Affine structures on nilmanifolds

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We investigate the existence of affine structures on nilmanifolds $\Gamma\backslash G$ in the case where the Lie algebra $\mathfrak{g}$ of the Lie group G is filiform nilpotent of dimension less or equal to 11 . Here we obtain examples of nilmanifolds without *any* affine structure in dimensions $10, 11$. These are new counterexamples to the Milnor conjecture. So far examples in dimension 11 were known where the proof is complicated, see [BGR] and [BEN]. Using certain 2 -cocycles we realize the filiform Lie algebras as deformation algebras from a standard graded filiform algebra. Thus we study the affine algebraic variety of complex filiform nilpotent Lie algebra structures of a given dimension ≤ 11 . This approach simplifies the calculations, and the counterexamples in dimension 10 are less complicated than the known ones. We also obtain results for the minimal dimension $\mu(\mathfrak{g})$ of a faithful $\mathfrak{g}$ -module for these filiform Lie algebras $\mathfrak{g}$.

1 Introduction

Let M denote an n -dimensional manifold (connected and without boundary). An *affine atlas* Φ on M is a covering of M by coordinate charts such that each coordinate change between overlapping charts in Φ is *locally affine*, i.e., extends to an affine automorphism $x \mapsto Ax + b$, $A \in \mathbf{GL}_n(\mathbb{R})$, of some n -dimensional real vector space E. A maximal affine atlas is an *affine structure* on M , and M together with an affine structure is called an *affine manifold*. An affine structure determines a differentiable structure and affine manifolds are *flat* – there is a natural correspondence between affine structures on M and *flat torsionfree affine connections* ∇ on M . Such an affine connection is a connection in the tangent bundle with zero torsion and zero curvature.

Subclasses of affine manifolds are *Riemannian-flat* and *Lorentz-flat* manifolds. A fundamental problem is the question of existence of affine structures. A closed surface admits affine structures if and only if its Euler characteristic vanishes; in fact, a closed surface with genus $g \geq 2$ does not possess *any* affine connection with curvature zero ([MI1]). The torus admits affine structures, even non-Riemannian ones: Let Γ be the set of transformations $(x, y) \mapsto (x + ny + m, y + n)$ of $E = \mathbb{R}^2$ where $n, m \in \mathbb{Z}$. Denote by $\mathbf{Aff}(E)$ the group of affine automorphisms,

$$\mathbf{Aff}(E) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in \mathbf{GL}(E),\ b \in E \right\}$$

Then Γ is a discrete subgroup of $\mathbf{Aff}(\mathbb{R}^2)$ acting properly discontinuously on $\mathbb{R}^2$. The quotient space $\Gamma\backslash\mathbb{R}^2$ is diffeomorphic to a torus and is a complete compact affine manifold. The flat torsionfree connection on $\mathbb{R}^2$ induces a flat torsionfree affine connection on $\Gamma\backslash\mathbb{R}^2$. It can be shown that the affine structure is not Riemannian.

In higher dimensions ($n \geq 3$) the existence question is open. There are only certain obstructions known ([SMI]): If M is compact and its fundamental group is build up out of finite groups by taking free products, direct products and finite extensions, then M does not admit affine structures.

Many examples of affine manifolds come from *left-invariant affine structures on Lie groups*: If G is a Lie group, an affine structure on G is *left-invariant*, if for each $g \in G$ the left-multiplication by g, $L_g : G \to G$, is an automorphism of the affine structure. (The affine connection then is left-invariant under left-translations). For G simply connected let $D : G \to E$ be the developing map. Then there is for each $g \in G$ a unique affine automorphism $\alpha(g)$ of E, such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{D} & E \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle \alpha(g)} \\ G & \xrightarrow{D} & E \end{array}$$

commutes. In that case $\alpha : G \to \mathbf{Aff}(E)$ is an affine representation.

It is not difficult to see ([FGH]) that G admits a complete left-invariant structure if and only if G acts *simply transitively* on E as affine transformations. In this case G must be solvable ([AUS]). If G has a left-invariant affine structure and Γ is a discrete subgroup of G, then the homogeneous space $\Gamma\backslash G$ of right cosets inherits an affine structure. If G is nilpotent, then $\Gamma\backslash G$ is called an *affine nilmanifold*. Any compact complete affine manifold with nilpotent fundamental group is already an affine nilmanifold ([FGH]).

Left-invariant affine structures play an important role in the study of affine crystallographic groups (in short *ACGs*), and of fundamental groups of affine manifolds, see [MI2]. A group $\Gamma \leq \mathbf{Aff}(E)$ is called *ACG* if it acts properly discontinuously on E with compact quotient. There is the following well-known conjecture (for details see [AMS]):

Auslander conjecture: *An ACG is virtually polycyclic.*

This may be restated as follows: The fundamental group of a compact complete affine manifold is virtually polycyclic. Milnor proved that a finitely generated torsionfree virtually polycyclic group Γ can be realized as a subgroup of $\mathbf{Aff}(E)$ acting properly discontinuously. Hence it is the fundamental group of a complete affine manifold.

If we assume that $\Gamma \subset \mathbf{Aff}(E)$ is a virtually polycyclic ACG acting on E, then there is a Lie group $G \subset \mathbf{Aff}(E)$ virtually containing Γ, acting simply transitively on E. The latter is equivalent to the fact that G admits a complete left-invariant affine structure. Indeed, Auslander's conjecture is equivalent to the following:

A compact complete affine manifold is finitely covered by quotients of solvable Lie groups with complete left-invariant affine structures.

Milnor asked in this context ([MI2]):

(1) *Which Lie groups admit left-invariant affine structures ?*

This question is particularly difficult for nilpotent Lie groups. There was much evidence that every nilpotent Lie group admits left-invariant affine structures, see Proposition 1. Milnor conjectured this to be true even for solvable Lie groups ([MI1]). Recently, however, counterexamples were discovered ([BGR] and [BEN]). There are nilmanifolds which are not affine. The key step here is to find n -dimensional nilpotent Lie algebras having no faithful representations in dimension $n+1$, hence no affine representation which could arise from a left-invariant affine structure on the Lie group G . In order to determine such Lie algebras we study the affine algebraic variety of all filiform Lie algebra structures in dimension ≤ 11 over $\mathbb{C}$. Every filiform nilpotent Lie algebra of dimension $n > 7$ is isomorphic to an *infinitesimal deformation* of the standard graded filiform $\mathfrak{g}$ by a 2 -cocycle from a certain subspace of $H^2(\mathfrak{g},\mathfrak{g})$, see [VER]. This description turns out to be useful for our question, i.e., to determine the minimal dimension $\mu(\mathfrak{g})$ of faithful $\mathfrak{g}$ -modules for such Lie algebras. We are led to new counterexamples in dimension 10 and 11 . If $\mathfrak{g}$ is a nilpotent filiform algebra of smaller dimension ($n \leq 9$) we have $\mu(\mathfrak{g}) = n$. Hence these algebras do not provide counterexamples with respect to this method.

2 Preliminaries

Let G be a finite-dimensional connected Lie group with Lie algebra $\mathfrak{g}$. We may assume that G is simply connected (otherwise consider $\widetilde{G}$).

Definition 1 An affine representation $\alpha : G \to \mathbf{Aff}(E)$ is called *étale*, if there exists a $v \in E$ whose stabilizer G_v is discrete in G , and whose G - orbit $G \cdot v$ is open in E .

Definition 2 A *left-symmetric algebra structure* (or *LSA–structure* in short) *on* $\mathfrak{g}$ *over a field* k is a k –bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x,y) \mapsto x \cdot y$ satisfying the conditions $x{\cdot}y - y{\cdot}x = [x,y]$ and $(x,y,z) = (y,x,z)$ for all x,y,z , where $(x,y,z) = x{\cdot}(y{\cdot}z) - (x{\cdot}y){\cdot}z$ denotes the associator of $x,y,z \in \mathfrak{g}$.

Lemma 1 *There is a canonical one-to-one correspondence between the following classes of objects (up to suitable equivalence):*

(a)	$\{$*Etale affine representations of* $G\}$
(b)	$\{$*Left-invariant affine structures on* $G\}$
(c)	$\{$*Flat torsionfree left-invariant affine connections* ∇ *on* $G\}$
(d)	$\{$*LSA–structures on* $\mathfrak{g}\}$

Under the bijection $(b),(d)$ *, bi-invariant affine structures correspond to associative LSA-structures.*

Proof: This is well known, see [SEG] or [KIM]. Here is a brief reminder of some of the arguments:
Suppose G admits a left-invariant flat torsionfree affine connection ∇ on G. Since the connection is left-invariant, for any two left-invariant vector fields $X, Y \in \mathfrak{g}$, the covariant derivative $\nabla_X Y \in \mathfrak{g}$ is left-invariant. It follows that covariant differentiation $(X, Y) \mapsto \nabla_X Y$ defines a bilinear multiplication $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted by $(X, Y) \mapsto XY$ in short. Since ∇ is locally flat and torsionfree, we have the following:

$$(i) \qquad [X, Y] = XY - YX$$

$$(ii) \qquad [X, Y]Z = X(YZ) - Y(XZ)$$

We can rewrite (ii) by using (i) as $(X, Y, Z) = (Y, X, Z)$. Thus $(\mathfrak{g}, \cdot)$ is a left-symmetric algebra (or LSA) with product $x \cdot y = \nabla_X Y$.
If we have any LSA–structure on $\mathfrak{g}$ with product $(x, y) \mapsto x \cdot y$, then denote by $\lambda : x \mapsto \lambda(x)$ the left-regular representation on the LSA $(\mathfrak{g}, \cdot)$: $\lambda(x)y = x \cdot y$. It is a Lie algebra representation:

$$\lambda : \mathfrak{g} \to \operatorname{End}(\mathfrak{g}), \qquad [\lambda(x), \lambda(y)] = \lambda([x, y])$$

Denote the corresponding $\mathfrak{g}$-module by $\mathfrak{g}_\lambda$. Furthermore, the identity map $\mathbf{1} : \mathfrak{g} \to \mathfrak{g}_\lambda$ is a 1-cocycle in $Z^1(\mathfrak{g}, \mathfrak{g}_\lambda)$: $\mathbf{1}([x, y]) = \mathbf{1}(x) \cdot y - \mathbf{1}(y) \cdot x$. Let $\mathfrak{aff}(\mathfrak{g})$ be the Lie algebra of $\mathbf{Aff}(G)$, i.e., $\mathfrak{aff}(\mathfrak{g}) = \left\{ \left(\begin{smallmatrix} A & b \\ 0 & 0 \end{smallmatrix}\right) \mid A \in \mathfrak{gl}(\mathfrak{g}),\ b \in \mathfrak{g} \right\}$ which we identify with $\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}$. Denote the linear part by $\ell(A, b) = A$ and the translational part by $t(A, b) = b$. Now we associate to the LSA $(\mathfrak{g}, \cdot)$ the map

$$\alpha = \lambda \oplus \mathbf{1} : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$$

It is an affine representation of $\mathfrak{g}$. We have $\lambda = \ell \circ \alpha$ and $t \circ \alpha = \mathbf{1}$. The corresponding affine representation of G is étale, see [SEG].
If $\alpha : G \to \mathbf{Aff}(E)$ is an étale affine representation, then its differential $\varrho : \mathfrak{g} \to \mathfrak{aff}(E)$ is a Lie algebra homomorphism such that the *evaluation map* $\operatorname{ev}_p : \mathfrak{g} \to E$, $x \mapsto \varrho(x)p = \theta(x)p + u(x)$ is an isomorphism for some $p \in E$, where $\theta : \mathfrak{g} \to \mathfrak{gl}(E)$ is a linear representation, say E_θ as module, and u is the translational part of ϱ. It suffices to look at $p = 0$. Then $u = \operatorname{ev}_0$ is a vector space isomorphism. Moreover $u \in Z^1(\mathfrak{g}, E_\theta)$, and hence $\lambda(a) = u^{-1} \circ \theta(a) \circ u$ defines an LSA–product via $a \cdot b = \lambda(a)b$ on $\mathfrak{g}$. $\square$

Milnor's question (1) now is equivalent to the algebraic question

(1') *Which Lie algebras admit LSA–structures ?*

Semisimple Lie algebras over characteristic zero do not admit LSA–structures. Certain reductive Lie algebras do, for example $\mathfrak{gl}(n)$. For more details see [BU1]. Milnor conjectured that every solvmanifold is affine, i.e., any solvable Lie algebra admits (complete) LSA–structures. It is the purpose of this paper to discuss counterexamples. We consider the nilpotent case. The following Proposition indicates where the support for Milnor's conjecture came from:

Proposition 1 *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of characteristic zero satisfying one of the following conditions:*

(1) $\dim \mathfrak{g} < 8$.
(2) $\mathfrak{g}$ *is* p *–step nilpotent with* $p < 4$.
(3) $\mathfrak{g}$ *is* $\mathbb{Z}$ *–graded.*
(4) $\mathfrak{g}$ *possesses a nonsingular derivation.*
(5) $\mathfrak{g}$ *is filiform nilpotent and a quotient of a higher–dimensional filiform nilpotent Lie algebra.*

Then $\mathfrak{g}$ *admits an LSA–structure.*

Proof: Condition (4) is a special case of the following: Let $\psi \in Z^1(\mathfrak{g}, \mathfrak{g}_\theta)$ be a nonsingular 1 –cocycle, where $\theta : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation.Then $\lambda(x) := \psi^{-1} \circ \theta(x) \circ \psi$ is defined and $\mathbf{1} \in Z^1(\mathfrak{g}, \mathfrak{g}_\lambda)$, i.e., λ defines an LSA–structure. If θ is the adjoint representation of $\mathfrak{g}$ then an invertible $\psi \in Z^1(\mathfrak{g}, \mathfrak{g})$ is a nonsingular derivation. Condition (3) implies condition (4) . The first two conditions are discussed in [BEN] and [MI2], the last one in [BU2]. □

The following Lemma is useful:

Lemma 2 *If* $\mathfrak{g}$ *admits an LSA–structure then* $\mathfrak{g}$ *has a faithful representation of dimension* $\dim \mathfrak{g} + 1$.

Proof: The LSA–structure on $\mathfrak{g}$ induces a *faithful* affine representation $\alpha : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$, called the *affine holonomy representation*, see [FGH]. If $\dim \mathfrak{g} = n$ then $\mathfrak{aff}(\mathfrak{g}) \subset \mathfrak{gl}(n+1)$ and we obtain a faithful linear representation of dimension $n+1$. □

Definition 3 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k . Define

$$\mu(\mathfrak{g}, k) := \min\{\dim_k M \mid M \text{ is a faithful } \mathfrak{g}\text{–module}\}$$

By Ado's Theorem (and Iwasawa's in prime characteristic) we know that μ is integer valued. Properties of μ are studied in [BU2]. We summarize here a few of them. Let us assume here $k = \mathbb{C}$.

Proposition 2 *Let* $\mathfrak{g}$ *be a Lie algebra of dimension* $n \geq 2$ *over* $\mathbb{C}$.

(1) *If* $\mathfrak{g}$ *is abelian then* $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1} \rceil$.
(2) *If* $\mathfrak{g}$ *has trivial center then* $\mu(\mathfrak{g}) \leq n$.
(3) *If* $\mathfrak{g}$ *is a Heisenberg Lie algebra* $\mathfrak{h}_{2m+1}$ *of dimension* $2m+1$ *, then* $\mu(\mathfrak{g}) = m+2$.
(4) *If* $\mathfrak{g}$ *is solvable then* $\mu(\mathfrak{g}) < \frac{\alpha}{\sqrt{n}} 2^n$ *with* $\alpha \sim 2.762872$
(5) *If* $\mathfrak{g}$ *is* p *–step nilpotent then* $\mu(\mathfrak{g}) < 1 + n^p$.
(6) *If* $\mathfrak{g}$ *is filiform nilpotent then* $n \leq \mu(\mathfrak{g}) < \frac{\sqrt{3}}{12} \exp(\pi \sqrt{2n/3})$.
(7) *If* $\mathfrak{g}$ *satisfies one of the conditions in Proposition* 1 *then* $\mu(\mathfrak{g}) \leq n+1$.

(8) *If* $\mathfrak{g}$ *is a quotient of a filiform nilpotent Lie algebra* $\mathfrak{g}'$ *with* $\dim \mathfrak{g}' > \dim \mathfrak{g} = n$ *then* $\mu(\mathfrak{g}) = n$.
(9) *If* $\mathfrak{g}$ *is filiform nilpotent with abelian commutator algebra then* $\mu(\mathfrak{g}) = n$.
(10) *If* $\mathfrak{g}$ *is filiform nilpotent of dimension* $n < 10$ *then* $\mu(\mathfrak{g}) = n$.

The key step for the construction of the counterexamples to the Milnor conjecture is to determine Lie algebras with

$$\mu(\mathfrak{g}) > \dim \mathfrak{g} + 1$$

In fact, we will construct filiform Lie algebras in dimensions $10, 11$ with that property. These Lie algebras are not quotients of any filiform Lie algebra of higher dimension.

3 Varieties of filiform Lie algebra structures

Let $\mathfrak{g}$ be a p-step nilpotent Lie algebra and let $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ and $\mathfrak{g}_0 = 0$, $\mathfrak{g}_k = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}_{k-1}\}$ for $k = 1, 2, \ldots$; the series

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \ldots \supset \mathfrak{g}^{p-1} \supset \mathfrak{g}^p = 0$$

is called *lower central series*, and the series

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_{p-1} \subset \mathfrak{g}_p = \mathfrak{g}$$

is called *upper central series*. They are both of length p. Define $F_i(\mathfrak{g}) = \mathfrak{g}^{i-1}$ and $M_i(\mathfrak{g}) = \mathfrak{g}_{p-i+1}$ for $i > 1$. Set $F_i(\mathfrak{g}) = M_i(\mathfrak{g}) = \mathfrak{g}$ for $i \leq 1$. It holds $[F_i(\mathfrak{g}), M_j(\mathfrak{g})] \subset M_{i+j}(\mathfrak{g})$ for $i, j \in \mathbb{Z}$. The series $F_i(\mathfrak{g})$ defines a filtration of $\mathfrak{g}$ and the series $M_i(\mathfrak{g})$ defines a filtration of the adjoint module $\mathfrak{g}$. These induce a filtration on the spaces of cochains, cocycles, coboundaries and cohomology.

Definition 4 A p-step nilpotent Lie algebra of dimension n is called *filiform nilpotent* if $p = n-1$. Let $L = L(n)$ be the Lie algebra generated by $e_0, \ldots, e_n$ with Lie brackets $[e_0, e_i] = e_{i+1}$ for $i = 1, 2, \ldots, n-1$ and the other brackets zero. L is called the *standard graded filiform of dimension* $n+1$.

Note that $F_i(\mathfrak{g}) = M_i(\mathfrak{g})$ for filiform Lie algebras. L is graded by

$$L = \bigoplus_{i \in \mathbb{Z}} L_i$$

where L_1 is generated by e_0 and e_1, L_i by e_i for $i = 2, 3, \ldots, n$ and the other subspaces are zero. Setting

$$C^j_q(L, L) = \{g \in C^j(L, L) \mid g(L_{i_1}, \ldots, L_{i_j}) \in L_{i_1 + \ldots + i_j + q} \;\; \forall \;\; 1 \leq i_1, \ldots, i_j < n\}$$

where $q \in \mathbb{Z}$ yields a $\mathbb{Z}$ –grading in the space $C^j(L, L)$ of j –cochains compatible with the coboundary operator d , i.e., $d(C^j_q(L, L)) \subset C^{j+1}_q(L, L)$. Hence we have assigned gradings to the spaces of cocycles and coboundaries compatible with the filtrations of the respective spaces:

$$F_k Z^j(L, L) = \bigoplus_{i \geq k} Z^j_i(L, L), \quad F_k B^j(L, L) = \bigoplus_{i \geq k} B^j_i(L, L),$$

$$F_k H^j(L, L) = \bigoplus_{i \geq k} H^j_i(L, L)$$

Denote by $\mathcal{L}_n(k)$ the affine algebraic variety of all Lie algebra structures in dimension n over k . A point in $\mathcal{L}_n(k)$ is a structural tensor $\{C^k_{ij}\}$ corresponding to a Lie algebra $\mathfrak{g}$ with basis $\{v_1, \ldots, v_n\}$ over k such that $[v_i, v_j] = \sum C^k_{ij} v_k$. The C^k_{ij} are called structure constants. They form a structural tensor $\gamma \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ which is identified with the bilinear skew-symmetric mapping $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ defining the Lie bracket on $\mathfrak{g}$. Nilpotent Lie algebras of class p form a subvariety $\mathcal{N}^p_n(k)$ of $\mathcal{L}_n(k)$. The group $\mathbf{GL}_n(k)$ acts on $\mathcal{L}_n(k)$ by $(g \cdot \gamma)(x, y) = g(\gamma(g^{-1}(x), g^{-1}(y)))$.
Two Lie algebras in $\mathcal{L}_n(k)$ are isomorphic iff they belong to the same orbit of $\mathbf{GL}_n(k)$. Let γ be a point of $\mathcal{L}_n(k)$ corresponding to the Lie algebra $\mathfrak{g}$. Then the Zariski tangent space to $\mathcal{L}_n(k)$ at the point γ coincides with $Z^2(\mathfrak{g}, \mathfrak{g})$, and the tangent space to the orbit of $\mathfrak{g}$ at γ coincides with $B^2(\mathfrak{g}, \mathfrak{g})$.

Let k be $\mathbb{R}$ or $\mathbb{C}$: A *deformation* in $\mathcal{L}_n(k)$ is a continuous curve $c : [0, \epsilon] \to \mathcal{L}_n(k)$, $t \mapsto \mathfrak{g}(t)$. For $t \in [0, \epsilon]$ denote by $\mathfrak{g}(t)$ the Lie algebra corresponding to the structural tensor $c(t) \in \mathcal{L}_n(k)$. If c is analytic we have the convergent series

$$c(t) = F_0 + F_1 \cdot t + F_2 \cdot t^2 + \ldots$$

where $F_m = \{F_m\}^k_{ij} = \frac{c^{(m)}(0)}{m!} \in k^{n^3}$. We may also consider formal series and formal deformations. The corresponding Lie algebra bracket is given by

$$[a, b]_t = F_0(a, b) + F_1(a, b)t + F_2(a, b)t^2 + \ldots$$

where $[a, b]_0 = [a, b]$ is the bracket in $\mathfrak{g}$. Then the Jacobi identity for $[\, ,]_t$ implies $F_1 \in Z^2(\mathfrak{g}, \mathfrak{g})$. These 2 –cocycles are called *infinitesimal deformations.* If F_1 corresponds to a deformation (which is not always the case) then F_1 is said to be *integrable.* If we have two equivalent (formal) deformations, i.e., if the corresponding Lie algebras are isomorphic, then F_1 and F'_1 corresponding to these deformations are cohomological, i.e., $F_1 - F'_1 \in B^2(\mathfrak{g}, \mathfrak{g})$.

The following Proposition describes filiform Lie algebra structures in the variety $\mathcal{N}^{n-1}_n(k)$ (see [VER]):

Proposition 3 (Vergne) *Every filiform nilpotent Lie algebra of dimension $n + 1 \geq 8$ is isomorphic to an infinitesimal deformation of the standard graded $n + 1$ –dimensional* *filiform L . More precisely it is isomorphic to an algebra $(L)_\psi$ where ψ is an integrable 2 –cocycle whose cohomology class lies in*

$$\begin{array}{lll} F_1H^2(L,L) & & \text{if} \quad n\equiv 0(2) \\ F_1H^2(L,L) \ + & <\psi_{\frac{n-1}{2},n}> & \text{if} \quad n\equiv 1(2) \end{array}$$

Here the algebra $\mathfrak{g}_\psi=(L)_\psi$ is defined by the bracket $[a,b]_\psi=[a,b]_L+\psi(a,b)$.
The fact that this bracket satisfies the Jacobi identity means that ψ is integrable, i.e., satisfies

$$(J) \qquad \psi(a,\psi(b,c))+\psi(b,\psi(c,a))+\psi(c,\psi(a,b))=0.$$

The *canonical* 2 –cocycles $\psi_{k,s}$ are defined by

$$\psi_{k,s}(e_i,e_{i+1})=\delta_{ik}e_s$$

for pairs (k,s) with $1\le k\le n-1$ and $2k\le s\le n$. Hakimjanov proved ([HAK]):

Proposition 4 *The cohomology classes of the cocycles $\psi_{k,s}$ with $1\le k\le n,\ 4\le s\le n,\ 2k+1\le s$ form a basis of $F_0H^2(L,L)$. This space has dimension $\frac{n^2-2n-3}{4}$ if n is odd, and dimension $\frac{n^2-2n-4}{4}$ if n is even.*

Now it is not difficult to see that the classes of $\psi_{k,s}$ with $1\le k\le [n/2]-1,\ 2k+2\le s\le n$ form a basis for $F_1H^2(L,L)$. Hence, with $\dim L=n+1$,

$$\dim F_1H^2(L,L)=\begin{cases}\dfrac{(n-2)^2}{4}, & n\equiv 0(2)\\[2ex] \dfrac{(n-3)(n-1)}{4}, & n\equiv 1(2)\end{cases}$$

We also have

$$(P) \qquad \psi_{k,s}(e_i,e_j)=(-1)^k\binom{j-k-1}{k-i}(\mathrm{ad}e_0)^{i+j-2k-1}e_s$$

for $1\le i<k<j-1\le n-1$. In case $i>k$, $\psi_{k,s}(e_i,e_j)=0$ and $\psi_{k,s}(e_k,e_j)=e_{s+j-k-1}$ for $k<j$.

4 Filiform Lie algebras in dimension 10

Let $L=L(9)=<e_0,e_1,\dots,e_9>$ be the standard graded filiform Lie algebra of dimension 10. According to Proposition 3 every filiform nilpotent Lie algebra of dimension 10 is isomorphic to $\mathfrak{g}_\psi = (L)_\psi$ for some $\psi \in F_1 H^2(L,L) + < \psi_{4,9} >$. In terms of the basis of this cohomology space we may write

$$\begin{aligned} \psi \quad = \quad & \alpha_1 \psi_{1,4} + \alpha_2 \psi_{1,5} + \ldots + \alpha_6 \psi_{1,9} \\ & + \alpha_7 \psi_{2,6} + \ldots + \alpha_{10} \psi_{2,9} \\ & + \alpha_{11} \psi_{3,8} + \alpha_{12} \psi_{3,9} \\ & + \alpha_{13} \psi_{4,9} \end{aligned}$$

For the convenience of the reader we will calculate ψ on the basis by using (P) . We only list nonzero values:

$$\begin{aligned}
\psi(e_1,e_2) &= \alpha_1 e_4 + \alpha_2 e_5 + \cdots + \alpha_6 e_9 \\
\psi(e_1,e_3) &= \alpha_1 e_5 + \alpha_2 e_6 + \cdots + \alpha_5 e_9 \\
\psi(e_1,e_4) &= (\alpha_1 - \alpha_7) e_6 + (\alpha_2 - \alpha_8) e_7 + \cdots + (\alpha_4 - \alpha_{10}) e_9 \\
\psi(e_1,e_5) &= (\alpha_1 - 2\alpha_7) e_7 + (\alpha_2 - 2\alpha_8) e_8 + (\alpha_3 - 2\alpha_9) e_9 \\
\psi(e_1,e_6) &= (\alpha_1 - 3\alpha_7 + \alpha_{11}) e_8 + (\alpha_2 - 3\alpha_8 + \alpha_{12}) e_9 \\
\psi(e_1,e_7) &= (\alpha_1 - 4\alpha_7 + 3\alpha_{11}) e_9 \\
\psi(e_1,e_8) &= -\alpha_{13} e_9 \\
\psi(e_2,e_3) &= \alpha_7 e_6 + \alpha_8 e_7 + \cdots + \alpha_{10} e_9 \\
\psi(e_2,e_4) &= \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9 \\
\psi(e_2,e_5) &= (\alpha_7 - \alpha_{11}) e_8 + (\alpha_8 - \alpha_{12}) e_9 \\
\psi(e_2,e_6) &= (\alpha_7 - 2\alpha_{11}) e_9 \\
\psi(e_2,e_7) &= \alpha_{13} e_9 \\
\psi(e_3,e_4) &= \alpha_{11} e_8 + \alpha_{12} e_9 \\
\psi(e_3,e_5) &= \alpha_{11} e_9 \\
\psi(e_3,e_6) &= -\alpha_{13} e_9 \\
\psi(e_4,e_5) &= \alpha_{13} e_9
\end{aligned}$$

The cocycle ψ is integrable iff (J) holds, i.e., iff $[a,b]_\psi = [a,b]_L + \psi(a,b)$ satisfies the Jacobi identity. This is equivalent to the following equations:

$$\begin{aligned}
&(1) \qquad \alpha_{13}(2\alpha_3 + \alpha_9) - \alpha_{12}(2\alpha_1 + \alpha_7) - 3\alpha_{11}(\alpha_2 + \alpha_8) + 7\alpha_7\alpha_8 = 0 \\
&(2) \qquad \alpha_{11}(2\alpha_1 + \alpha_7) - 3\alpha_7^2 = 0 \\
&(3) \qquad \alpha_{13}(2\alpha_1 - \alpha_7 - \alpha_{11}) = 0
\end{aligned}$$

Since the integrability conditions are pleasantly simple we obtain the following filiform Lie algebras $\mathfrak{g}_\psi$ with bracket $[a,b]_\psi$:

Case A: $2\alpha_1+\alpha_7\neq 0$:

$(A1)$	$\alpha_1\neq 0,$	$\alpha_7=-\alpha_1,\ \alpha_{11}=3\alpha_1,$ $\alpha_{12}=(\alpha_{13}\alpha_1^{-1})(2\alpha_3+\alpha_9)-(9\alpha_2+16\alpha_8)$
$(A2)$	$\alpha_1\neq 0,$	$\alpha_{11}=\alpha_7=\alpha_1,$ $\alpha_{12}=-(\alpha_1(3\alpha_2-4\alpha_8)-\alpha_{13}(2\alpha_3+\alpha_9))/(3\alpha_1)$
$(A3)$	$\alpha_1\neq 0,\ \alpha_7^2\neq\alpha_1^2,$	$\alpha_{13}=0,\ \alpha_{11}=3\alpha_7^2/(2\alpha_1+\alpha_7),$ $\alpha_{12}=\alpha_7(14\alpha_1\alpha_8-9\alpha_2\alpha_7-2\alpha_7\alpha_8)/(2\alpha_1+\alpha_7)^2$

Case B: $2\alpha_1+\alpha_7=0$:

$(B1)$ $\qquad \alpha_{13}=\alpha_7=\alpha_1=0,\ \alpha_{11}(\alpha_2+\alpha_8)=0$

$(B2)$ $\qquad \alpha_{13}\neq 0,\ \alpha_{11}=\alpha_7=\alpha_1=0,\ \alpha_9=-2\alpha_3$

We have the following result for the minimal dimension of faithful modules for these classes of Lie algebras:

Proposition 5 *If* $\mathfrak{g}_\psi$ *is a filiform Lie algebra of class* $A3, B1, B2$ *then* $\mu(\mathfrak{g}_\psi)=10$ *; if* $\mathfrak{g}_\psi$ *is of class* $A1,\ A2$ *satisfying the additional condition* $3\alpha_2+\alpha_8=0$ *in case of class* $A1$ *, then* $\mu(\mathfrak{g}_\psi)=10$ *or* 11.

Proof: The above Lie algebras are generated by e_0, e_1 and have one–dimensional center $\mathfrak{z}=<e_9>$. Let $\varrho:\mathfrak{g}_\psi\to\mathfrak{gl}(M)$ be a faithful representation. Then $\dim M\geq 10$ and the faithfulness of ϱ is equivalent to $\varrho(e_9)\neq 0$ (see [BGR]). Define $E_i=\varrho(e_i)$. The module M is generated by E_0 and E_1. We call such a module M a Δ *–module* if $\dim M=11$ and if M is nilpotent, i.e., every $\varrho(x)$ is nilpotent. We will now construct Δ –modules M :
There is a basis $\{f_1,f_2,\dots,f_{11}\}$ for M such that E_0,E_1 are simultaneously strictly upper triangular matrices and, moreover, such that there is in each row and each column of E_0 at most one nonzero entry (see [BGR]). Note that the *center* Z of a Δ –module is $\ker(E_0)\cap\ker(E_1)$ and contains f_1 . Any subspace U of Z is a submodule, and the quotient module will be faithful iff f_1 is not in $U.$ Since $\mu(\mathfrak{g}_\psi)\geq 10$ the dimension of Z is at most 2.
Define the *first layer* of E_0 to be the first upper diagonal, say $\{\lambda_1,\lambda_2,\dots,\lambda_{10}\}$, the *second layer* the second upper diagonal $\{\lambda_{11},\lambda_{12},\dots,\lambda_{19}\}$ and so forth. Let N_1 denote the set of indices i such that $\lambda_i=0$ in the first layer of E_0 , N_j the set of indices i such that $\lambda_i=1$ in the j –th layer of E_0 for $j=2,3,\dots,10$.

Definition 5 Define the *combinatorical type* of a Δ –module M (the *type* of E_0) to be

$$\text{type}(M)=\{N_1\mid N_2\mid\dots\mid N_{10}\}.$$

Of course, this notation generalizes to n –dimensional filiform Lie algebras and their Δ –modules of dimension $n+1$.
Empty sets N_i are omitted in this notation. If E_0 is of full Jordan block form, i.e., if $N_j = \emptyset$ for all j then we set $\text{type}(M) = \emptyset$. Not all types are faithful. It is easy to see that the faithfulness of M depends only on the first and second layers of E_0, E_1 (the formulas for E_9 contain only those elements from E_0, E_1 , see [BGR]). It is possible to give a list of all faithful types. Moreover we can reduce the number of types as follows:

Lemma 3 *Let M be a Δ - module for $\mathfrak{g}_\psi$. Then we may assume that the type of M is one of the following:*

(1)	$\emptyset$		
(2)	$\{i\}$	$i = 5, ..., 10$	
(3)	$\{i, i+1\}$	$i = 5, ..., 9$	
(4)	$\{i, i+1 \mid 10+i\}$	$i = 5, ..., 9$	
(5)	$\{i, i+1, j \mid 10+i\}$	$i = 5, 6, 7$	$j > i+2$
(6)	$\{i, j, j+1 \mid 10+j\}$	$j = 6, 7, 8, 9$	$i < j-1$

The proof is exactly the same as for Lemma 3.2 in [BGR]. Note that types with additional entries r in the third or higher layer can also be reduced to one of the types listed (by constructing a module with $r = 0$). This was not mentioned in [BGR].

To construct the Δ –modules M we specify the type of M . We obtain equations in the entries of E_1 . It turns out that, once solved the equations involving the first and second layer of E_1 , the remaining equations can always be easily solved by substitutions of certain x_i appearing as *linear* monomials. Thus we will describe the Δ –modules (which prove the claim of the Proposition) by specifying the *type of* E_0 and the *first and second layer* of E_1 . The complete solution may be found in the appendix [APP].
Denote by $\overline{\mathfrak{g}}_\psi$ the graded filiform algebra *associated* to $\mathfrak{g}_\psi$ (induced by the natural filtration of $\mathfrak{g}_\psi$ by degree). Also, for a $\mathfrak{g}_\psi$ –module M denote the associated $\overline{\mathfrak{g}}_\psi$ –module by $\overline{M}$. It is obtained from M by considering the filtration $M^0 = M^1 = M$ and $M^{i+1} = E_0 M^i + E_1 M_{i-1}$ and forming the associated graded module. The first two layers of E_0, E_1 describe $\overline{M}$. Using the coefficients of $\operatorname{ad} e_1$ of $\overline{\mathfrak{g}}_\psi$ we construct the second layer of E_1 for M. For a more precise statement, see Remark 1 below. Let $f = f_{11}, U =< f >$, and $E_9(i,j)$ denote the entry of E_9 at position (i,j) . The constructed modules are as follows:

Case $B1$, $\alpha_{11} = 0$:

E_0 is of type $\{1, 10\}$
First layer of E_1 : $\{1,0,0,0,0,0,0,0,0,0\}$
Second layer of E_1 : $\{0,0,0,0,0,0,0,1,0\}$
The center of M is generated by f_1, f and $E_9(1,11) = 1$. We obtain a faithful module M/U of dimension 10 .

Case $B1$, $\alpha_2 + \alpha_8 = 0$:

E_0 is of type $\{1,10\}$
First layer of E_1 : $\{1,0,0,0,0,0,0,0,0,0\}$
Second layer of E_1 : $\{0,0,0,0,0,0,-\frac{\alpha_{11}}{2},-\frac{8\alpha_{11}}{5},0\}$
The center of M is generated by f_1, f and $E_9(1,11)=1$. M/U is a faithful module of dimension 10 .

Case $B2$:

E_0 is of type $\{1,10\}$
First layer of E_1 : $\{1,0,0,0,0,0,0,0,-\frac{\alpha_{13}}{2},0\}$
Second layer of E_1 : $\{0,0,0,0,0,0,0,0,0\}$
The center of M is generated by f_1, f and $E_9(1,11)=1$. M/U is a faithful module of dimension 10 .

Case $A3$:

E_0 is of type $\{10\}$
First layer of E_1 : $\{1,0,0,0,0,0,0,0,0,0\}$
Second layer of E_1 : $\{0,\ -\alpha_1,\ -\alpha_1,\ \alpha_7-\alpha_1,\ 2\alpha_7-\alpha_1,\ \frac{\alpha_1(5\alpha_7-2\alpha_1)}{2\alpha_1+\alpha_7},\ \frac{(5\alpha_7-2\alpha_1)(\alpha_1-\alpha_7)}{2\alpha_1+\alpha_7},$ $(5\alpha_7^3-2\alpha_1^3+10\alpha_1^2\alpha_7+16\alpha_1\alpha_7^2)/2(\alpha_1^2-\alpha_7^2),\ 0\}$
The center of M is generated by f_1, f and $E_9(1,11)=1$. M/U is a faithful module of dimension 10 .

Case $A2$ *satisfying* $\alpha_{13}\neq 0$ *and* $33\alpha_2-20\alpha_8=0$:

E_0 is of type $\{9\}$
First layer of E_1 : $\{-\frac{10\alpha_{13}}{11},0,0,0,0,0,0,0,0,0\}$
Second layer of E_1 : $\{-\frac{23\alpha_2\alpha_{13}}{22\alpha_1},-\alpha_1,-\alpha_1,0,\alpha_1,\alpha_1,0,1,-2\}$
The center of M is generated by f_1 and $E_9(1,11)=-10$. M is a faithful module of dimension 11 .

Case $A2$ *satisfying* $\alpha_{13}=0$ *and* $33\alpha_2-20\alpha_8=0$:

E_0 is of type $\{9\}$
First layer of E_1 : $\{0,0,0,0,0,0,0,0,0,0\}$
Second layer of E_1 : $\{0,-\alpha_1,-\alpha_1,0,\alpha_1,\alpha_1,0,1,-2\}$
The center of M is generated by f_1 and $E_9(1,11)=-10$. M is a faithful module of dimension 11 .

Case $A2$ *satisfying* $\alpha_{13}\neq 0$, $\gamma=33\alpha_2-20\alpha_8\neq 0$ *and* $726\alpha_1^2-\gamma\alpha_{13}=0$:

E_0 is of type $\{1,9,10 \mid 19\}$

First layer of E_1 : $\{1,0,0,0,0,0,0,\frac{\alpha_{13}}{11},0,\frac{1}{\alpha_1^2}\}$

Second layer of E_1 : $\{0,-\alpha_1,-\alpha_1,0,\alpha_1,\alpha_1,0,\frac{2\alpha_1^2\alpha_{13}^2}{121},-\frac{2\alpha_{13}}{11}\}$

The center of M is generated by f_1 and $E_9(1,11)=1$. M is a faithful module of dimension 11 .

Case A2 satisfying $\alpha_{13}\neq 0$, $\gamma=33\alpha_2-20\alpha_8\neq 0$ *and* $726\alpha_1^2-\gamma\alpha_{13}\neq 0$:

E_0 is of type $\{9,10 \mid 19\}$

First layer of E_1 : $\{-\frac{660\alpha_1^2\alpha_{13}}{726\alpha_1^2-\gamma\alpha_{13}},0,0,0,0,0,0,\frac{66\alpha_1^2}{\gamma},0,\frac{8712\alpha_1^4}{\gamma^2}\}$

Second layer of E_1 : $\{0,-\alpha_1,-\alpha_1,0,\alpha_1,\alpha_1,0,1,-\frac{132\alpha_1^2}{\gamma}\}$

The center of M is generated by f_1 and

$E_9(1,11)=-\frac{479160\alpha_1^4}{\gamma(726\alpha_1^2-\gamma\alpha_{13})}$. M is a faithful module of dimension 11 .

Case A2 satisfying $\alpha_{13}=0$ *and* $\gamma=33\alpha_2-20\alpha_8\neq 0$:

E_0 is of type $\{9,10 \mid 19\}$

First layer of E_1 : $\{0,0,0,0,0,0,0,\frac{66\alpha_1^2}{\gamma},0,\frac{8712\alpha_1^4}{\gamma^2}\}$

Second layer of E_1 : $\{0,-\alpha_1,-\alpha_1,0,\alpha_1,\alpha_1,0,1,-\frac{132\alpha_1^2}{\gamma}\}$

The center of M is generated by f_1 and

$E_9(1,11)=-\frac{660\alpha_1^2}{\gamma}$. M is a faithful module of dimension 11 .

Case A1 satisfying $3\alpha_2+\alpha_8=0,\ \alpha_2\neq 0$ *and* $22\alpha_1^2-\alpha_2\alpha_{13}=0$:

E_0 is of type $\{1,9,10 \mid 19\}$

First layer of E_1 : $\{1,0,0,0,0,0,0,\frac{\alpha_{13}}{11},0,\frac{1}{\alpha_1^2}\}$

Second layer of E_1 : $\{0,7\alpha_1,3\alpha_1,2\alpha_1,\alpha_1,\alpha_1,0,\frac{2\alpha_1^2\alpha_{13}^2}{121},-\frac{2\alpha_{13}}{11}\}$

The center of M is generated by f_1 and $E_9(1,11)=1$. M is a faithful module of dimension 11 .

Case A1 satisfying $3\alpha_2+\alpha_8=0,\ \alpha_{13},\alpha_2\neq 0$ *and* $\gamma=22\alpha_1^2-\alpha_2\alpha_{13}\neq 0$:

E_0 is of type $\{9,10 \mid 19\}$

First layer of E_1 : $\{-\frac{20\alpha_1^2\alpha_{13}}{\gamma},0,0,0,0,0,0,\frac{2\alpha_1^2}{\alpha_2},0,\frac{8\alpha_1^4}{\alpha_2^2}\}$

Second layer of E_1 : $\{0,7\alpha_1,3\alpha_1,2\alpha_1,\alpha_1,\alpha_1,-\frac{7\alpha_1\gamma}{5\alpha_2\alpha_{13}},1,-\frac{4\alpha_1^2}{\alpha_2}\}$

The center of M is generated by f_1 and $E_9(1,11)=-\frac{440\alpha_1^4}{\gamma\alpha_2}$. M is a faithful module of dimension 11 .

Case $A1$ *satisfying* $3\alpha_2+\alpha_8=0$, $\alpha_{13}=0$ *and* $\alpha_2\neq 0$:

E_0 is of type $\{9,10 \mid 19\}$

First layer of E_1 : $\{0,0,0,0,0,0,0,\frac{2\alpha_1^2}{\alpha_2},0,\frac{8\alpha_1^4}{\alpha_2^2}\}$

Second layer of E_1 : $\{14\alpha_1,7\alpha_1,3\alpha_1,2\alpha_1,\alpha_1,\alpha_1,0,1,-\frac{4\alpha_1^2}{\alpha_2}\}$

The center of M is generated by f_1 and $E_9(1,11)=-\frac{20\alpha_1^2}{\alpha_2}$. M is a faithful module of dimension 11 .

Case $A1$ *satisfying* $3\alpha_2+\alpha_8=0$, $\alpha_{13}\neq 0$ *and* $\alpha_2=0$:

E_0 is of type $\{9\}$

First layer of E_1 : $\{-\frac{10\alpha_1}{11},0,0,0,0,0,0,0,\alpha_{13},0,2\alpha_{13}\}$

Second layer of E_1 : $\{0,7\alpha_1,3\alpha_1,2\alpha_1,\alpha_1,\alpha_1,-\frac{77\alpha_1}{5},\alpha_{13},-2\alpha_{13}\}$

The center of M is generated by f_1 and $E_9(1,11)=-10\alpha_{13}$. M is a faithful module of dimension 11 .

Case $A1$ *satisfying* $3\alpha_2+\alpha_8=0$ *and* $\alpha_{13},\alpha_2=0$:

E_0 is of type $\{9\}$

First layer of E_1 : $\{0,0,0,0,0,0,0,0,0,0,0\}$

Second layer of E_1 : $\{14\alpha_1,7\alpha_1,3\alpha_1,2\alpha_1,\alpha_1,\alpha_1,0,1,-2\}$

The center of M is generated by f_1 and $E_9(1,11)=-10$. M is a faithful module of dimension 11 .

It is easy to check that we have indeed constructed faithful modules of dimension 10 or 11 for *all* Lie algebras $\mathfrak{g}_\psi$ except for those of class $A1$ satisfying $3\alpha_2+\alpha_8\neq 0$. $\square$

Remark 1 For the graded algebra $\overline{\mathfrak{g}}_\psi$ we have $[e_1,e_i]_{gr}=\beta_i e_{2+i}$. Consider the set $\{\beta_2,\dots,\beta_6\}$. As an example, for $\overline{\mathfrak{g}}_\psi$ of class $A1$ we obtain

$$\{\beta_2,\dots,\beta_6\}=\{\alpha_1,\alpha_1,2\alpha_1,3\alpha_1,7\alpha_1\}.$$

Let $\{y_1,\dots,y_{10}\}$ be the second layer of E_1 for M . The set of coefficients $\{y_2,\dots,y_6\}$ (or their negative ones) coincides with $\{\beta_2,\dots,\beta_6\}$.

The next Proposition shows that the "missing" case indeed provides counterexamples to the Milnor conjecture:

Proposition 6 *Let* $\mathfrak{g}_\psi=\mathfrak{g}(\alpha_1,\dots,\alpha_{13})$ *be a Lie algebra of class* $A1$ *, satisfying* $3\alpha_2+\alpha_8\neq 0$. *Then* $12\leq\mu(\mathfrak{g}_\psi)\leq 22$.

Proof: Let $\mathfrak{g}_\psi$ be any filiform nilpotent Lie algebra of dimension 10 . Suppose there is *any* faithful module M of dimension $m < 12$. By Lemma 3.2. in [BEN] we may assume that M is nilpotent and is of dimension 11. Then is has to be isomorphic to one of the Δ –modules listed in Lemma 3 . We have to check, for *every* type of this list, if the equations for the coefficients of E_1 have a solution or not. Many of the types (especially (5), (6)) are done after just solving a few linear equations. The result of the computations then is exactly Proposition 5 and 6 . In fact, the solutions for the classes of Proposition 5 were determined just by this procedure. In [BGR] we have written up the computations in detail for 11 –dimensional algebras. The computations here are similar but a great deal simpler. As an example for a more difficult type, assume that E_0 is of type $\{10\}$,

First layer of E_1 : $\{x_1, x_2, \dots, x_{10}\}$

Second layer of E_1 : $\{x_{11}, \dots, x_{19}\}$

Examining the module-equations we find (assume $x19 \neq 0$): $x_8 = 2x_7,\, x_7 = 0,\, x_5x_6 = 0,\, 3x_6 = 2x_5$ and $x_3 = x_4 = 0,\, 7x_2 = 2x_1 + \alpha_{13}$. Then $x_1(2x_1+\alpha_{13}) = \alpha_{13}(9x_1+8\alpha_{13}) = 0$. This implies $x_1 = x_2 = \alpha_{13} = 0$. Furthermore $x_{17} = 2x_{16} - \alpha_1$, $x_{16} = 3x_{15} - 2x_{14} + \alpha_1$ and $x_{15} = 4x_{14} - 5x_{13} + 2x_{12} - 3\alpha_1$. Then we find eight equations in $x_{11}, x_{12}, x_{13}, x_{14}$ and α_1 (see [APP]). They have only the trivial solution, i.e., $\alpha_1 = 0$, contradiction. If $x_{19} = 0$ it follows $\alpha_1 = 0$ by similar computations.

If $\mathfrak{g}_\psi$ is of class $A1$ (with $\alpha_{13} \neq 0$) then we obtain very soon a contradiction except for the following types:

$$\{9\},\ \{1,9,10 \mid 19\},\ \{1,8,9 \mid 18\},\ \{9,10 \mid 19\},\ \{8,9 \mid 18\}$$

However, in these cases it follows $3\alpha_2 + \alpha_8 = 0$.

Let $\mathfrak{g}$ denote $\mathfrak{g}_\psi$ of class $A1$ satisfying $3\alpha_2 + \alpha_8 \neq 0$. The universal enveloping algebra $U(\mathfrak{g})$ has a basis of ordered monomials $e^\alpha = e_9^{\alpha_9} \cdots e_0^{\alpha_0}$ with an order function. We have $\mathrm{ord}(e_0) = 1$ and $\mathrm{ord} e_i = i$ for $i \geq 1$ (for details see [BGR]). Let

$$U^m(\mathfrak{g}) = \{T \in U(\mathfrak{g}) \ :\ \mathrm{ord}(T) \geq m\}$$

$U^m(\mathfrak{g})$ is an ideal of $U(\mathfrak{g})$ of finite codimension. Define $V = U(\mathfrak{g})/U^m(\mathfrak{g})$.
One can show that V is a faithful $\mathfrak{g}$ –module if m is greater than the nilpotency class of $\mathfrak{g}$. Take $m = 10$. Then V has a vector space basis above has the vector space basis

$$\{e_9^{\alpha_9} \cdots e_0^{\alpha_0} \mid 9\alpha_9 + \cdots + 2\alpha_2 + \alpha_1 + \alpha_0 \ \leq\ 9\}$$

The elements e_i of $\mathfrak{g}$ act on V by $e_ie_j = [e_i, e_j] + e_je_i$ for $i < j$. Consider the following quotient module $\widehat{V}$ of V with vector space basis:

$$\{e_9, e_8, e_4^2, e_7, e_4e_3, e_3e_2^2, e_6, e_4e_2, e_4e_1^2, e_3^2, e_3e_2e_1, e_3e_1^3, e_2^3, e_2^2e_1^2, e_5,$$
$$e_4e_1, e_3e_2, e_3e_1^2, e_2^2e_1, e_2e_1^3, e_1^5, e_4, e_3e_1, e_2^2, e_2e_1^2, e_1^4, e_3, e_2e_1, e_1^3, e_2, e_1^2, e_1, 1\}$$

We have constructed a faithful $\mathfrak{g}$ – module $\widehat{V}$ of dimension 33 ; Passing succesively to faithful quotient modules we obtain a faithful representation of dimension 22. $\square$

Remark 2 Let G be a connected simply connected Lie group of dimension 10 with filiform nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}(\alpha_1, \dots, \alpha_{13})$. If $\mathfrak{g}$ is of class $A1$ with $3\alpha_2 + \alpha_8 \neq 0$ then it follows from Proposition 6 that G does not admit any left-invariant affine structure.
A natural question is, whether all other classes actually do admit left-invariant affine structures. We don't believe this to be true, i.e., there should be Lie algebras $\mathfrak{g}$ with $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$ without any LSA–structure (e.g. some of the algebras of class A_1, A_2). On the other hand we may construct left-invariant affine structures (i.e., LSA–structures) for certain subclasses of $A3, B1, B2$.

Proposition 7 *Let $\mathfrak{g}(\alpha_1, \dots, \alpha_{13})$ be a filiform nilpotent Lie algebra of dimension 10 . If $\mathfrak{g}$ satisfies on of the following conditions, then $\mathfrak{g}$ admits an LSA-structure.*

(1) $\mathfrak{g}$ *is of class* $A3$ *with* $\mathfrak{g} \neq \mathfrak{g}(\alpha_1, \dots, \alpha_6, 0, 0, 0, \alpha_{10}, 0, 0, 0)$ *where* $\alpha_1, \alpha_{10} \neq 0.$
(2) $\mathfrak{g}$ *is of class* $B1$ *with* $\alpha_{11} \neq 0$ *and* $\alpha_2 = \alpha_3 = \alpha_8 = 0$
(3) $\mathfrak{g}_\psi$ *is of class* $B1$ *with* $\alpha_{11} = 0$ *and* $\alpha_2(\alpha_2 - 8\alpha_8 + 6\alpha_{12}) = 0$
(4) $\mathfrak{g}_\psi$ *is of class* $B2$ *and* $2\alpha_8 = 5\alpha_2$ *and* $50\alpha_4 = \alpha_2(42\alpha_{12} - 133\alpha_2)/\alpha_{13}$

Proof: If $\mathfrak{g}$ is a quotient of a filiform nilpotent Lie algebra $\mathfrak{h}$ of dimension 11 , then $\mathfrak{g}$ admits an LSA–structure by Proposition $1(5)$. Then there is a surjective homomorphism $\mathfrak{h} \to \mathfrak{g}$ with one–dimensional kernel. This property is well suited for computations. The above algebras are precisely those admitting such a homomorphism.

There is another construction to obtain LSA-structures, only depending on $\operatorname{ad} e_1$ of the Lie algebra.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension m over k with basis $\{e_0, e_1, \dots, e_{m-1}\}$ such that $[e_0, e_i] = e_{i+1}$ and $\operatorname{ad} e_1$ maps $k\{e_i, \dots, e_{m-1}\}$ into $k\{e_{i+1}, \dots, e_{m-1}\}$. Let $\mathfrak{g}$ be generated by e_0, e_1 . Define linear maps $\lambda(e_i) : \mathfrak{g} \to \mathfrak{g}$ by

$$\lambda(e_0)e_i = \begin{cases} 0, & i = 0 \\ \dfrac{(i-1)}{i} e_{i+1}, & i \geq 1 \end{cases}$$

and $\lambda(e_1) = \operatorname{ad} e_1$, $\lambda(e_{i+1}) = [\lambda(e_0), \lambda(e_i)]$. They induce a linear map $\lambda : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$. It follows $\lambda(e_0)e_i - \lambda(e_i)e_0 = e_{i+1}$ for $i \geq 1$ by induction. In fact, $\lambda(e_i)e_0 = -\frac{1}{i}e_{i+1}$. Define a map $\alpha : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$ by

$$\alpha(e_0) = \begin{pmatrix} \lambda(e_0) & e_0 \\ 0 & 0 \end{pmatrix}, \quad \alpha(e_1) = \begin{pmatrix} \lambda(e_1) & e_1 \\ 0 & 0 \end{pmatrix}, \quad \alpha(e_{i+1}) = [\alpha(e_0), \alpha(e_i)],$$

where $e_0 = (1, 0, \dots, 0)$, $e_1 = (0, 1, 0, \dots, 0), \dots, e_{m-1} = (0, \dots, 0, 1)$. Again by induction (using $\lambda(e_0)e_{i-1} - \lambda(e_{i-1})e_0 = e_i$) we obtain

$$\alpha(e_i) = [\alpha(e_0), \alpha(e_{i-1})] = \begin{pmatrix} \lambda(e_i) & e_i \\ 0 & 0 \end{pmatrix}.$$

Hence the image of $\mathfrak{g}$ under α is indeed inside of $\mathfrak{aff}(\mathfrak{g})$. *However, is α an affine representation of $\mathfrak{g}$?* Note that α is an affine representation iff λ is a linear representation. In this case, $\lambda(b) = a \cdot b$ defines an LSA–structure on $\mathfrak{g}$. In general, λ need not be a representation. In low dimensions however, it is very often a reprensentation. The following result is true for filiform Lie algebras of dimension 10 :

Proposition 7 *Let $\mathfrak{g}_\psi$ be a filiform nilpotent Lie algebra of dimension 10. Then the map α defined above is an affine representation of $\mathfrak{g}_\psi$ iff*

(1) $\mathfrak{g}_\psi$ *is of class* $A3$ *and* $10\alpha_7 = \alpha_1$, $22\alpha_8 = 3\alpha_2$, $\alpha_9 = 3(385\alpha_1\alpha_3 + 23\alpha_2^2)/(8085\alpha_1)$
(2) $\mathfrak{g}_\psi$ *is of class* $B1$ *with* $\alpha_{11} \neq 0$ *and* $\alpha_2 = \alpha_3 = \alpha_8 = 0$
(3) $\mathfrak{g}_\psi$ *is of class* $B1$ *with* $\alpha_{11} = 0$ *and* $\alpha_2(\alpha_2 - 8\alpha_8 + 6\alpha_{12}) = 0$
(4) $\mathfrak{g}_\psi$ *is of class* $B2$ *and* $2\alpha_8 = 5\alpha_2$ *and* $50\alpha_4 = \alpha_2(42\alpha_{12} - 133\alpha_2)/\alpha_{13}$

Proof: From the matrix equation $[E_1, E_2] = \alpha_1 E_4 + \cdots + \alpha_6 E_9$ we obtain immediately $\alpha_1(10\alpha_7 - \alpha_1) = 0$, compute the entry at position $(8,3)$. Likewise calculations prove the result. $\square$

5 Filiform Lie algebras in dimension 11

Let $L = L(10) = < e_0, e_1, \dots, e_{10} >$ be the standard graded filiform Lie algebra of dimension 11. Then every filiform nilpotent Lie algebra of dimension 11 is isomorphic to $\mathfrak{g}_\psi = (L)_\psi$ for some $\psi \in F_1 H^2(L, L)$. In terms of the basis of this cohomology space we may write

$$\begin{aligned} \psi \;=\; & \alpha_1\psi_{1,4} + \alpha_2\psi_{1,5} + \ldots + \alpha_7\psi_{1,10} \\ & + \alpha_8\psi_{2,6} + \alpha_9\psi_{2,7} \ldots + \alpha_{12}\psi_{2,10} \\ & + \alpha_{13}\psi_{3,8} + \ldots + \alpha_{15}\psi_{3,10} \\ & + \alpha_{16}\psi_{4,10} \end{aligned}$$

The integrability of ψ is determined by four equations, see [APP]. We are interested here in the case $\alpha_1 \neq 0$. This includes the algebras $\mathfrak{a}(r,s,t)$ of [BEN] and [BGR], choose

$$\psi = \psi_{1,4} + (1-r)\psi_{2,6} - s\psi_{2,7} - t\psi_{2,8} + \beta_1\psi_{3,8} + \beta_2\psi_{3,9} + \beta_3\psi_{3,10} + \beta_4\psi_{4,10}$$

with certain β_i , see [APP]. The integrability conditions then imply $2\alpha_1 + \alpha_8 \neq 0$ and $\alpha_1^2 \neq \alpha_8^2$. We have the following result:

Proposition 8 *Let $\mathfrak{g}_\psi$ be a filiform nilpotent Lie algebra of dimension 11 satisfying $\alpha_1 \neq 0$. Then $\mu(\mathfrak{g}_\psi) \leq 12$ if and only if $\alpha_8 = 0$ or $10\alpha_8 = \alpha_1$ or $5\alpha_8^2 = 2\alpha_1^2$ or $4\alpha_1^2 - 4\alpha_1\alpha_8 + 3\alpha_8^2 = 0$.*

Proof: For $\mathfrak{g}_\psi = \mathfrak{a}(r,s,t)$ we obtain exactly Theorem B of [BGR], substituting $\alpha_1 = 1$ and $\alpha_8 = 1 - r$ in the above equations. The proof is the same as for Theorem B. By avoiding polynomials in r, s, t the computations here are easier. $\square$

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