An introduction to Lie algebras and the theorem of Ado

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Introduction

Lie groups and Lie algebras are of great importance in modern physics, particularly in the context of (continuous) symmetry transformations. The Lie algebra of a Lie group is defined as the tangent space to the neutral element of the group and its elements can be seen as "infinitesimal transformations". The Lie algebra of a Lie group is uniquely determined (the converse is not true unless the group is simply connected) and many questions about the group can be reduced to questions about the Lie algebra, which are usually easier to handle. It is particularly pleasant if the algebra can be represented by matrices and an important result in this area is given by Ado's theorem, which states that any finite-dimensional Lie algebra can be represented by (finite) matrices. In this thesis we will prove Ado's theorem for nilpotent Lie algebras and provide a method to construct such matrix representations. It is also worthwhile to mention that, although Lie algebras historically arose as a means to study Lie groups, they are meanwhile often studied in their own right.

The first chapter contains the basic definitions and some helpful examples. In the second chapter a short introduction to representation theory is given as well as the proofs to Engel's theorem and Lie's theorem. The third chapter is reserved for the proof of Ado's theorem for nilpotent Lie algebras and the theory needed for it (also, the explicit construction of faithful representations is shown in two examples). Finally, the fourth chapter contains the proof to Ado's theorem for arbitrary Lie algebras as well as the needed theory.

The proofs to Engel's and Lie's theorems are, for the most part, based on the proofs given in [1]. The proof of Ado's theorem for nilpotent Lie algebras (section 3) is the same as given in [3], the proof of Ado's theorem for arbitrary Lie algebras is based on the one given in [5].

In the following, with the exception of the construction of the universal enveloping algebra in chapter 3.2., all vector spaces are assumed to be finite-dimensional. Furthermore, unless mentioned otherwise, the underlying fields are of characteristic zero and algebraically closed. Note however that Ado's theorem is valid for Lie algebras over fields of arbitrary characteristic (and, in this context, is sometimes called Iwasawa's theorem).

1 Lie algebras

Definition. *A Lie algebra over a field $\mathbb{F}$ is a vector space L over $\mathbb{F}$, together with an operation $L \times L \to L$, $(x, y) \to [x, y]$, which fulfills the following axioms:*

(i) The operation $[.,.]$ is bilinear.

(ii) $\forall x \in L : \ [x, x] = 0.$

(iii) $\forall x, y, z \in L : \ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

The operation $[.,.]$ is usually called the bracket operation. Property *(iii)* is called the Jacobi identity. Note that *(ii)* implies

(ii′) $\forall x, y \in L : \ [x, y] = -[y, x].$

If $char\mathbb{F} \neq 2$, *(ii)* and *(ii′)* are equivalent.

An important example for a Lie agebra is the set of all linear transformation $V \to V$ (where V is any vector space of a field $\mathbb{F}$), denoted by $EndV$ or, in the context of Lie algebras, $\mathfrak{gl}(V)$. $\mathfrak{gl}(V)$ is itself a vector space with dimension $(dimV)^2$ and a ring w.r.t. the composition of maps. The bracket operation is defined by $[x, y] = xy - yx \ \forall x, y \in \mathfrak{gl}(V)$, where xy is the composition of x and y. After a basis of V has been chosen, the elements of $\mathfrak{gl}(V)$ can be represented as $n \times n$ matrices and we write $\mathfrak{gl}_n(\mathbb{F})$. The standard basis consists of matrices e_{ij}, having a one in the (i, j) position and zeros everywhere else. The Lie bracket is then given by

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{jk}.$$

Every vector space V with the bracket operation defined as $[x, y] = 0 \ \forall x, y \in V$ is a Lie algebra. Such Lie algebras are called abelian.

If L is one-dimensional, it has exactly one basis vector x with the commutation relation $[x, x] = 0$, so any one-dimensional Lie algebra is abelien.

If L is two-dimensional with basis vectors x, y, it is either abelien or

$$[x, y] = \alpha x + \beta y.$$

Now define a new basis by $x' = \alpha x + \beta y$ and take y' to be an orthogonal vector. It follows that

$$[x', y'] = \gamma x'$$

and by scaling $y' \to \gamma^{-1}y'$ we get

$$[x', y'] = x'.$$

Therefore, up to isomorphism, there are exactly two two-dimensional Lie algebras, one of which is abelian.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is the vector space of all real 2×2-matrices with trace zero with the commutator as Lie bracket. It is spanned by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The commutation relations are

$$[e, f] = g, \quad [e, g] = -2e, \quad [f, g] = 2f.$$

More generally, $\mathfrak{sl}_n(\mathbb{R})/\mathfrak{sl}_n(\mathbb{C})$ is the space of all real/complex $n \times n$-matrices with trace zero.

The Heisenberg Lie algebra $\mathfrak{h}_n$ is a $(2n+1)$-dimensional real vector space with a basis $\{x_1, \ ..., \ x_n, \ y_1, \ ..., \ y_n, \ z\}$ and the Lie bracket defined by

$$[x_i, y_i] = z$$

and all other brackets equal to zero. For example, $\mathfrak{h}_1$ can be identified with (i.e. "a faithful representation of $\mathfrak{h}_1$ is given by", see the definitions below) the space of real matrices spanned by

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the Lie bracket is now given by the commutator.

The following example shows the connection between a Lie algebra and the corresponding Lie group (by a naive approach). Let $Aff(\mathbb{R})$ be the group of all invertible affine transformations of the line, i.e.

$$Aff(\mathbb{R}) = \{L_{a,b} : \mathbb{R} \to \mathbb{R} |\ L_{a,b}x = ax + b,\ a, b \in \mathbb{R},\ a \neq 0\}.$$

The elements of $Aff(\mathbb{R})$ can be written as matrices of the form

$$L_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$$

and the transformation is given by their action on vectors of the form $\begin{pmatrix} x \\ 1 \end{pmatrix}$. Now let l be a 2×2-matrix, such that $1 + \epsilon l \in Aff(\mathbb{R})$ (where 1 denotes the identity matrix and $\epsilon \in \mathbb{R}$). Obviously l has to be of the form

$$l = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

These elements form the Lie algebra $\mathfrak{aff}(\mathbb{R})$ (with the commutator as the Lie bracket). Note that in general one has to use the exponential map to get from the Lie algebra to the Lie group (and to define the Lie algebra of the Lie group). Another downside of this approach is that it doesn't explain how the group multiplication leads to the Lie bracket.

The affine group $Aff(\mathbb{R}^n)$ is the group of all invertible affine transformations $\mathbb{R}^n \to \mathbb{R}^n$. As above, it is formed by elements of the form

$$L_{A,b} = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, A \in GL_n(\mathbb{R}), b \in \mathbb{R}^n$$

and the corresponding Lie algebra is given by

$$\mathfrak{aff}(\mathbb{R}^n) = \{\begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}_n(\mathbb{R}),\ b \in \mathbb{R}^n\}.$$

Definition. *A Lie algebra homomorphism between two Lie algebras L_1 and L_2 is a linear map $\phi : L_1 \to L_2$, such that*

$$\forall x, y \in L_1 :\ \phi([x, y]) = [\phi(x), \phi(y)].$$

1.1 Subalgebras, ideals, quotients

Definition. *A subspace K of a Lie algebra L is called a Lie subalgebra, if*

$$x, y \in K \implies [x, y] \in K.$$

Obviously, every one-dimensional subspace of a Lie algebra is a subalgebra. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is a subalgebra of $\mathfrak{gl}_2(\mathbb{C})$.

The normalizer of a subspace K of L is defined as

$$N_L(K) = \{x \in L | [x, K] \subset K\}.$$

$N_L(K)$ is a subalgebra of L, because for $x \in K$ and $x_1,\ x_2 \in N_L(K)$, the Jacobi identity implies

$$\big[[x_1, x_2], x\big] = \big[x_1, [x_2, x]\big] - \big[x_2, [x_1, x]\big]$$

and therefore $[x_1, x_2] \in N_L(K)$. If K is a subalgebra, $N_L(K)$ is the largest subalgebra of L which includes K as an ideal.

Definition. *A subspace I of a Lie algebra L is called an ideal, if*

$$\forall x \in L\ \forall i \in I :\ [x, i] \in I.$$

Obviously every ideal is also a subalgebra. An example for an ideal is the center $Z(L) = \{x \in L | [x, l] = 0 \, \forall l \in L\}$. If $Z(L) = L$, L is called abelian. Another example is $[L, L] = \{[l_1, l_2] | \, l_1, l_2 \in L\}$.

Consider the Heisenberg Lie algebra $\mathfrak{h}_n$ with the basis $\{x_1, \; ..., \; x_n, \; y_1, \; ..., \; y_n, \; z\}$ and the Lie bracket defined by $[x_i, y_i] = z$. Then the one-dimensional subspace spanned by z is an ideal of $\mathfrak{h}_n$, namely $[\mathfrak{h}_n, \mathfrak{h}_n] = \langle z \rangle$.

Proposition 1. *The kernel $Ker(\phi)$ of a Lie algebra homomorphism $\phi : L_1 \to L_2$ is an ideal of L_1, its image $Im(\phi)$ is a subalgebra of L_2.*

Proof. That the kernel and the image are subspaces is a simple consequence of the linearity of homomorphisms. Since

$$[\phi(x), \phi(y)] = \phi([x, y]),$$

$Im(\phi)$ is a subalgebra. Now assume $x \in Ker(\phi)$. Then, for every $y \in L_1$,

$$\phi([x, y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0$$

and therefore $[x, y] \in Ker(\phi)$. □

Definition. *Given a Lie algebra L and an ideal $I \subset L$, we can construct the quotient algebra L/I. Seen as a vector space, L/I is simply the quotient space, i.e.*

$L/I = \{x + I \mid x \in L\}$, *where*

$x + I = \{x + i \mid i \in I\}$.

The bracket operation on L/I is then defined by $[x + I, y + I] = [x, y] + I$. It is easy to check that this is well defined.

1.2 Nilpotent, solvable, simple and semisimple Lie algebras

Definition. *A Lie algebra L is called simple, if $[L, L] \neq 0$ and L has no ideals except itself and 0.*

Note that this implies $[L, L] = L$ and $Z(L) = 0$.

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is simple:

Proof. Assume I is a non-zero ideal of $\mathfrak{sl}_2(\mathbb{R})$. Take $x \in I, \; x \neq 0$, then $x = \alpha e + \beta f + \gamma g$, where $\{e, \; f, \; g\}$ is the basis of $\mathfrak{sl}_2(\mathbb{R})$ introduced above, $\alpha, \; \beta, \; \gamma \in \mathbb{R}$ and at least one coefficient is non-zero. Now note that

$[[x, e], e] = -2\beta e$ and

$[[x, f], f] = 2\alpha f$.

Thus, if either α or β are non-zero, $I = \mathfrak{sl}_2(\mathbb{R})$. On the other hand, if $\alpha = \beta = 0$, then $\gamma \neq 0$ and, because $[x, e] = 2\gamma x$ and $[x, f] = -2\gamma y$, $I = \mathfrak{sl}_2(\mathbb{R})$. □

Definition. *Given a Lie algebra L, we define a sequence of ideals by $L^{(0)} = L$, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$. The Lie algebra L is called solvable if $L^{(i)} = 0$ for some i.*

A simple algebra is not solvable (since $L^{(i)} = L \; \forall i$).

Definition. *Given a Lie algebra L, we define a sequence of ideals by $L^0 = L$, $L^{i+1} = [L, L^i]$. The Lie algebra L is called nilpotent if $L^i = 0$ for some i.*

Note that, since $L^{(i)} \subset L^i \; \forall i$, all nilpotent Lie algebras are solvable. A simple algebra is never nilpotent (because it is not even solvable). Abelian algebras are nilpotent and therefore also solvable.

Consider again the Heisenberg Lie algebra $\mathfrak{h}_n$. We have already seen that $\mathfrak{h}_n^1 = \mathfrak{h}_n^{(1)} = [\mathfrak{h}_n, \mathfrak{h}_n] = \langle z \rangle$ is one-dimensional and therefore abelian, i.e. $\mathfrak{h}_n$ is solvable with $\mathfrak{h}_n^{(2)} = 0$. Since $[x_i, z] = [y_i, z] = 0$, it is also nilpotent with $\mathfrak{h}_n^2 = 0$.

The non-abelian two-dimensional Lie algebra L with the basis $\{x, \; y\}$ and $[x, y] = x$ is an example for a solvable Lie algebra, which is not nilpotent:

$L^{(1)} = L^1 = [L, L] = \langle x \rangle$,

$L^{(2)} = [L^1, L^1] = 0$, but

$L^n = L^1 \ \forall n \geq 1.$

Definition. *A nilpotent Lie algebra of dimension n is called filiform, if $L^{n-2} \neq 0$ (and thus $L^{n-1} = 0$).*

Let L be a vector space with basis $\{x_1,\ ...,\ x_n\}$. If we define a bracket operation by

$[x_1, x_i] = x_{i+1}$ for $i \in \{2,\ ...,\ n-1\}$,

L becomes a filiform Lie algebra and is called the standard filiform Lie algebra of dimension n.

Definition. *A Lie algebra L is called semisimple, if it has no non-zero solvable ideals. This is equivalent to the condition, that L is a direct sum of simple Lie algebras (hence the name).*

Proposition 2. *If L is a Lie algebra and $L/Z(L)$ is nilpotent, then L is also nilpotent.*

Proof. If $L/Z(L)$ is nilpotent, $L^n \subset Z(L)$ for some n. It follows that $L^{n+1} = 0$. □

Proposition 3. *Let L be a Lie algebra.*

(i) If S_1 and S_2 are two solvable ideals in L, then $S_1 + S_2$ is a solvable ideal in L.

(ii) If N_1 and N_2 are two nilpotent ideals in L, then $N_1 + N_2$ is a nilpotent ideal in L.

Proof.
(i): Clearly the sum of two ideals is another ideal. Now note that $(S_1 + S_2)/S_2 \cong S_1/(S_1 \cap S_2)$ (isomorphism theorem). The right side is a homomorphic image of S_1 and therefore solvable, which implies that the left side is also solvable. It remains to show that, if I is a solvable ideal in a Lie algebra K and K/I is solvable, K is solvable as well. Assume $(K/I)^{(n)} = 0$, i.e. $K^{(n)} \subset I$. Since I is solvable, $I^{(m)} = 0$ for some m. It follows that $K^{(n+m)} = 0$.
The proof for *(ii)* is analogous. □

Definition. *Let L be a Lie algebra. The preceeding proposition implies the existence of a maximal solvable ideal in L and a maximal nilpotent ideal in L. The maximal solvable ideal is called the radical of L and denoted by $Rad(L)$. The maximal nilpotent ideal in L is called the nilradical and denoted by $Nil(L)$.*

2 The representation theory of Lie algebras

Definition. *A representation of a Lie algebra L on a vector space V is a Lie algebra homomorphism $\phi : L \to \mathfrak{gl}(V)$.*

Definition. *A representation is called faithful, if it is one-to-one, i.e. if the homomorphism $\phi : L \to \mathfrak{gl}(V)$ is injective.*

2.1 Examples

An important example is the so called adjoint representation, defined by

$ad : L \to \mathfrak{gl}(L),\ x \to ad_x$, where

$ad_x : L \to L,\ ad_x(y) = [x, y].$

If L is a Lie algebra and $x \in L$, then x is called ad-nilpotent if ad_x is nilpotent, i.e. $(ad_x)^n = 0$ for some n. The kernel of ad is the center of L. This means that, if $Z(L) = 0$, ad is injective and therefore isomorphic to its image, which is a subalgebra of $\mathfrak{gl}(L)$. So, any simple Lie algebra is isomorphic to a linear Lie algebra.

Consider again $\mathfrak{sl}_2(\mathbb{R})$ with its basis $\{e,\ f,\ g\}$ and the commutation relations

$$[e, f] = g,\quad [e, g] = -2e,\quad [f, g] = 2f.$$

We are interested in the (faithful) representations of $\mathfrak{sl}_2(\mathbb{R})$ on vector spaces V of different dimensions. For $dimV = 2$, this is easy: a faithful representation is given by the identity map. Now consider $V = \mathbb{R}^3 \cong \mathfrak{sl}_2(\mathbb{R})$ with basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \cong e,\quad \begin{pmatrix}0\\1\\0\end{pmatrix} \cong -g,\quad \begin{pmatrix}0\\0\\1\end{pmatrix} \cong f$$

and the adjoint representation. Since $\mathfrak{sl}_2(\mathbb{R})$ is simple, the adjoint representation is faithful, its matrix form is given by

$$ad_e = \begin{pmatrix}0&2&0\\0&0&1\\0&0&0\end{pmatrix},\quad ad_f = \begin{pmatrix}0&0&0\\1&0&0\\0&2&0\end{pmatrix},\quad ad_g = \begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}.$$

As a last example consider the non-abelian two-dimensional Lie algebra with the basis $\{x,\ y\}$ and $[x, y] = x$. Note that even though this algebra is solvable, its center is zero and we can again use the adjoint representation. The matrix form is

$$ad_x = \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad ad_y = \begin{pmatrix}-1&0\\0&0\end{pmatrix}.$$

2.2 Modules, submodules, quotient modules

In the literature one often finds the notion of (left) L-modules, which is (in this context) basically just another way of saying "representation of L". For completeness, even though in this thesis we will always speak of representations, a short summary of L-modules is presented in the following.

Definition. *Let L be a Lie algebra. A vector space V, together with an operation $L \times V \to V$, $(l, v) \to l.v$, is called a (left) L-module, if*

(i) $(\alpha l_1 + \beta l_2).v = \alpha(l_1.v) + \beta(l_2.v)$,

(ii) $l.(\alpha v_1 + \beta v_2) = \alpha(l.v_1) + \beta(l.v_2)$,

(iii) $[l_1, l_2].v = l_1.l_2.v - l_2.l_1.v$.

Given a representation $\phi : L \to \mathfrak{gl}(V)$, the vector space V is an (left) L-module with

$l.v = \phi(l)(v).$

Conversely, given an L-module V, the above equation defines a representation $\phi : L \to \mathfrak{gl}(V)$.

A L-submodule U is simply a subspace of V, which is closed w.r.t. the inherited operation, i.e. $L.U \subset U$. Equivalentely, if $\phi : L \to \mathfrak{gl}(V)$ is a representation of L and U is a subspace of L which is left invariant under $\phi(L)$, i.e. $\phi(L)(U) \subset U$, then $\phi : L \to \mathfrak{gl}(U)$ is called a subrepresentation.

$V \neq 0$ is called irreducible, if its only submodules are 0 and itself. V is called completely reducible, if it is a direct sum of irreducible submodules. Again, irreducible/completely reducible representations are defined equivalentely.

Note that any Lie algebra L is an L-module for the adjoint representation and its L-submodules are its ideals. Therefore, any simple Lie algebra, seen as an L-module, is irreducible. A Lie algebra is called reductive, if its adjoint representation is completely reducible, i.e. the Lie algebra is a direct sum of ideals. An equivalent definition is that a Lie algebra L is reductive, if every ideal I has a complementery ideal K, i.e. $L = I \oplus K$.

A linear map $\psi : V \to W$, where V and W are L-modules, is called a homomorphism of L-modules, if

$$\psi(l.v) = l.\psi(v).$$

2.3 Structure theorems: Lie and Engel

The aim of this subsection is to prove the theorems of Lie and Engel, but first, a few preliminary results will be derived.

Theorem 1. *Let $x \in \mathfrak{gl}(V)$ be nilpotent. Then ad_x is also nilpotent.*

Proof. Define

$l_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V),\ l_x(y) = xy$ and

$r_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V),\ r_x(y) = yx.$

If x is nilpotent (e.g. $x^n = 0$), then so are l_x and r_x:

$$(l_x)^n = x^n y = 0,$$

$$(r_x)^n = yx^n = 0.$$

Therefore, $ad_x = l_x - r_x$ is nilpotent as well (note that $l_x \circ r_x = r_x \circ l_x$):

$$(ad_x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (l_x)^{2n-k}(-r_x)^k = 0.$$

□

Theorem 2. *Let $V \neq 0$ be a finite-dimensional vector space and L a subalgebra of $\mathfrak{gl}(V)$. If all elements of L are nilpotent, then there exists a nonzero $v \in V$, such that $l(v) = 0\ \forall l \in L$.*

Proof. The proof uses induction on $dimL$:

$dimL = 1$: Denote the unique basis vector of L by l_B. Then choose an arbitrary vector $v \in V$ and define the sequence $v_0 = v,\ v_{i+1} = l_B(v_i)$. Since l_B is nilpotent, $v_n = 0$ for some n. If we choose n minimal, $v_{n-1} \neq 0$ and $l(v_{n-1}) = 0\ \forall l \in L$.

$dimL > 1$: Let K be a maximal proper subalgebra of L (such an algebra exists since L is finite-dimensional and nilpotent). Let us now show that K is an ideal of L. Since all elements of L are nilpotent and $ad_k \in \mathfrak{gl}(L)\ \forall k \in K$, it follows from theorem 1 that

$\forall k \in K\ \exists n : (ad_k)^n = 0$, i.e. the image of $ad : K \to \mathfrak{gl}(L)$ is nilpotent.

ad_k also acts on the quotient space L/K by $ad_k(l+K) = ad_k(l)+K = [k,l]+K$. The induction hypothesis guarantees us the existence of an vector $x \in L,\ x \notin K$, such that

$ad_k(x+K) = 0\ \forall k \in K$, i.e. $[x,k] \in K\ \forall k \in K$.

This means K is properly included in $N_L(K)$ and since K is maximal, $N_L(K) = L$, i.e. K is an ideal in L.

$K + \langle x \rangle$, where $\langle x \rangle$ denotes the span of x, is a subalgebra of L and, since K is already a maximal proper subalgebra and $x \notin K$, it follows that

$L = K + \langle x \rangle.$

Due to the induction hypothesis there is a $v \in V$, s.t. $K(v) = 0$, i.e. $k(v) = 0 \; \forall k \in K$. Define

$W = \{v \in V | K(v) = 0\} \neq 0.$

It now suffices to show the existence of a $v \in W - \{0\}$ with $x(v) = 0$. Note that W is stable under L, i.e. $L(W) = \{l(w) | l \in L, \; w \in W\} \subset W$, since $k\big(l(w)\big) = l\big(k(w)\big) - [l, k](w) = 0$.

In particular, W is stable under $\langle x \rangle$ and therefore $\langle x \rangle|_W \subset \mathfrak{gl}(W)$. It follows (again by the induction hypothesis) that

$\exists v \in W: \; \langle x \rangle(v) = 0$ and therefore $L(v) = 0$.

□

With these results we are now prepared to prove Engel's theorem.

Engel's theorem. *If all elements of a Lie algebra L are ad-nilpotent, then L is nilpotent.*

Proof. Again, by induction on $dimL$, the case $L = 0$ being trivial. The image of $ad : L \to \mathfrak{gl}(L)$ consists of nilpotent endomorphisms and therefore satisfies the requirements of theorem 2. It follows that $Z(L) \neq 0$. Now $L/Z(L)$ consists of ad-nilpotent elements and $dim\big(L/Z(L)\big) < dimL$. By the induction hypothesis, $L/Z(L)$ is nilpotent. Due to proposition 2 L is also nilpotent. □

Proposition 4. *Under the assumptions of theorem 2, there exists a basis of V, relative to which the matrices of L are strictly upper triangular.*

Proof. Let $v_0 \in V$ be a non-zero vector, such that $L(v_0) = 0$ (the existence of such a vector is guaranteed by theorem 2. Define $W = V/\langle x_0 \rangle$. Note that $L|_W$ still consists of nilpotent endomorphisms. Using induction on $dimV$ (the case $dimV = 0$ being trivial), W has a basis $\{v_1, \; ..., \; v_{n-1}\}$ (mod v_0) relative to which $L|_W$ consists of strictly upper triangular matrices and $\{v_0, \; ..., \; v_{n-1}\}$ is the basis of V we were looking for. □

Theorem 3. *Let $V \neq 0$ be a finite-dimensional vector space over the field $\mathbb{F}$ and L a solvable subalgebra of $\mathfrak{gl}(V)$. Then V contains a common eigenvector for all elements of L.*

Proof. The proof is done by induction on $dimL$ and is similar to the proof of theorem 2 .

$dimL = 1$: There is one basis vector l_B with an eigenvalue λ ($\mathbb{F}$ is algebraically closed). The corresponding eigenvector is an eigenvector for all elements $\alpha \cdot l_B$ in L with eigenvalues $\alpha \cdot \lambda$.

$dimL > 1$: Since L is solvable, $[L, L]$ is a proper ideal in L.

Therefore, a subspace K of codimension one containing $[L, L]$ exists, i.e. K is an ideal and

$\exists x \in L - K \; : \; \; L = K + \langle x \rangle.$

The induction hypothesis guarantees the existence of a common eigenvector for all elements in K, i.e. there exist $v \in W$ and a linear function $\lambda : K \to \mathbb{F}$, such that $k(v) = \lambda(k) \cdot v$ for every $k \in K$. Therefore

$W = \{v \in V | \; \exists \lambda : K \to \mathbb{F} : \; k(v) = \lambda(k) \cdot v \; \forall k \in K\} \neq 0.$

Assume for a moment that $L(W) \subset W$. It follows, that $\langle x \rangle|_W \subset \mathfrak{gl}(W)$. Since the field is algebraically closed, x has an eigenvector $v \in W$, which is thus also an eigenvector for every $l \in L = K + \langle x \rangle$.

It remains to show that L leaves W invariant. Note that for arbitrary $k \in K$, $l \in L$, $w \in W$

$k\big(l(w)\big) = (l \circ k)(w) - [l, k](w)$

and thus, since K is an ideal in L and $[l, k] \in K$,

$k\big(l(w)\big) = \lambda(k) \cdot l(w) - \lambda([l, k]) \cdot w.$

It therefore suffices to show that $\lambda([l, k]) = 0$ for all $k \in K$, $l \in L$. Fix $w \in W$, $l \in L$ and choose $n \in \mathbb{Z}^+$ minimal, such that $w, \; l(w), \; ..., \; l^n(w)$ are linearly dependent. Define

$W_0 = 0, \;\; W_i = \langle w, \; ..., \; l^{i-1}(w) \rangle.$

By construction $l(W_n) \subset W_n$. We will now show that, for $k \in K$, $(k \circ l^i)(w) = \lambda(k) \cdot l^i(w) + w'$ with $w' \in W_i$, i.e. that k is an upper triangular matrix with the diagonal entries $\lambda(k)$ relative to the basis $\{w, \ ..., \ l^{n-1}(w)\}$.

The proof is done by induction on i, the case $i = 0$ being trivial. Note that

$(k \circ l^i)(w) = (k \circ l \circ l^{i-1})(w) = (l \circ k \circ l^{i-1})(w) - ([l, k] \circ l^{i-1})(w)$.

Due to the induction hypothesis,

$(k \circ l^{i-1})(w) = \lambda(k) \cdot l^{i-1}(w) + w'$ and $([l, k] \circ l^{i-1})(w) = \lambda([l, k]) \cdot l^{i-1}(w) + w''$ with $w', \ w'' \in W_{i-1}$.

Therefore,

$(k \circ l^i)(w) = \lambda(k) \cdot l^i(w) + l(w') + \lambda([l, k]) \cdot l^{i-1}(w) + w''$.

Since $w' \in W_{i-1}$, it follows (by construction of the W_i) that $l(w') \in W_i$. Also, $\lambda([l, k]) \cdot l^{i-1}(w) \in W_i$ and $w'' \in W_{i-1} \subset W_i$.

In particular $K(W_n) \subset W_n$. It furthermore follows that $n \cdot \lambda(k) = tr(k)$ (the trace is to be taken over W_n, i.e. $k \in K|_{W_n} \subset \mathfrak{gl}(W_n)$). If k is a commutator, the trace equals zero, in particular $\lambda([l, k]) = 0 \ \forall k \in K$. Therefore, since the above construction can be done for arbitrary $l \in L$,

$\lambda([l, k]) = 0 \ \forall k \in K \ \forall l \in L$.

□

Lie's theorem. *Let $V \neq 0$ be a finite-dimensional vector space with $dim(V) = n$ and L a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a basis of V, such that the elements of L are upper triangular matrices with respect to that basis.*

Proof. Induction on n, the case $n = 1$ being trivial. Due to theorem 3 all elements of L have a common eigenvector $v_0 \in V$. The quotient space $V/\langle v_0 \rangle$ has dimension $dimV - 1$ and due to the induction hypothesis a basis $\{v_1 + \langle v_0 \rangle, ..., v_{n-1} + \langle v_0 \rangle\}$, s.t. the elements of $L|_{V/\langle v_0 \rangle}$ are upper triangular matrices. Therefore $\{v_0, ..., v_{n-1}\}$ is a basis of V in which every element of L is an upper triangular matrix. □

3 The theorem of Ado for nilpotent Lie algebras. How can a faithful representation be constructed?

3.1 Ad hoc examples

We have already seen that the adjoint representation of a Lie algebra L is faithful if the center of L is trivial (i.e. L is simple/semisimple). Since $L = Z(L) \oplus L/Z(L)$ (as direct sum of vector spaces), if one finds a representation of L which is faithful on $Z(L)$, one can then take the direct sum of that representation and the adjoint representation to construct a representation which is faithful on L.

Next, consider an abelian Lie algebra L (i.e. the center of L is L) and the map

$\phi : L \to \mathfrak{gl}(L \times \mathbb{F}),\ x \to \phi_x$, where

$\phi_x : L \times \mathbb{F} \to L \times \mathbb{F},\ (y, \alpha) \to (\alpha x, 0).$

If the elements of $L \times \mathbb{F}$ are written as vectors $\begin{pmatrix} y \\ \alpha \end{pmatrix}$, ϕ_x is the matrix $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. This gives us a faithful representation of every abelian Lie algebra (and therefore of the center of an arbitrary Lie algebra).

3.2 The universal enveloping algebra

3.2.1 Tensor products and the tensor algebra

Let V and W be two vector spaces and $V \times W = \{(v, w) |\ v \in V,\ w \in W\}$. The free vector space $F(V \times W)$ on $V \times W$ is the vector space in which the elements of $V \times W$ form a basis, i.e.

$F(V \times W) = \{\sum_{i=1}^{n} \alpha_i e_{(v_i, w_i)} |\ v_i \in V,\ w_i \in W,\ \alpha_i \in \mathbb{F},\ n \in \mathbb{N}\},$

where the $e_{(v,w)}$ are per definition linearly independent for different $(v, w) \in V \times W$. Now define R to be the space generated by

$e_{(v_1+v_2,w)} - e_{(v_1,w)} - e_{(v_2,w)},$

$e_{(v,w_1+w_2)} - e_{(v,w_1)} - e_{(v,w_2)},$

$e_{(\alpha v,w)} - \alpha e_{(v,w)},$

$e_{(v,\alpha w)} - \alpha e_{(v,w)}$

for $v,\ v_1,\ v_2 \in V$, $w,\ w_1,\ w_2 \in W$ and $\alpha \in \mathbb{F}$.

Definition. *The tensor product $V \otimes W$ of V and W is defined as the quotient space $F(V \times W)/R$ and its elements are denoted by $v \otimes w$.*

By construction,

$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$

$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$

$(\alpha v) \otimes w = v \otimes (\alpha w) = \alpha(v \otimes w).$

It follows that, if $\{v_i\}$ and $\{w_i\}$ are bases of V and W, $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$ and that the dimension of $V \otimes W$ is $dimV \cdot dimW$.

Definition. *For a vector space V define*

$T^i V = V^{\otimes i} = V \otimes\ ...\ \otimes V$ *(i times).*

The tensor algebra of V is then defined as

$TV = \bigoplus_{i=0}^{\infty} T^i V = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus ....$

Multiplication in TV is defined by the tensor product, i.e. for $x \in T^k V$ and $y \in T^l V$,

$xy = x \otimes y \in T^{(k+l)} V$

and TV is thus an associative algebra. From now on, since it is also customary in the literature, we will write xy instead of $x \otimes y$.

3.2.2 The universal enveloping algebra of a Lie algebra

Definition. *The universal enveloping algebra $U(L)$ of a Lie algebra L is defined as the quotient of TL by the ideal generated by $xy - yx - [x, y]$ for $x, y \in L$.*

Note that L acts on $U(L)$ by left multiplication, i.e. we have a map:

$\phi : L \to gl\big(U(L)\big),\ x \to \phi_x$, where

$\phi_x : U(L) \to U(L),\ \phi_x(y) = xy$.

Since

$$\phi_{[x,y]}(z) = [x, y]z = (xy - yx)z = xyz - yxz = (\phi_x \circ \phi_y - \phi_y \circ \phi_x)(z) \text{ for } z \in U(L),$$

$\phi([x, y]) = [\phi(x), \phi(y)]$ and ϕ is a Lie algebra homomorphism. Since each ϕ_x maps $1 \in U(L)$ into x, ϕ is injective and therefore a faithful representation of L.

3.2.3 The Poincare-Birkhoff-Witt theorem

Let L be a Lie algebra with an ordered basis $\{x_1,\ ...,\ x_n\}$. The Poincare-Birkhoff-Witt theorem states that the universal enveloping algebra $U(L)$ is spanned by monomials of the form $x_1^{\alpha_1} ... \ x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{N}_0$. A proof can be found in e.g. [1, 17.4].

3.3 Constructing a faithful representation of $\mathfrak{h}_1$

In this subsection we will construct a faithful representation of the Heisenberg Lie algebra $\mathfrak{h}_1$. The proof of Ado's theorem for nilpotent Lie algebras, presented in the next subsection, will be a direct generalisation of the method used here. Let $\{x_1,\ x_2,\ x_3\}$ be the basis of $\mathfrak{h}_1$ with $[x_2, x_3] = x_1$ and all other commutators zero. Note that

$$\mathfrak{h}_1^0 = \mathfrak{h}_1,\ \ \mathfrak{h}_1^1 = [\mathfrak{h}_1, \mathfrak{h}_1] = \langle x_1 \rangle,\ \ \mathfrak{h}_1^2 = 0.$$

Now consider the universal enveloping algebra $U(\mathfrak{h}_1)$ with its basis formed by the monomials

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

with $\alpha \in \mathbb{Z}_+^3$. Define $U^3(\mathfrak{h}_1)$ as the subspace of $U(\mathfrak{h}_1)$ spanned by the monomials with

$$2\alpha_1 + \alpha_2 + \alpha_3 \geq 3$$

and denote the quotient space $U(\mathfrak{h}_1)/U^3(\mathfrak{h}_1)$ by V. This quotient space is now spanned by the monomials (mod $U^3(\mathfrak{h}_1)$) with $2\alpha_1 + \alpha_2 + \alpha_3 < 3$ and therefore finite. More precisely, a basis of V is given by

$$\{x_1,\ x_2^2,\ x_2x_3,\ x_3^2,\ x_2,\ x_3,\ 1\}.$$

Now define, as before (see section 4.2.2), a representation ϕ of $\mathfrak{h}_1$ on V by left multiplication, i.e. $\phi_x(v) = xv$. Again, since each ϕ_x maps $1 \in V$ into $x \neq 0$, ϕ is injective and therefore a faithful representation of $\mathfrak{h}_1$ (but now on an finite vector space). In matrix form, w.r.t. the above basis, this representation is given by:

$$\phi_{x_1} = \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},\ \ \phi_{x_2} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},\ \ \phi_{x_3} = \begin{pmatrix} 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}.$$

However, we can do better (i.e. we can find a representation in a smaller dimension), but first, let's see if we can find a lower limit for the dimension of faithful representations of $\mathfrak{h}_1$. We will use that, if L is a nilpotent Lie algebra and $Z(L) \subset [L, L]$, there exists a minimal faithful representation ϕ of L, such that $Im(\phi)$ consists of nilpotent endomorphisms (see e.g. [4, Corollary 2.8]). Thus, due to proposition 3, there is a basis of V relative to which the image of ϕ consists of strictly upper triangular matrices. This implies that any faithful representation of $\mathfrak{h}_1$ has to be at least of dimension 3.

There is another proof for this using Lie's theorem instead of Engel's theorem: Let ϕ be a faithful representation of $\mathfrak{h}_1$ on a vector space V. Then $Im(\phi)$ is a solvable subalgebra of $\mathfrak{gl}(V)$ and, due to Lies theorem, there is a basis of V such that $Im(\phi)$ consists of upper triangular matrices. This obviously excludes faithful representations of dimension 1. Now assume that we have a faithful representation on a vector space of dimension 2. Note that, since the commutator of two upper triangular matrices is strictly upper triangular, $\phi(x_1)$ would be a strictly upper triangular matrix (w.r.t. to some basis). But then the other commutation relations, $[\phi(x_2), \phi(x_1)] = [\phi(x_3), \phi(x_1)] = 0$, imply that $\phi(x_1) = 0$, as can be verified by a straightforward calculation. Therefore, no 2-dimensional faithful representation of $\mathfrak{h}_1$ exists and any faithful representation of $\mathfrak{h}_1$ has to be at least of dimension 3.

Now we are going to construct a lower-dimensional faithful representation of $\mathfrak{h}_1$ by "downsizing" the representation given above. First note, that the subspace $W \subset V$ spanned by x_2^2, x_2x_3, x_3^2 is invariant under $Im(\phi)$, since $xw = 0$ for $x \in \mathfrak{h}_1$ and $w \in W$ and that ϕ is still faithful on $U = V/W = \langle x_1,\ x_2,\ x_3,\ 1\rangle$ (mod W)[1], giving us a four-dimensional representation.
The matrices are given by

$$\phi_{x_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \phi_{x_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \phi_{x_3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, we can still do better. It is easy to check that $P = \langle x_3 \rangle$ is invariant under ϕ and that ϕ is still faithful on U/P. W.r.t. the basis $\{x_1,\ x_2,\ 1\}$, the matrices are

$$\phi_{x_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \phi_{x_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ \phi_{x_3} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have thus found a faithful representation of dimension 3 and therefore, as shown above, a minimal faithful representation.

3.4 Ado's theorem for nilpotent Lie algebras

Ado's theorem for nilpotent Lie algebras. *Every finite-dimensional nilpotent Lie algebra L has a faithful representation $\phi : L \to \mathfrak{gl}(V)$ on a finite-dimensional vector space V.*

Proof. Since L is nilpotent, there exists a $k \in \mathbb{N}$, s.t. $L^k = 0$ and $L^{k-1} \neq 0$. Choose a basis $\{x_1,\ ..,\ x_n\}$ of L s.t. the first n_1 elements span L^{k-1}, the first n_2 span L^{k-2} and so on. Now consider the universal enveloping algebra $U(L)$ with its basis formed by the monomials

$$x^\alpha = x_1^{\alpha_1} \ ...\ x_n^{\alpha_n}$$

with $\alpha \in \mathbb{Z}_+^n$. Let $t = \sum_\alpha c_\alpha x^\alpha$ be an element of $U(L)$ with only finitely many $c_\alpha \neq 0$ and define an order function on $U(L)$ as follows:

$ord(t) = min\{ord(x^\alpha)|\ c_\alpha \neq 0\}$[2], where

$ord(x^\alpha) = \sum_{i=1}^n \alpha_i ord(x_i)$ and

$ord(x_i) = max\{s|\ x_i \in L^{s-1}\}$.

Furthermore, set

$ord(1) = 0$ and $ord(0) = \infty$.

Let $U^m(L) = \{t \in U(L)|\ ord(t) \geq m\}$. It is easy to show that this is an ideal in $U(L)$ and $V = U(L)/U^m(L)$ is finite. The representation ϕ of L on V is now simply given by left multiplication, i.e. $\phi_x(v) = xv$. If $m > k$, $\phi_x(1) = x \neq 0\ \forall x \in L - \{0\}$ and ϕ is faithful. □

[1] The faithful representation $\mathfrak{h}_1 \to V/W$ is the composition of $\phi : \mathfrak{h}_1 \to V$ and the canonical map $\pi : V \to V/W$. In slight abuse of notation, we also denote it by ϕ, i.e. $\pi \circ \phi \to \phi$.

[2] Compare this definition to the example in the last subsection, e.g., $ord(-x_1^{\alpha_1} + 5x_2^{\alpha_2}x_3^{\alpha_3}) = min\{2\alpha_1, \alpha_2 + \alpha_3\}$

3.5 Constructing a faithful representation for the standard filiform Lie algebra of dimension 4

Consider the four-dimensional Lie algebra $L = \langle x_1,\ x_2,\ x_3,\ x_4\rangle$ with the Lie bracket defined by

$$[x_1, x_2] = x_3,$$

$$[x_1, x_3] = x_4$$

and all other brackets equal to zero. The center of L is thus spanned by x_4 and L is 3-step nilpotent, i.e.

$$L^1 = \langle x_3,\ x_4\rangle,$$

$L^2 = Z(L) = \langle x_4\rangle$ and

$$L^3 = 0.$$

Now define an order function on $U(L)$ as shown in the last section, i.e.

$$ord(x^\alpha) = ord(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4,$$

$$ord(\textstyle\sum_\alpha c_\alpha x^\alpha) = min\{ord(x^\alpha)|\ c_\alpha \neq 0\},$$

and define $U^4(L)$ as the space spanned by all monomials with $ord(x^\alpha) \geq 4$. $U(g)/U^4(g)$ is finite with dimension 14, an (ordered) basis is given by

$$\{x_4,\ x_1x_3,\ x_2x_3,\ x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,$$

$$x_3,\ x_1^2,\ x_1x_2,\ x_2^2,$$

$$x_1,\ x_2,$$

$$1\}.$$

Note that the monomials in the first row have order 3, the ones in the second row order 2, and so on. As above, the faithful representation ϕ is given by left multiplication. We could now go ahead and compute the matrix representation as in the second to last section, but since our representation is now 14-dimensional, we refrain from doing so for all matrices. We will however give a short example of how to do it and provide the matrix form of ϕ_{x_2}:

$$\phi_{x_2}(x_1^2) = x_2x_1^2$$

$$= (x_1x_2 - [x_1, x_2])x_1$$

$$= x_1^2x_2 - x_1[x_1, x_2] - [x_1, x_2]x_1$$

$$= x_1^2x_2 - x_1x_3 - x_3x_1$$

$$= x_1^2x_2 - 2x_1x_3 + x_4.$$

This short calculation determines 14 entries of the matrix form of ϕ_{x_2} (w.r.t. the above basis), three of which are non-zero. The complete matrix (w.r.t. the above basis) is

$$\phi_{x_2} = \begin{pmatrix}
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

3.6 Constructing a faithful representation for an abelian Lie algebra

Let's do another easy example to end this chapter. We have already seen how to construct a faithful representation of an abelian Lie algebra in section 3.1, but since an abelian Lie algebra is trivially nilpotent, we can also use the above construction.

Consider the Lie algebra $L = \mathbb{C}^2$ with the basis $\{x_1,\ x_2\}$ and the Lie bracket $[x_1, x_2] = 0$. Note that $L^0 = L$ and $L^1 = 0$. Define the order function and $U(L),\ U^m(L)$ as above and consider the space

$$U(L)/U^2(L) = \langle x_1,\ x_2,\ 1\rangle.$$

The representation $\phi : L \to \mathfrak{gl}\big(U(L)/U^2(L)\big)$ is given by left multiplication. In matrix form,

$$\phi_{x_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \ \phi_{x_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the construction shown in section 3.1 would lead to the same representation.

4 The theorem of Ado

In this section we will prove the general version of Ado's theorem, i.e. that every Lie algebra has a faithful finite-dimensional representation. We will also give some examples of the construction of such representations.

4.1 Derivations

Definition. *A derivation of a Lie algebra L is a linear map $\delta : L \to L$, s.t.*

$$\delta\big([x,y]\big) = \big[x, \delta(y)\big] + \big[\delta(x), y\big] \ \ \forall x, y \in L.$$

The space of all derivations of L, denoted by $DerL$, is a Lie subalgebra of $\mathfrak{gl}(V)$. We have already seen an example of a derivation, namely the adjoint representation: Due to the Jacobi identity, ad_x is a derivation (for arbitrary $x \in L$), i.e. $ad_L \subset DerL$.

4.2 Direct and semidirect sums of Lie algebras

Definition. *Given two Lie algebras L_1 and L_2, define L to be the direct sum of L_1 and L_2 as vector spaces, i.e.*

$$L = L_1 \oplus L_2.$$

The Lie bracket on L is then defined componentwise, i.e.

$$\Big[\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \Big] = \begin{pmatrix} [l_1, k_1] \\ [l_2, k_2] \end{pmatrix}$$

for $l_1, k_1 \in L_1$ and $l_2, k_2 \in L_2$. With this bracket L is a Lie algebra and called the (external) direct sum of L_1 and L_2. Note that $\begin{pmatrix} L_1 \\ 0 \end{pmatrix} \cong L_1$ and $\begin{pmatrix} 0 \\ L_2 \end{pmatrix} \cong L_2$ are ideals in L.

Definition. *Let L be a Lie algebra and let I_1 and I_2 be ideals in L, such that $L = I_1 \oplus I_2$ as vector spaces. Then L is called the internal direct sum of I_1 and I_2, also denoted by $L = I_1 \oplus I_2$.*

Definition. *Given two Lie algebras L_1, L_2 and a Lie algebra homomorphism $D : L_1 \to DerL_2$, define L to be the direct sum of L_1 and L_2 as vector spaces, i.e.*

$$L = L_1 \oplus L_2.$$

Now define the bracket operation as

$$\Big[\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \Big] = \begin{pmatrix} [l_1, k_1] \\ [l_2, k_2] + D(l_1)k_2 - D(k_1)l_2 \end{pmatrix}.$$

L, together with this Lie bracket, becomes a Lie algebra and is called the (external) semidirect sum of L_1 and L_2, denoted as

$$L = L_1 \ltimes_D L_2.$$

Note that $\begin{pmatrix} L_1 \\ 0 \end{pmatrix} \cong L_1$ is a subalgebra in L and $\begin{pmatrix} 0 \\ L_2 \end{pmatrix} \cong L_2$ is an ideal in L.

Definition. *Let L be a Lie algebra, K a subalgebra of L and I an ideal in L, such that $L = K \oplus I$ as vector spaces. Then L is called the internal semidirect sum of K and I, denoted*

$$L = K \ltimes I.$$

Note that an internal semidirect sum is an external semidirect sum with $D(k) = ad_k$ for $k \in K$. Furthermore, an internal semidirect sum is direct, if and only if K is an ideal in L.

4.3 Proof of Ado's theorem

The following result, which is also interesting in its own right, is of central importance for the proof:

Lemma (Neretin). *Every finite-dimensional Lie algebra L can be embedded into a Lie algebra*

$$\mathfrak{g} = \mathfrak{p} \ltimes \mathfrak{n},$$

such that $\mathfrak{p}$ is a reductive subalgebra and $\mathfrak{n}$ is a nilpotent ideal.

The proof of this result will be postponed to the next subsection. Note that, since $\mathfrak{p}$ is reductive, we can write

$$\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{q} \ltimes \mathfrak{n}),$$

where $\mathfrak{k}$ is the kernel of the adjoint action of $\mathfrak{p}$ on $\mathfrak{n}$, $\mathfrak{q}$ is an ideal in $\mathfrak{p}$, $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{q}$ and $\mathfrak{q} \cong \mathfrak{p}/\mathfrak{k}$ now acts faithfully on $\mathfrak{n}$. Moreover, $\mathfrak{k}$ and $\mathfrak{q}$ are both reductive.

To prove Ado's theorem we have to find a faithful representation of both $\mathfrak{k}$ and $\mathfrak{q} \ltimes \mathfrak{n}$. The following proposition does the trick for the former.

Proposition 5. *Let L be a reductive Lie algebra. Then $L = Z(L) \oplus [L, L]$.*

Proof. Obviously $Z(L)$ and $[L, L]$ are both left invariant by the adjoint action of L. Furthermore, $Z(L) \cap [L, L] = 0$. Because $[L, L]$ already contains all the commutators, there can't be another complementary ideal, i.e. $L = Z(L) \oplus [L, L]$. □

Therefore, to find a faithful representation of a reductive Lie algebra, we can take the direct sum of two faithful representations, one of an abelian Lie algebra, the other one of a Lie algebra with trivial center. We have already seen how to construct such representations.

Ado's theorem. *Every finite-dimensional Lie algebra L has a faithful representation $\phi : L \to \mathfrak{gl}(V)$ on a finite-dimensional vector space V.*

Proof. After the discussion above, it remains to show that the semidirect product $\mathfrak{q} \ltimes \mathfrak{n}$ has a faithful representation if $\mathfrak{q}$ is reductive, $\mathfrak{n}$ is nilpotent and $\mathfrak{q}$ acts faithfully on $\mathfrak{n}$.

Consider the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. As before, $\mathfrak{n}$ acts on $U(\mathfrak{n})$ by left multiplication. Furthermore, the adjoint action of $\mathfrak{q}$ on $\mathfrak{n}$ extends to an action on $U(\mathfrak{n})$ (let $y_1 \,...\, y_n$ be an arbitrary monomial in $U(\mathfrak{n})$):

$$[q, 1] = 0,$$

$$[q, y_1 \,...\, y_n] = \textstyle\sum_{i=1}^{n} y_1 \,...\, y_{i-1}[q, y_i]y_{i+1} \,...\, y_n \quad \text{for } q \in \mathfrak{q}.$$

We can now combine the two actions to an action of $\mathfrak{q} \ltimes \mathfrak{n}$ on $U(\mathfrak{n})$:

$$\begin{pmatrix} q \\ n \end{pmatrix} \times y_1 \,...\, y_n = ny_1 \,...\, y_n + [q, y_1 \,...\, y_n] \quad \text{for } n \in \mathfrak{n} \text{ and } q \in \mathfrak{q}.$$

It can be verified that this defines a $(\mathfrak{q} \ltimes \mathfrak{n})$-module structure on $U(\mathfrak{n})$. The rest of the proof almost mirrors the proof of section 3.4.:

Let $k \in \mathbb{N}$, s.t. $\mathfrak{n}^k = 0$ and $\mathfrak{n}^{k-1} \neq 0$. Define an ordered basis $\{x_1, \,...,\, x_n\}$, s.t. the first n_1 elements span $\mathfrak{n}^{k-1}$, the first n_2 span $\mathfrak{n}^{k-2}$ and so on. Let $t = \sum_\alpha c_\alpha x^\alpha$ (where $x^\alpha = x_1^{\alpha_1} \,...\, x_n^{\alpha_n}$ with $\alpha \in \mathbb{Z}_+^n$) be an element of $U(\mathfrak{n})$ with only finitely many $c_\alpha \neq 0$ and define an order function on $U(\mathfrak{n})$ as follows:

$$ord(t) = min\{ord(x^\alpha) | \; c_\alpha \neq 0\}, \text{ where}$$

$$ord(x^\alpha) = \textstyle\sum_{i=1}^{n} \alpha_i ord(x_i),$$

$$ord(x_i) = max\{s | \; x_i \in \mathfrak{n}^{s-1}\},$$

$$ord(1) = 0 \text{ and } ord(0) = \infty.$$

Let $U^m(\mathfrak{n}) = \{t \in U(\mathfrak{n}) | \; ord(t) \geq m\}$ and note that it is left invariant by the action of $\mathfrak{q} \ltimes \mathfrak{n}$ $\Big($because $ord\Big[\begin{pmatrix} q \\ n \end{pmatrix} \times t\Big] \geq ord(t)$ for $t \in U(\mathfrak{n})\Big)$, i.e. the action of $\mathfrak{q} \ltimes \mathfrak{n}$ on $U(\mathfrak{n})/U^m(\mathfrak{n})$ is well defined. It remains to show that this action is faithful for some m. Let $m > k$ from now on (which implies in particular that $\mathfrak{n}$ is fully embedded in $U(\mathfrak{n})/U^m(\mathfrak{n})$) and assume there exist $n \in \mathfrak{n}$, $q \in \mathfrak{q}$, s.t. $\begin{pmatrix} q \\ n \end{pmatrix} \times t = 0$ for all $t \in U(\mathfrak{n})/U^m(\mathfrak{n})$. It follows that

$n = \binom{q}{n} \times 1 = 0$, which also implies that

$[q, n'] = \binom{q}{0} \times n' = 0$ for all $n' \in \mathfrak{n} \subset U(\mathfrak{n})/U^m(\mathfrak{n})$.

Since the action of $\mathfrak{q}$ on $\mathfrak{n}$ is faithful, $q = 0$. □

4.4 Proof of Neretin's lemma

For the proof we will need the following two theorems. A proof can be found in [1] (Jordan-Chevalley decomposition) and [2] (Levi's theorem).

Jordan-Chevalley decomposition. *Let V be a vector space and $x \in \mathfrak{gl}(V)$. There exist unique $x_s,\ x_n \in \mathfrak{gl}(V)$, s.t.*

$$x = x_s + x_n,$$

x_s is diagonizable, x_n is nilpotent, x_s and x_n commute.

Levi's theorem. *Let L be a Lie algebra and denote its radical by $\mathfrak{r}$. There exists a semisimple subalgebra $\mathfrak{h}$, s.t.*

$$L = \mathfrak{h} \ltimes \mathfrak{r}.$$

Such a subalgebra $\mathfrak{h}$ is usually called Levi subalgebra, Levi complement or Levi part of L.

We will now introduce the so called elementary expansions. Let L be a Lie algebra and assume it has an ideal I of codimension one. Let $L - I = \langle x \rangle$ and $d = ad_x : I \to I$ with the Jordan-Chevalley decomposition $d = d_s + d_n$. Now define

$$\mathfrak{g} = \mathbb{C}^2 + I \ \ \text{(direct sum of vector spaces)}$$

and let $\{y,\ z\}$ be a basis of $\mathbb{C}^2$. With the following commutator relations, $\mathfrak{g}$ becomes a Lie algebra:

$$[y, z] = 0, \ \ [y, i] = d_s(i), \ \ [z, i] = d_n(i),$$

where $i \in I$ and the commutator of $i_1, i_2 \in I$ being the same as before. Now note that the subalgebra $\langle y+z \rangle + I$ is isomorphic to L and $[L, L] = [\mathfrak{g}, \mathfrak{g}]$. The Lie algebra $\mathfrak{g}$ is called an elementary expansion of L.

We are now ready to prove Neretin's lemma: Let L be an arbitrary (finite-dimensional) Lie algebra with the Levi decomposition

$$L = \mathfrak{h} \ltimes Rad(L).$$

Note that $\mathfrak{h}$ is semisimple (and therefore also reductive), which also implies that its action on $L = \mathfrak{h} \ltimes Rad(L)$ is completely reducible. Define $\mathfrak{n} = Nil(L)$ and note that

$[L, Rad(L)] \subset \mathfrak{n}$ and

$$[L, L] = \mathfrak{h} \ltimes [L, Rad(L)].$$

We remark here that any nilpotent ideal containing $[L, Rad(L)]$ can be used instead of the nilradical. If $L = \mathfrak{h} \ltimes \mathfrak{n}$, we are done. Assume from now on that $\mathfrak{h} \ltimes \mathfrak{n}$ is a proper subalgebra of L. This implies the existence of a subspace I of codimension one containing $\mathfrak{h} \ltimes \mathfrak{n}$ and therefore also $[L, L]$, i.e. I is an ideal. Now let $L - I = \langle x \rangle$ and construct the elementary expansion $\mathfrak{g}$ of L as above, i.e. $\mathfrak{g} = \mathbb{C}^2 + I = \langle y, z \rangle + I$. Define

$$\mathfrak{h}' = \langle y \rangle + \mathfrak{h},$$

$$\mathfrak{n}' = \langle z \rangle + \mathfrak{n}.$$

By construction, $\mathfrak{n}'$ is a nilpotent ideal in $\mathfrak{g}$. Now note that, since the action of $\mathfrak{h}$ on L is completely reducible, $\langle x \rangle$ is invariant under the adjoint action of $\mathfrak{h}$, i.e. (recall that $[L, L] \subset I$) x commutes with $\mathfrak{h}$. This implies that y commutes with $\mathfrak{h}$ as well, i.e.

$$\mathfrak{h}' = \langle y \rangle \oplus \mathfrak{h}$$

and $\mathfrak{h}'$ is reductive. By construction, since ad_y is diagonizable, the action of $\mathfrak{h}'$ on $\mathfrak{g}$ is fully reducible. Note that $\mathfrak{n}$ is a proper subalgebra of $\mathfrak{n}'$, $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{h}'$ and L is a proper subalgebra of $\mathfrak{g}$, all with codimension one. Therefore,

$$dim(\mathfrak{g}) - \big[dim(\mathfrak{h}') + dim(\mathfrak{n}')\big] = dim(L) - \big[dim(\mathfrak{h}) + dim(\mathfrak{n})\big] - 1.$$

This process can now be repeated with $\mathfrak{g},\ \mathfrak{h}',\ \mathfrak{n}'$ instead of $L,\ \mathfrak{h},\ \mathfrak{n}$ until, after finitely many repitions, we get the desired algebra.

□

4.5 Constructing a faithful representation of the 2-dimensional upper triangular matrices

Upper triangular matrices are a standard example for a solvable Lie algebra. Consider the space L of all complex 2-dimensional upper triangular matrices, spanned by

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \ x_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Lie bracket is then given by

$$[x_1, x_2] = 2x_2,$$

all other commutators being zero. A trivial representation is given by the identity map, so let's see if we can find another one. Note that

$$L^1 = L^{(1)} = [L, L] = \langle x_2 \rangle,$$

$$L^2 = L^1 \text{ and}$$

$$L^{(2)} = 0,$$

i.e. L is solvable but not nilpotent. The center $Z(L)$ is spanned by x_3 and the nilradical is

$$\mathfrak{n} = Z(L) \oplus [L, L] = \langle x_2,\ x_3 \rangle.$$

Note that $\mathfrak{n}$ is abelian. Furthermore,

$$L = \langle x_1 \rangle \ltimes \mathfrak{n}$$

and the action of $\langle x_1 \rangle$ on $\mathfrak{n}$ is faithful. We can now apply the construction of section 4.3 to this example. Note that

$$U(\mathfrak{n})/U^2(\mathfrak{n}) = \langle x_2,\ x_3,\ 1 \rangle.$$

If we denote the representation of L on $U(\mathfrak{n})/U^2(\mathfrak{n})$ by ϕ, then

$$\phi_{x_1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \ \phi_{x_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \ \phi_{x_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

4.6 Constructing a faithful representation of an abstract Lie algebra

Consider the abstract 4-dimensional Lie algebra L with basis $\langle x_1,\ x_2,\ x_3,\ x_4 \rangle$ and the Lie brackets

$$[x_1, x_2] = x_2,$$

$$[x_1, x_4] = x_2$$

and all other commutators equal to zero. L is solvable (therefore the Levi subalgebra $\mathfrak{h} = 0$) and the (abelian) nilradical is

$$Nil(L) = \langle x_2,\ x_3,\ x_2 - x_4 \rangle.$$

We could now proceed as in the last subsection to obtain a 4-dimensional representation of L. However, this time we would like to apply Neretin's lemma. Therefore, note that

$$\mathfrak{n} = Z(L) \oplus [L, L] = \langle x_2,\ x_3 \rangle$$

is a nilpotent ideal in L and contains $[L, Rad(L)] = [L, L]$. It can thus be used as a starting point for elementary expansions (instead of the nilradical). An ideal containing $\mathfrak{n}$ is given by

$$I = \mathfrak{n} + \langle x_4 \rangle.$$

Now construct the elementary expansion

$\mathfrak{g} = \mathbb{C}^2 + I = \langle y, z \rangle + I$ with

$$[y, x_2] = x_2,$$

$$[y, x_4] = x_2$$

and define

$$\mathfrak{h}' = \langle y \rangle,$$

$$\mathfrak{n}' = \langle z \rangle + \mathfrak{n}.$$

Note that $\mathfrak{h}' \ltimes \mathfrak{n}'$ is of codimension one in $\mathfrak{g}$, i.e. we need to repeat the procedure one more time. Define

$\mathfrak{g}' = \mathbb{C}^2 + \mathfrak{h}' \ltimes \mathfrak{n}' = \langle y', z' \rangle + \mathfrak{h}' \ltimes \mathfrak{n}'$ with

$$[z', y] = -x_2.$$

We now have

$$\mathfrak{h}'' = \langle y', y \rangle,$$

$\mathfrak{n}'' = \langle z' \rangle + \mathfrak{n}'$ and

$$\mathfrak{g}' = \mathfrak{h}'' \ltimes \mathfrak{n}'' = \langle y' \rangle \oplus \big(\langle y \rangle \ltimes \mathfrak{n}''\big).$$

We can now apply the construction of section 4.3 to get a representation ϕ of $\mathfrak{g}'$, spanned by $\{\phi_{y'},\ \phi_y,\ \phi_{z'},\ \phi_z,\ \phi_{x_2},\ \phi_{x_3}\}$. The representation of L is then given by

$\langle \phi_{x_1}, \phi_{x_2}, \phi_{x_3}, \phi_{x_4} \rangle$, where

$\phi_{x_1} = \phi_y + \phi_z$ and $\phi_{x_4} = \phi_{y'} + \phi_{z'}$.

Now note that

$\mathfrak{n}''^1 = \langle x_2 \rangle$ and

$$\mathfrak{n}''^2 = 0.$$

The quotient algebra $U(\mathfrak{n}'')/U^3(\mathfrak{n}'')$ is thus spanned by

$$\{x_2,\ z'z,\ z'x_3,\ zx_3,\ z'^2,\ z^2,\ x_3^2,$$

$$z',\ z,\ x_3,$$

$$1\}.$$

The identity map is an obvious representation of $\langle y' \rangle$ and the matrix forms of the ϕ_{x_i} w.r.t. the basis

$$\{y',\ x_2,\ z'z,\ z'x_3,\ zx_3,\ z'^2,\ z^2,\ x_3^2,\ z',\ z,\ x_3,\ 1\}$$

are

$$\phi_{x_1} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \phi_{x_2} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\phi_{x_3} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \phi_{x_4} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

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