 2, \dots, n$ and complete if and only if $\alpha_1 = 0$. Then they are special. They are simple if $a_j \neq 0$ for $j > 2$. The Lie algebra is two–step solvable. It is not difficult to prove that this family contains infinitely many nonisomorphic complete simple LSAs for $n \geq 5$.

2. Deformations of LSAs

A method to derive new LSAs is the study of *infinitesimal deformations* of LSAs.

Let $(A, .)$ be an LSA and $f_t : A \times A \to A$ be a bilinear function defined by

$$f_t(a, b) = a.b + tF_1(a, b) + t^2 F_2(a, b) + t^3 F_3(a, b) + \cdots$$

where F_i are bilinear functions with $F_0(a, b) = a.b$. The family of algebras (A_t, f_t) is left–symmetric if

$$f_t(a, f_t(b, c)) - f_t(f_t(a, b), c) = f_t(b, f_t(a, c)) - f_t(f_t(b, a), c)$$

for all $a, b, c \in A$. This is equivalent to the following equations, valid for all nonnegative integers ν :

$$\sum_{\lambda+\mu=\nu} F_\lambda(a, F_\mu(b, c)) - F_\lambda(F_\mu(a, b), c) + F_\lambda(F_\mu(b, a), c) - F_\lambda(b, F_\mu(a, c)) = 0$$

where $\lambda, \mu \geq 0$. For $\nu = 0$ this means that $A = A_0$ is left–symmetric. For $\nu = 1$ we obtain an equation for F_1 :

$$F_1(a, b.c) - F_1(a.b, c) + F_1(b.a, c) - F_1(b, a.c) + a.F_1(b, c) - F_1(a, b).c + F_1(b, a).c - b.F_1(a, c) = 0$$

According to deformation theory we regard the infinitesimal deformation F_1 as an element of $Z^2(A, A)$. This is a motivation to define cohomology groups $H^n(A, A)$ for LSAs as follows (see [BU4]):

Definition 2.1. Let A be an LSA and $C^n(A, A) = \{f : A \times \cdots \times A \to A \mid f \text{ is multilinear}\}$ be the space of n–cochains, A being a bimodule for A . Define the coboundary operator $\delta^n : C^n(A, A) \to C^{n+1}(A, A)$ by

$$
\begin{aligned}
(\delta^n f)(x_1,\ldots,x_{n+1}) &= \sum_{i=1}^{n}(-1)^{i+1}x_i.f(x_1,\ldots,\hat{x}_i,\ldots,x_n,x_{n+1})\\
&+\sum_{i=1}^{n}(-1)^{i+1}f(x_1,\ldots,\hat{x}_i,\ldots,x_n,x_i).x_{n+1}\\
&-\sum_{i=1}^{n}(-1)^{i+1}f(x_1,\ldots,\hat{x}_i,\ldots,x_n,x_i.x_{n+1})\\
&+\sum_{i<j\le n}(-1)^{i+j}f([x_i,x_j],x_2,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1})
\end{aligned}
$$

In particular

$$
\begin{aligned}
(\delta^1 f)(x_1,x_2) &= x_1.f(x_2)+f(x_1).x_2-f(x_1.x_2)\\
(\delta^2 f)(x_1,x_2,x_3) &= f(x_1,x_2.x_3)-f(x_1.x_2,x_3)+f(x_2.x_1,x_3)-f(x_2,x_1.x_3)\\
&+x_1.f(x_2,x_3)-f(x_1,x_2).x_3+f(x_2,x_1).x_3-x_2.f(x_1,x_3)
\end{aligned}
$$

Since $\delta^2=0$ we obtain cohomology groups $H^n(A,A)$. Note that $Z^1(A,A)=\mathrm{Der}(A)$.

Example 2.1. Consider the LSA $A=I_4$ of the above table. We will determine all simple LSAs which can be obtained by infinitesimal deformations of A . With respect to the basis of A the second cohomology is given by

$$
H^2(A,A)=\left\{g\,:\,A\times A\to A \text{ bilinear } \mid g(e_1)=\begin{pmatrix}0&0&0&0\\0&0&-\alpha&-\beta\\0&\alpha&0&-\gamma\\0&\beta&\gamma&0\end{pmatrix},g(e_i)=0\right\}
$$

where we set $g(e_i,e_j)=g(e_i)e_j$. These 2–cocycles are integrable, i.e., the product $x\cdot y=x.y+g(x,y)$ is left–symmetric and defines LSAs $I_4^d(\alpha,\beta,\gamma)$ depending on 3 parameters. The left multiplications of this algebras are given by the matrices $L(e_1)=\mathrm{diag}(2,1,1,1)+g(e_1)$, $L(e_i)=e_{1i}$ for $i=2,3,4$.

algebra	basis	nonzero products
I_4^d	$e_1,\ldots,e_4$	$e_1.e_2=e_2+\alpha e_3+\beta e_4,\ e_1.e_3=-\alpha e_2+e_3+\gamma e_4,$ $e_1.e_4=-\beta e_2-\gamma e_3+e_4,\ e_2.e_2=e_3.e_3=e_4.e_4=e_1,$ $e_1.e_1=2e_1$ with $\alpha^2+\beta^2+\gamma^2\neq 1$

The algebra $I_4^d(\alpha,\beta,\gamma)$ is simple if and only if $\alpha^2+\beta^2+\gamma^2\neq -1$: The ideal I generated by e_1 equals I_4^d if the vectors $e_1.e_2,\, e_1.e_3,\, e_1.e_4$ are linear independent, i.e., if the

determinant of $L(e_1) = 2(1+\alpha^2+\beta^2+\gamma^2)$ is nonzero. On the other hand every nonzero ideal will contain e_1 .
It is not difficult to classify the algebras $I_4^d(\alpha,\beta,\gamma)$: Over $\mathbb{C}$ any of them is isomorphic to one of the following algebras: $I_4^d(0,0,c), I_4^d(0,1,i), I_4^d(0,1,-i)$. These algebras are different except for $I_4^d(0,0,c) \simeq I_4^d(0,0,-c)$. They have trivial radical. For $c^2 \neq -1$ we obtain infinitely many simple LSAs.

Remark 2.1. We may also obtain new simple *complete* LSAs by deformations, but not for $n < 5$. There exists only one complete simple LSA in dimension 3 and only one in dimension 4 , see sections 4 and 5 .

3. The Lie algebra of a simple LSA

Let A be a nontrivial LSA, i.e., of dimension $n \geq 2$. Its Lie algebra $\mathfrak{g}$ is an important invariant. We will study the algebraic properties of $\mathfrak{g}$. For convenience we will assume that $K = \mathbb{C}$, although many results are true for other fields of characteristic zero.

The examples show that there are simple LSAs with solvable and reductive Lie algebra. It is well known that the Lie algebra of an LSA cannot be semisimple [BU1].

Lemma 3.1. *$\mathfrak{g}$ is abelian if and only if A is associative and commutative.*

Proof: If A is commutative then $\mathfrak{g}$ is abelian by definition. Assume that $\mathfrak{g}$ is abelian. Then $x.y = y.x$ for all $x,y \in A$ and using left–symmetry, $0 = [xz].y = x.(z.y) - z.(x.y) = x.(y.z) - (x.y).z = (x,y,z)$. $\square$

This implies that the Lie algebra of a simple LSA is not abelian. Otherwise it would be 1 –dimensional over $\mathbb{C}$. In fact, a more general statement holds:

Proposition 3.1. *If A is a simple LSA then $\mathfrak{g}$ cannot be nilpotent.*

Proof: Assume that $\mathfrak{g}$ is nilpotent. Then we claim that $\mathrm{rad}(A)$ is a nonzero ideal in A . By lemma 1.2. we know that $\mathrm{rad}(A)$ is an ideal of A . In the solvable non–nilpotent case it might be trivial, as I_n shows. However for a nilpotent Lie algebra, $\mathrm{rad}(A)$ is not zero: Since $\mathfrak{g}$ is nilpotent there exists a finite number of different weights $\lambda_1,\dots,\lambda_s$ for L such that

$$A_i = \{x \in A \mid \exists\, n_i \text{ s.t. } (L(y)-\lambda(y)Id)^{n_i}x = 0 \quad \forall\, y \in A\}$$

is a left ideal and $A = \bigoplus_{i=1}^s A_i$. There is a basis $\{a_{ij}\}$ of A_i such that all $L(x)$ have upper–triangular form on A_i and $\lambda_i(x)$ on the diagonal. Denoting by $\{x_{i,j} \mid i = 1,\dots,s;\ j = 1,\dots,n_i\}$ the coefficients of $x \in A$ with respect to the base $\{a_{ij}\}$ one shows that $N = \{x \in A \mid x_{i,n_i} = 0\, \forall\, i\}$ is an ideal of A with $[A,A] \subseteq N \subseteq \mathrm{rad}(A)$.

Since A is simple it follows that $\mathrm{rad}(A)$ must coincide with A . Hence A is complete and A is right nil. We can apply the following lemma [KIM]:

Lemma 3.2. *Let A be an LSA with Lie algebra $\mathfrak{g}$. Then the following conditions are equivalent:*

(a) A is left nil, i.e., $L(x)$ is a nilpotent linear transformation, for all $x \in A$.
(b) A is right nil and $\mathfrak{g}$ is nilpotent.

This implies that there is a basis $\{e_1, \ldots, e_n\}$ of A such that all $L(x)$ are simultaneously in strict upper-triangular form. Then we have $e_1 \in \ker(L)$. Since A is nontrivial $\ker(L)$ is a proper two–sided ideal in A , see lemma 1.2 . This is a contradiction. $\square$

Note that the Lie algebra of a complete LSA is always solvable [AUS]. Hence the Lie algebra of a complete simple LSA is solvable, non–nilpotent. The stronger result holds for the Lie algebra of a special (hence complete) LSA:

Proposition 3.2. *The Lie algebra of a special LSA has trivial center.*

Proof: Let $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}^\lambda$ be the canonical decomposition and $e_0 \in \mathfrak{g}^0$ with $e_0 \neq 0$. Let e_λ the a base vector of $\mathfrak{g}^\lambda$ with $\lambda \in \Lambda$. With respect to the basis $\mathcal{B} = \{e_\lambda \mid \lambda \in \Lambda\}$ the algebra A has multiplication $e_\lambda . e_\mu = c_{\lambda,\mu} e_{\lambda+\mu}$. This implies that $R(e_0)$ is diagonalizable. Assume that there is a $z \neq 0$ in the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. Since $Z(\mathfrak{g}) \subseteq \mathfrak{h}$ for any Cartan subalgebra, $z \in \mathfrak{h} = \mathfrak{h}^0 = \mathfrak{g}^0$ and $R(z)$ is diagonalizable since $\dim \mathfrak{g}^0 = 1$ and z is a multiple of e_0 . On the other hand, $R(z)$ is nilpotent by Lemma 1.1. , hence $R(z) = 0$. This is a contradiction. $\square$

In particular, the Lie algebra of a simple special LSA has trivial center.

Example 3.1. It is not true in general that the center of the Lie algebra of a simple LSA is trivial. Consider the following example:

algebra	basis	nonzero products
B_4	$e_1, \ldots, e_4$	$e_1.e_2 = e_2.e_1 = e_4,\ e_2.e_3 = 2e_1,\ e_3.e_2 = e_1,$ $e_4.e_2 = -e_2,\ e_4.e_3 = e_3,\ e_4.e_4 = -e_4$

The Lie algebra of B_4 is given by $\mathfrak{g} =< e_i \mid [e_2, e_3] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = -e_3 >$ It is 3–step solvable and has a 1–dimensional center.

Proposition 3.3. *The Lie algebra of a n–dimensional simple LSA admits a faithful linear representation of degree n .*

Proof: The left multiplication L is faithful since $\ker(L)$ is zero. $\square$

Remark 3.1. In the solvable case, the above proposition is really a condition on the Lie algebra $\mathfrak{g}$. But it is by no means easy to find solvable Lie algebras without any faithful representation of degree n [BU3].

Example 3.2. There exist solvable Lie algebras without any simple LSA–structure. An example is $\mathfrak{r}_3(\mathbb{C}) =< e_1, e_2, e_3 \mid [e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3 >$, see Proposition 4.2. . On the other side, $\mathfrak{r}_3(\mathbb{C})$ nevertheless has a faithful representation of degree 3, namely the adjoint representation.

4. Simple LSAs in dimension 2,3

The classification of 2–dimensional LSAs with basis $\{x, y\}$ is as follows:

Proposition 4.1. *Any 2–dimensional LSA with nonabelian Lie algebra over $\mathbb{C}$ is isomorphic to exactly one of the following algebras:*

$$\mathfrak{b}_{1,\alpha}: \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \qquad\qquad \mathfrak{b}_4: \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{b}_{2,\beta\neq 0}: \quad \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \beta-1 & 0 \\ 0 & \beta \end{pmatrix} \qquad\qquad \mathfrak{b}_5: \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{b}_3: \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The algebras are described by the left multiplications $L(x), L(y)$. The only associative algebras are $\mathfrak{b}_{1,-1}, \mathfrak{b}_{2,1}$ and the only complete algebra is $\mathfrak{b}_{1,0}$.

Proof: Since $\mathfrak{g}$ is solvable we may assume (by Lie's theorem) that the left multiplications are simultaneously in upper–triangular form, i.e., $L(x) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_3 \end{pmatrix}$, $L(y) = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & \beta_3 \end{pmatrix}$. This algebra is left–symmetric if and only if $\beta_1(\alpha_1 - \alpha_3) = \alpha_1\alpha_2$, $\alpha_3(\beta_1 - \beta_3 - \alpha_2) = 0$ and $\alpha_1\beta_2 + \alpha_2\beta_3 - 2\alpha_3\beta_2 = \alpha_2^2$. The result follows by a staightforward case by case analysis.

Corollary 4.1. *Any 2–dimensional complex simple LSA is isomorphic to $\mathfrak{b}_4$.*

Note that $\mathfrak{b}_4$ is the LSA I_2. The corollary is not true for $K = \mathbb{R}$: The 2–dimensional associative algebra $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is a simple LSA different from $\mathfrak{b}_4$.

Proof: $\ker(L)$ is a 1-dimensional ideal in $\mathfrak{b}_{1,\alpha}, \mathfrak{b}_3$ and $< x >$ is one in $\mathfrak{b}_{2,\beta}, b_5$.

Proposition 4.2. *Let A be a simple 3–dimensional LSA over $\mathbb{C}$. Then its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{r}_{3,\lambda} =< e_1, e_2, e_3 \mid [e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3 >$ with $|\lambda| \leq 1, \lambda \neq 0$, and A is isomorphic to exactly one of the following algebras:*

algebra	basis	nonzero products	conditions
A_1^λ	e_1, e_2, e_3	$e_1.e_1 = (\lambda+1)e_1,\ e_1.e_2 = e_2,\ e_1.e_3 = \lambda e_3$ $e_2.e_3 = e_3.e_2 = e_1$	$\lambda \neq 0,\ \lvert\lambda\rvert \leq 1$
A_2	e_1, e_2, e_3	$e_1.e_1 = \frac{3}{2}e_1,\ e_1.e_2 = e_2,\ e_1.e_3 = \frac{1}{2}e_3$ $e_2.e_3 = e_3.e_2 = e_1,\ e_3.e_3 = -e_2$	$\lambda = \frac{1}{2}$

Corollary 4.2. *Let A be a complete simple LSA of dimension 3 over $\mathbb{C}$. Then A is isomorphic to A_1^{-1} with Lie algebra $\mathfrak{r}_{3,-1}(\mathbb{C})$.*

Proof of Proposition 4.2. Let A be a simple LSA with basis $\{e_1, e_2, e_3\}$. Its Lie algebra $\mathfrak{g}$ cannot be semisimple or nilpotent. According to the classification of complex Lie algebras of dimension 3 it must be solvable, i.e., equal to $r_{3,\lambda}(\mathbb{C})$ or $r_3(\mathbb{C}) =< e_i \mid [e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3 >$ or $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C} =< e_i \mid [e_1, e_2] = e_1 >$.
Let $L(e_k) = (x_{ij}^k)$, where $1 \leq i, j, k \leq 3$. Since L is a representation of a solvable complex Lie algebra we can apply Lie's theorem. We may assume that the $L(e_k)$ have simultaneously lower–triangular form, i.e., $x_{12}^k = x_{13}^k = x_{23}^k = 0$. This product is left–symmetric if certain equations (LSA–equations) of order 2 hold in the variables x_{ij}^k.

Claim 4.1. *We may assume that $(x_{11}^1, x_{11}^2, x_{11}^3) = (0, 0, 1)$.*

Proof: First observe that $(x_{11}^2, x_{11}^3) \neq (0, 0)$. Otherwise $I =< e_2, e_3 >$ would be a 2–dimensional ideal in A. Assume $x_{11}^3 = 0$. Then $x_{11}^2 \neq 0$ so that the LSA–equations imply $x_{21}^3 = x_{22}^3 = 0$. Then $I =< e_3 >$ is a 1–dimensional ideal, which is a contradiction. Hence x_{11}^3 is nonzero. We can assume that $x_{11}^2 = 0$. Otherwise we apply the base change $e_1' = e_1,\ e_2' = e_2 + \alpha e_3,\ e_3' = e_3$ with $\alpha = -x_{11}^2/x_{11}^3$. This preserves the lower triangular form, fixes (x_{11}^1, x_{11}^3) and translates x_{11}^2 to 0. Similarly we may assume $x_{11}^1 = 0$ by applying the base change $e_1' = e_1 + \alpha e_3,\ e_i' = e_i,\ i > 1$ with $\alpha = -x_{11}^1/x_{11}^3$. This preserves (x_{11}^2, x_{11}^3) with $x_{11}^2 = 0$. Finally we obtain $x_{11}^3 = 1$ by normalization.

As a consequence the LSA–equations yield $x_{32}^3 = x_{33}^2,\ x_{31}^3 = x_{33}^1,\ x_{31}^2 = x_{32}^1$. The Lie algebra of A satisfies $[e_1, e_2] = (x_{22}^1 - x_{21}^2)e_2,\ [e_1, e_3] = -e_1 - x_{21}^3 e_2,\ [e_2, e_3] = -x_{22}^3 e_2$. The Jacobi identity implies $x_{22}^1 = x_{21}^2$.

Claim 4.2. *The Lie algebra $\mathfrak{g}$ has trivial center.*

Proof: Clearly $\mathfrak{g} =< e_i \mid [e_1, e_3] = -e_1 - x_{21}^3 e_2,\ [e_2, e_3] = -x_{22}^3 e_2 >$ has trivial center if and only if $x_{22}^3 \neq 0$. Assume $x_{22}^3 = 0$. Then $[L(e_1), L(e_3)] = L([e_1, e_3])$ implies $x_{22}^1 = -x_{22}^2 x_{21}^3,\ x_{21}^1 = x_{22}^1 x_{21}^3$ and $I =< e_1 + x_{21}^3 e_2, e_3 >$ is a 2–dimensional ideal. This is a contradiction.

Since $x_{22}^3 \neq 0$ the LSA–equations imply $x_{22}^1 = x_{33}^1 = x_{22}^2 = x_{33}^2 = 0$. Then $(x_{32}^1, x_{32}^2) \neq (0, 0)$, otherwise $L(e_2) = 0$ and $\ker(L)$ would be a proper ideal. There are two cases: $x_{32}^1 \neq 0$ or $x_{32}^1 = 0$ and $x_{32}^2 \neq 0$.

Case 1: $x^1_{32} \neq 0$. We obtain $x^1_{21} = 0, x^3_{33} = 2x^3_{22}$ and $x^3_{21} = x^1_{32}(x^3_{22} - 1)/x^1_{32}$. The algebra is an LSA if and only if $(x^3_{22} - 1)(x^1_{31}x^2_{32} - (x^1_{32})^2) = 0$. The second factor must be nonzero, otherwise $L(s) = 0$ for $s = e_1 - x^1_{32}/x^2_{32}e_2$. Hence $x^3_{22} = 1$ and we obtain the following LSA:

$$e_1.e_1 = x^1_{31}e_3,\ e_1.e_2 = e_2.e_1 = x^1_{32}e_3,\ e_2.e_2 = x^2_{32}e_3, e_3.e_1 = e_1,\ e_3.e_2 = e_2,\ e_3.e_3 = 2e_3\ .$$

This LSA is simple iff $x^1_{31}x^2_{32} \neq (x^1_{32})^2$: If $e_3 \in I$ then $I = A$. If $e_3 \notin I$ then $I = 0$ or $x^1_{31}x^2_{32} = (x^1_{32})^2$. The algebra is isomorphic to A^1_1 with $\lambda = 1$.

Case 2: $x^1_{32} = 0$ and $x^2_{32} \neq 0$. Then it follows $x^3_{33} = x^3_{22} + 1$ with $x^3_{22} \neq 0$. If $x^3_{22} = 1$ then we obtain the LSA A^1_1 as in the case above. Otherwise apply the base change $e'_1 = e_3, e'_2 = e_1 - x^3_{21}e_2/(x^3_{22} - 1), e'_3 = e_2$. With respect to this basis, the algebra is left–symmetric iff $x^1_{21}(x^3_{22} - 2) = 0$ and has Lie algebra $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ with $\lambda = x^3_{22}$. In case $x^1_{21} \neq 0$ we obtain $\lambda = 2$ and the simple LSA

$$e_1.e_1 = 3e_1,\ e_1.e_2 = e_2,\ e_1.e_3 = 2e_3,\ e_2.e_2 = x^1_{21}e_3,\ e_2.e_3 = e_3.e_2 = x^1_{32}e_1.$$

Here $x^1_{32} \neq 0$ and the LSA is isomorphic to A_2 . Note that $\mathfrak{r}_{3,2}(\mathbb{C}) \simeq \mathfrak{r}_{3,1/2}(\mathbb{C})$.
In the other case $x^1_{21} = 0$ and we obtain the simple LSAs

$$e_1.e_1 = (\lambda + 1)e_1,\ e_1.e_2 = e_2,\ e_1.e_3 = \lambda e_3,\ e_2.e_3 = e_3.e_2 = x^1_{32}e_1.$$

They are isomorphic to A^λ_1 . This finishes the proof. □

5. Complete simple LSAs

The classification of 4–dimensional simple LSAs is already quite complicated. It is feasible however for complete LSAs. In fact, there is only *one* complete simple LSA over $\mathbb{C}$ in dimension 4 :

Proposition 5.1. *Let A be a complete simple 4–dimensional LSA over $\mathbb{C}$. Then A is isomorphic to the following algebra:*

algebra	basis	nonzero products
B_4^{-2}	$e_1, \dots, e_4$	$e_1.e_2 = e_2.e_1 = e_4,\ e_3.e_2 = e_4.e_1 = e_1$ $e_2.e_3 = 2e_1,\ e_4.e_2 = -e_2,\ e_4.e_3 = 2e_3$

with Lie algebra $\mathfrak{g}^{-2}_{4,8} =< e_i \mid [e_1, e_4] = -e_1,\ [e_2, e_3] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = -2e_3 >$.

Proof: Since A is complete, its Lie algebra $\mathfrak{g}$ is solvable. Let $L(e_k) = (x^k_{ij})$, where $1 \leq i, j, k \leq 4$. By Lie's theorem we can assume that the $L(e_k)$ have lower–triangular form, that is $x^k_{ij} = 0$ for $i < j$.

Claim 5.1. *We may assume that* $(x_{11}^1, x_{11}^2, x_{11}^3, x_{11}^4) = (0,0,0,1)$.

Proof: If $(x_{11}^2, x_{11}^3, x_{11}^4) = (0,0,0)$ then $I =< e_2, e_3, e_4 >$ would be a 3–dimensional ideal. Using this it follows that $(x_{11}^3, x_{11}^4) = (0,0)$ is also impossible. Otherwise $x_{11}^2 \neq 0$ and the LSA–equations show that $I =< e_3, e_4 >$ would be an ideal. Now we have two cases: First, if $x_{11}^4 = 0$ then $x_{11}^3 \neq 0$ and we may assume after suitable basis change as in the proof of claim 4.1. , $(x_{11}^1, x_{11}^2, x_{11}^3, x_{11}^4) = (0,0,1,0)$. In case $(x_{21}^4, x_{22}^4) = (0,0)$ we will obtain an ideal $I =< e_4 >$ and otherwise the ideal $< e_2, e_4 >$. Hence, the second case must hold: $x_{11}^4 \neq 0$. Again, by basis changes the claim follows.

Claim 5.2. *The rank of* $\mathfrak{g}$ *is equal to* 1 .

Proof: Recall that $\mathrm{rank}(\mathfrak{g}) \geq 1$ is the dimension of a Cartan subalgebra of $\mathfrak{g}$. The characteristic polynomial of $\mathrm{ad}(e_4)$ is $t^4 - t^3(x_{22}^4 + x_{33}^4 + 1) + t^2(x_{22}^4 x_{33}^4 + x_{22}^4 + x_{33}^4) - t x_{22}^4 x_{33}^4$. If x_{22}^4 and x_{33}^4 are nonzero, then the minimal index i such that the monomial t^i appears in the polynomial is equal to 1 . Clearly this would imply $\mathrm{rank}(\mathfrak{g}) = 1$. In fact, x_{22}^4 and x_{33}^4 cannot be zero:

Case 1: Suppose $x_{33}^4 = 0$. If x_{22}^4 is also zero, then it follows $x_{22}^3 = 0$ or $L(e_2) = 0$ by using the LSA equations. Since the LSA is simple, $\ker(L) \neq 0$. Also $x_{43}^3 = 0$, otherwise $I =< e_1 + x_{21}^4 e_2, e_3, e_4 >$ would be a 3–dimensional ideal. If $x_{22}^2 \neq 0$ then the LSA equations would imply $L(e_1) = 0$. But for $x_{22}^2 = 0$ it follows again that I is a proper ideal. If $x_{22}^4 \neq 0$ then $x_{33}^1 = x_{44}^1 = x_{33}^2 = x_{44}^2 = 0$. If $x_{22}^3 \neq 0$ then $L(e_2)$ would be zero. Hence $x_{22}^3 = 0$ and $x_{22}^1 = x_{22}^2 = 0$. Then $x_{32}^1 = 0$, otherwise $L(e_3) = 0$. If $x_{21}^3 \neq 0$ then $x_{22}^4 = 1$ and $e_1 + x_{21}^4 e_2 + x_{31}^4 e_3$ would be a nontrivial element of $\ker(L)$. Hence $x_{21}^3 = 0$ and $x_{43}^2 = 0$. Then it follows $x_{43}^3 \neq 0$, otherwise $L(e_3) = 0$. The LSA equations imply $x_{22}^4 = -1$ or $L(e_2 - x_{43}^2 / x_{43}^3 e_3) = 0$. It follows that $I =< e_4.e_1, e_4.e_2, e_4 >$ is a 3–dimensional ideal. This is a contradiction.

Case 2: Suppose $x_{22}^4 = 0$. Then we may assume $x_{33}^4 \neq 0$, since the other case was treated above. Then $x_{43}^2 = 0$. If $x_{43}^1 \neq 0$ then $I =< e_1 + x_{21}^4 e_2, e_3, e_4 >$ would be a proper ideal. Hence $x_{43}^1 = 0$ and we are left with two alternatives: Either $L(e_3) = 0$ or $I =< e_1, e_3, e_4 >$ would be a proper ideal. This is a contradiction.

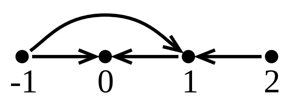
The classification now goes as follows: Since $x_{33}^4 \neq 0$, the LSA–equations imply $x_{22}^3 = x_{33}^3 = x_{43}^3 = 0$. Then also $x_{21}^3 = 0$, otherwise $L(e_2 + x_{32}^4 e_3) = 0$. Furthermore we obtain $L(e_2) = 0$ if and only if $(x_{43}^1, x_{43}^2) = (0,0)$. In the first case, where $x_{43}^2 \neq 0$ we obtain $x_{32}^1 \neq 0$ (otherwise $\ker(L) \neq 0$) and $x_{22} = -1/2$. Then the LSA equations can be easily satisfied and the resulting LSA is isomorphic to B_4^{-2} . In the second case, where $x_{43}^2 = 0$ and $x_{43}^1 \neq 0$, we obtain $x_{22}^4 = -2, x_{33}^4 = -1, x_{32}^2 = x_{42}^2 = 0$. This leads to a simple LSA which is again isomorphic to B_4^{-2} . □

Remark 5.1. The algebra B_4^{-2} is part of a family of simple LSAs B_4^{γ} with Lie algebra $\mathfrak{g}_{4,8}^{\gamma} =< e_i \mid [e_1, e_4] = (1+\gamma) e_1,\ [e_2, e_3] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = \gamma e_3 >,\ \gamma \neq 0$

algebra	basis	nonzero products
$B_4^{-\gamma}$	$e_1,\ldots,e_4$	$e_1.e_2=e_2.e_1=e_4,\ e_3.e_2=e_1,\ e_4.e_1=-(\gamma+1)e_1$ $e_4.e_2=-e_2,\ e_4.e_3=-\gamma e_3,\ e_4.e_4=-(\gamma+2)e_4$

The algebra B_4^0 is not simple. The infinitesimal deformations of B_4 (see example 3.1.) yield exactly the algebras B_4^γ . Here $H^2(B_4,B_4)=\{g\mid g(e_4)=\operatorname{diag}(\alpha,0,\alpha,\alpha),g(e_i)=0\}$. For $\alpha=-1-\gamma\neq-1$ we obtain the simple LSAs B_4^γ .

Remark 5.2. The algebra B_4^{-2} is special with weights $\Lambda=\{-1,0,1,2\}$ and $\mathfrak{g}^{-1}=<e_2>,\mathfrak{g}^0=<e_4>,\mathfrak{g}^1=<e_1>,\mathfrak{g}^2=<e_3>$. The graph Γ is given by



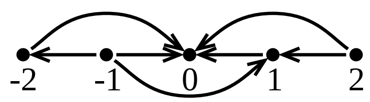
It is natural to ask whether all complete simple LSAs are special. This is not the case, as the following example of a complete simple LSA in dimension 6 shows:

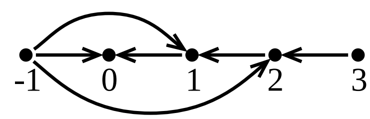
algebra	basis	nonzero products
A	$e_1,\ldots,e_6$	$e_1.e_3=e_3.e_1=e_5+e_6,\ e_2.e_4=e_4.e_2=e_5-e_6$ $e_5.e_1=e_1,\ e_5.e_3=-e_3,\ e_6.e_2=e_2,\ e_6.e_4=-e_4$

The vectors e_5,e_6 span a two–dimensional Cartan algebra of the Lie algebra $\mathfrak{g}$ of A . Since $\operatorname{rank}(\mathfrak{g})=2$, the LSA is not special. There is also an example of a simple complete LSA in dimension 5 which is not special and has a Lie algebra $\mathfrak{g}$ with $\operatorname{rank}(\mathfrak{g})=1$. In the following we list all complete simple special LSAs of dimension 5 over $\mathbb{C}$, see [GIV]. There is a family of complete simple LSAs containing infinitely many nonisomorphic LSAs (see also the algebras $A_5(\alpha_1,\ldots,\alpha_5)$ in the first table after example 1.1.).

algebra	basis	nonzero products
$A_{5,1}^\lambda$	$e_0,e_1,e_{-1},e_\lambda,e_{-\lambda}$	$e_{-\lambda}.e_\lambda=e_\lambda.e_{-\lambda}=e_0,\ e_{-1}.e_1=e_1.e_{-1}=e_0$ $e_0.e_\mu=\mu e_\mu$ for $\mu=-\lambda,\lambda,-1,1$
$A_{5,2}^{\alpha,\beta,\gamma}$	$e_\mu,\ \mu=-2,-1,0,1,2$	$e_{-2}.e_2=e_2.e_{-2}=e_0,\ e_{-1}.e_1=e_1.e_{-1}=e_0,\ e_0.e_\mu=\mu e_\mu$ $e_2.e_{-1}=\alpha e_1,\ e_{-1}.e_2=\beta e_1,\ e_{-1}.e_{-1}=(2\alpha-\beta)e_{-2}$
$A_{5,3}$	$e_\mu,\ \mu=-1,0,1,2,3$	$e_0.e_\mu=\mu e_\mu,\ e_{-1}.e_k=ke_{k-1}$ $e_k.e_{-1}=e_{k-1}$ for $k=1,2,3$

The graph of $A_{5,1}^{\lambda}$ consists of five vertices $-\lambda, -1, 0, 1, \lambda$ and four edges directed inwards towards 0. The graphs of the algebras $A_{5,2}, A_{5,3}$ are as follows:





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