FACULTY OF SCIENCE

Almost-inner derivations of Lie algebras

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Preface

This master thesis is about almost-inner derivations of Lie algebras, written to filfill the requirements for becoming a Master of Science in Mathematics. I started studying the subject in September 2015. The process of researching and writing this thesis took me till the start of June 2016. During this period, there are a lot of people who helped me making this thesis. I want to thank them for their help and support.

I would first like to express my gratitude to Prof. dr. Karel Dekimpe, for giving me already the taste of doing research, for the excellent guidance, for reading all preliminary versions and giving advice and comments to improve this thesis. Thank you for always making time to help, both in Leuven and in Kortrijk. I could not have imagined having a better promotor.

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Abstract

The purpose of this thesis is to study almost-inner derivations of Lie algebras in an algebraic way. An almost-inner derivation φ of a Lie algebra $\mathfrak{g}$ is a derivation φ such that $\varphi(X) \in [X, \mathfrak{g}]$ for all $X \in \mathfrak{g}$. This kind of derivations arises in the study of spectral geometry, in particular in the construction of isospectral and non-isometric manifolds. The notion has not been much studied algebraically and only in some specific cases.

In the first chapter, the geometric connection between Lie groups and Lie algebras is explained. Further, there is an algebraic introduction to Lie algebra-theory. Those preliminaries will be useful in the following chapters.

The second chapter is devoted to the concept of almost-inner derivations. First, some special types of derivations are introduced. This concerns in particular the almost-inner ones, which form a generalisation of the inner derivations. These notions permit to understand the underlying geometric motivation, which is worked out in the second section. It illustrates why almost-inner derivations are interesting to study. To investigate the concept more thoroughly, there is need of some theory concerning the definition. This is elaborated in the last section. There is a procedure to compute the set of all almost-inner derivations. This working method is explained and illustrated with an example. It is used as a manual for the calculations in the rest of the thesis.

In the third chapter, special classes of Lie algebras are introduced. The first class consists of the complex Lie algebras of dimension at most four. Next, the metabelian filiform Lie algebras are studied. The following class concerns two-step nilpotent Lie algebras which are defined by graphs. Further, also free nilpotent Lie algebras and Lie algebras of strictly uppertriangular and uppertriangular matrices are treated. For these classes (and sometimes only for specific cases), the set of all almost-inner derivations is computed. For the metabelian filiform Lie algebras, it turns out that the dimension of the set of almost-inner derivations is at most one more than the dimension of the set of inner derivations. In all other cases, both sets are equal. The question for the free nilpotent Lie algebras is only solved when the nilindex is two or three. In the context of almost-inner derivations, nilpotent Lie algebras are geometrically of most importance. Therefore, the introduced classes (except for the first and last ones) only consist of nilpotent Lie algebras.

The appendix contains some computer programs implemented in Matlab. Those algorithms ease the computations. The first part checks whether or not given structure constants define a Lie algebra. The second section concerns algorithms computing a basis for the space of all derivations of a given Lie algebra.

Samenvatting

In deze thesis worden bijna-inwendige derivaties van Lie algebra's bestudeerd op een algebraïsche manier. Een bijna-inwendige derivatie φ van een Lie algebra $\mathfrak{g}$ is een derivatie φ zodat $\varphi(X) \in [X, \mathfrak{g}]$ voor alle $X \in \mathfrak{g}$. Dit soort derivaties duikt op in de spectrale meetkunde, in het bijzonder bij de constructie van isospectrale en niet-isometrische variëteiten. De bijna-inwendige derivaties zijn nog niet vaak algebraïsch bestudeerd en enkel in bepaalde specifieke gevallen.

In het eerste hoofdstuk wordt het meetkundig verband tussen Lie groepen en Lie algebra's uitgewerkt. Verder is er ook een algebraïsche inleiding op Lie algebra-theorie. De ingevoerde concepten zullen in de latere hoofdstukken terugkomen.

Het tweede hoofdstuk handelt over bijna-inwendige derivaties. Eerst worden verschillende soorten derivaties ingevoerd. In het bijzonder zijn dat de bijna-inwendige derivaties, een veralgemening van de inwendige derivaties. Deze noties laten het toe om de achterliggende meetkundige motivatie te begrijpen. De tweede sectie illustreert waarom bijna-inwendige derivaties interessant zijn om onderzoek naar te doen. Om deze concepten verder te kunnen bestuderen, is er meer theorie nodig. Dit is uitgewerkt in het laatste deel. Het berekenen van de verzameling van alle bijna-inwendige derivaties gebeurt aan de hand van een vast stappenplan. Deze werkwijze is uitgelegd en geïllustreerd aan de hand van een voorbeeld. Het vormt de leidraad bij de berekeningen in de rest van de thesis.

In het derde hoofdstuk worden speciale klassen van Lie algebra's ingevoerd. Het eerste type bestaat uit de complexe Lie algebra's waarvan de dimensie hoogstens vier is. Vervolgens worden meta-abelse filiforme Lie algebra's bestudeerd. De derde klasse bevat de twee-staps nilpotente Lie algebra's die door grafen gedefinieerd zijn. Ook worden de vrije nilpotente Lie algebra's en de (strikte) bovendriehoeksmatrices behandeld. Voor al deze klassen (in sommige gevallen enkel voor specifieke voorbeelden) is de verzameling van bijna-inwendige derivaties berekend. Voor de meta-abelse filiforme Lie algebra's blijkt dat de dimensie van de verzameling bijna-inwendige derivaties hoogstens één meer is dan de dimensie van de verzameling inwendige derivaties. In alle andere gevallen zijn beide verzamelingen gelijk. Voor de vrije nilpotente Lie algebra's is het resultaat enkel gekend als de nilindex gelijk is aan twee of drie. In de context van bijna-inwendige Lie algebra's zijn de nilpotente Lie algebra's meetkundig het meest van belang. Daarom bestaan de ingevoerde klassen (op de eerste en laatste na) enkel uit nilpotente Lie algebra's.

De appendix bevat een aantal computerprogramma's die geïmplementeerd zijn in Matlab. Deze algoritmes maken de berekeningen eenvoudiger. In het eerste deel wordt nagegaan

of gegeven structuurconstanten al dan niet een Lie algebra definiëren. Het volgende deel gaat over algoritmes die een basis berekenen voor de verzameling van derivaties voor een gegeven Lie algebra.

Contents

Chapter 1

Basic theory of Lie algebras

This chapter introduces the basic theory of Lie algebras which will be used in later chapters. First, the geometric connection between Lie groups and Lie algebras is explained. Then, there is an algebraic introduction to Lie algebras. This contains some well-known examples and terminology of Lie algebras. Those preliminaries are important for the rest of this thesis.

1.1 Lie groups and Lie algebras

In this section, the concepts of a Lie group and a Lie algebra are introduced. Lie groups are important for geometrical purposes, but Lie algebras are easier to handle for many computations. Therefore, to gain insight about Lie groups, the corresponding Lie algebras are studied. The material in this section is based on [4].

Definition 1.1.1 (Lie group)**.** *A (real) Lie group G is a group which is a smooth manifold and for which the multiplication μ and inversion ν are smooth maps.*

In this definition, the maps $\mu : G \times G \to G : (x, y) \mapsto xy$ and $\nu : G \to G : x \mapsto x^{-1}$ are defined as expected.

Example 1.1.2. *Denote $M_n(\mathbb{R})$ the space of all real $(n \times n)$-matrices. Consider this space as $\mathbb{R}^{n^2}$. The determinant function $\det : \mathbb{R}^{n^2} \to \mathbb{R} : A \mapsto \det(A)$ is a smooth map. Hence, the general linear group*

$$GL(n, \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

is open in $\mathbb{R}^{n^2}$ and therefore a manifold. Further, $GL(n, \mathbb{R})$ is obviously a group. Moreover, the entries of the product of two matrices are polynomials in the entries of the two matrices. Therefore, the matrix multiplication and inversion are smooth maps, which means that $GL(n, \mathbb{R})$ defines a Lie group.

The left translation is the following notion to be defined.

Definition 1.1.3 (Left translation)**.** *Let G be a Lie group and $g \in G$. The left translation $\lambda_g : G \to G$ is defined by $\lambda_g(h) = \mu(g, h) = gh$.*

Since the multiplication map of a Lie group is smooth by definition, λ_g defines a diffeomorphism for the manifold G, where the inverse is given by $\lambda_{g^{-1}}$.

Let $f : M \to N$ be a smooth map between two smooth manifolds M and N. The differential of f at x is denoted by $\mathrm{d}f_x : T_xM \to T_{f(x)}N$. Let $X \in T_eG$ and $g \in G$, then $L_X(g)$ is defined as $\mathrm{d}(\lambda_g)_e(X) \in T_gG$. It is thus the differential of the left translation in the neutral element. The map $L_X : G \to TG$ is a vector field on G, since $L_X(g) \in T_gG$ and L_X is smooth.

The notion $\mathfrak{X}(M)$ is used for the set of all smooth vector fields on the manifold M. For arbitrary vector fields on a smooth manifold, the Lie bracket can be defined.

Definition 1.1.4 (Lie bracket)**.** *Let M be a smooth manifold and let $\xi, \eta \in \mathfrak{X}(M)$ be vector fields. The Lie bracket $[\xi, \eta] \in \mathfrak{X}(M)$ of ξ and η is the unique vector field for which $[\xi, \eta] \cdot f = \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f)$ holds for all smooth functions $f : M \to \mathbb{R}$.*

It is clear that the Lie bracket is bilinear and skew-symmetric.

Let M be a smooth manifold and let ξ, η and $\zeta \in \mathfrak{X}(M)$ be vector fields on M. By calculation, it is easy to show that the Jacobi-identity

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$$

holds. Another well-known concept in differential geometry is that of the pullback.

Definition 1.1.5 (Pullback)**.** *Let $f : M \to N$ be a local diffeomorphism for manifolds M and N and let $\xi \in \mathfrak{X}(N)$ be a vector field on N. The pullback $f^*\xi \in \mathfrak{X}(M)$ is defined by*

$$f^*\xi(x) := (\mathrm{d}f_x)^{-1} \circ \xi(f(x)) \quad \textit{for all } x \in M.$$

Let G be a Lie group with $g \in G$ and let $\xi \in \mathfrak{X}(G)$ be a vector field, then $(\lambda_g)^*\xi \in \mathfrak{X}(G)$. A vector field is called left invariant if it is preserved by the pullback of the left translation by g, where g is an arbitrary element of the Lie group.

Definition 1.1.6 (Left invariant vector field)**.** *Let G be a Lie group and let $\xi \in \mathfrak{X}(G)$. Then ξ is left invariant if and only if $(\lambda_g)^*\xi = \xi$ for all $g \in G$.*

The space of left invariant vector fields of a Lie group G is denoted by $\mathfrak{X}_L(G)$. Next proposition reveals a relation between the tangent space at the neutral element and the set of all left invariant vector fields of a Lie group.

Proposition 1.1.7. *Let G be a Lie group and denote $\mathfrak{g} = T_eG$. The vector field L_X is left invariant, where $X \in \mathfrak{g}$. Moreover, $\mathfrak{g}$ is isomorphic with $\mathfrak{X}_L(G)$.*

Proof. Let $g, h \in G$ be arbitrary and $X \in \mathfrak{g}$. By definition,

$$((\lambda_g)^*L_X)(h) = (\mathrm{d}(\lambda_g)_h)^{-1} \circ L_X(\lambda_g(h)) = \mathrm{d}(\lambda_{g^{-1}})_{gh} \circ \mathrm{d}(\lambda_{gh})_e(X) = \mathrm{d}(\lambda_h)_e(X) = L_X(h)$$

holds, which means that L_X is left invariant. Hence, there are linear maps between $\mathfrak{g}$ and $\mathfrak{X}_L(G)$. Define

$$\varphi : \mathfrak{g} \to \mathfrak{X}_L(G) : X \mapsto L_X \quad \text{and} \quad \psi : \mathfrak{X}_L(G) \to \mathfrak{g} : \xi \mapsto \xi(e).$$

By definition, $L_X(e) = X$, such that $\psi \circ \varphi = \mathrm{Id}_{\mathfrak{g}}$. On the other hand, if $\xi \in \mathfrak{X}_L(G)$ and $X = \xi(e)$, then

$$\xi(g) = ((\lambda_{g^{-1}})^*\xi)(g) = (\mathrm{d}(\lambda_{g^{-1}})_g)^{-1} \circ \xi(\lambda_{g^{-1}}(g)) = \mathrm{d}(\lambda_g)_e \circ \xi(g^{-1}g) = \mathrm{d}(\lambda_g)_e(X) = L_X(g)$$

is satisfied. This means that $\xi = L_X$ and $\varphi \circ \psi$ is the identity on $\mathfrak{X}_L(G)$. Hence, the linear maps φ and ψ are inverse linear isomorphisms between $\mathfrak{g}$ and $\mathfrak{X}_L(G)$. $\square$

It is easy to check that for every local diffeomorphism $f : M \to N$ (where M and N are manifolds), the equality $[f^*\xi, f^*\eta] = f^*([\xi,\eta])$ holds for all $\xi,\eta \in \mathfrak{X}(N)$. In particular, let G be a Lie group with $g \in G$ and let $\xi,\eta \in \mathfrak{X}_L(G)$ be left invariant vector fields, then

$$\lambda_g^*[\xi,\eta] = [\lambda_g^*\xi, \lambda_g^*\eta] = [\xi,\eta]$$

holds, which means that $[\xi,\eta] \in \mathfrak{X}_L(G)$.

Definition 1.1.8 (Lie algebra)**.** *Let G be a Lie group. The Lie algebra of G is the tangent space $\mathfrak{g} = T_eG$ with the map*

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X,Y) \mapsto [X,Y] = [L_X, L_Y](e).$$

Due to the last proposition, this is well-defined and $[L_X, L_Y] = L_{[X,Y]}$ holds. In this definition, the Lie bracket is skew-symmetric and satisfies the Jacobi-identity, since these are properties of the Lie bracket of vector fields. Next example gives the Lie algebra of the general linear group.

Example 1.1.9. *Denote the Lie algebra of $GL(n,\mathbb{R})$ with $\mathfrak{gl}(n,\mathbb{R})$. It is clear that, as a vector space, $\mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$. Let $A \in GL(n,\mathbb{R})$ and $B \in M_n(\mathbb{R})$. One can show that the Lie bracket on $\mathfrak{gl}(n,\mathbb{R})$ is given by*

$$[A,B] = [L_A, L_B](e) = AB - BA,$$

for all $A,B \in M_n(\mathbb{R})$.

For the rest of this thesis, the Lie algebras will be studied algebraically instead of geometrically.

1.2 Basic theory of Lie algebras

As showed in the first section, Lie algebras can be defined as the tangent space at the neutral element of a Lie group. So, for every Lie group, there is a corresponding Lie algebra. In his third fundamental theorem, Lie showed that the converse is also true for finite-dimensional real Lie algebras. However, it is also interesting to study Lie algebras without the underlying Lie group. In this section, an algebraic definition of a Lie algebra is given. The notion has the same properties as before, but makes it possible to work without the background from differential geometry. Since the Lie groups from Section 1.1 were real manifolds, the corresponding Lie algebras were real too. However, there is no need to restrict to the case where $K = \mathbb{R}$. The same reasoning as before can be done similarly for other fields. When the field is not denoted, the result holds for arbitrary fields.

1.2.1 Definitions

This section elaborates the terminology which will be used in later chapters. First, the definition of an algebra is recalled.

Definition 1.2.1 (Algebra)**.** *Let K be a field. An algebra over a field K is a K-vector space A together with a bilinear map $\cdot : A \times A \to A$.*

A Lie algebra is an algebra, for which the bilinear map is called the Lie bracket. This bracket satisfies certain conditions, motivated by the corresponding properties of the Lie bracket for vector fields.

Definition 1.2.2 (Lie algebra)**.** *Let K be a field. A Lie algebra $\mathfrak{g}$ over K is an algebra where the bilinear map*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto [X, Y]$$

satisfies the following properties:

- *The bracket $[X, X] = 0$ holds for all $X \in \mathfrak{g}$;*
- *For all X, Y and $Z \in \mathfrak{g}$, the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ is fulfilled.*

The first condition implies skew-symmetry, since for all $X, Y \in \mathfrak{g}$

$$0 = [X + Y, X + Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y] = [X, Y] + [Y, X]$$

holds by bilinearity of the bracket. When $\mathrm{char}(K) \neq 2$, both concepts are even equivalent.

Since a Lie algebra is a vector space, every element can be expressed uniquely as linear combination of the basis vectors. Although a Lie algebra can have infinite dimension, only finite-dimensional Lie algebras are studied in this thesis.

Definition 1.2.3 (Structure constants)**.** *Let $\mathfrak{g}$ be an n-dimensional Lie algebra with basis $\mathcal{B} = \{X_1, X_2, \ldots, X_n\}$. Then there exist $c_{ij}^k \in K$ such that*

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

These values c_{ij}^k (with $1 \leq i, j, k \leq n$) are the structure constants of $\mathfrak{g}$ with respect to the basis $\mathcal{B}$.

The conditions imposed on the structure constants by the Jacobi identity can be calculated as follows. Consider an n-dimensional Lie algebra $\mathfrak{g}$ with basis $\mathcal{B} = \{X_1, \ldots, X_n\}$. Let X_i, X_j and X_k be three arbitrary basis vectors, so $1 \leq i, j, k \leq n$. Then

$$\begin{aligned}
0 &= [X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] \\
&= \sum_{m=1}^{n} c_{jk}^m [X_i, X_m] + \sum_{m=1}^{n} c_{ki}^m [X_j, X_m] + \sum_{m=1}^{n} c_{ij}^m [X_k, X_m] \\
&= \sum_{m=1}^{n} (c_{jk}^m [X_i, X_m] + c_{ki}^m [X_j, X_m] + c_{ij}^m [X_k, X_m]) \\
&= \sum_{m=1}^{n} (c_{jk}^m \sum_{l=1}^{n} c_{im}^l X_l + c_{ki}^m \sum_{l=1}^{n} c_{jm}^l X_l + c_{ij}^m \sum_{l=1}^{n} c_{km}^l X_l) \\
&= \sum_{m=1}^{n} \sum_{l=1}^{n} (c_{jk}^m c_{im}^l + c_{ki}^m c_{jm}^l + c_{ij}^m c_{km}^l) X_l
\end{aligned}$$

holds. This means that

$$\sum_{m=1}^{n} (c_{jk}^m c_{im}^l + c_{ki}^m c_{jm}^l + c_{ij}^m c_{km}^l) = 0 \tag{1.1}$$

has to be satisfied for all $1 \leq i, j, k, l \leq n$.

It suffices to specify the Lie bracket $[X_i, X_j]$ and hence the structure constants only when $i < j$, since $[X_i, X_i] = 0$ and $[X_j, X_i] = -[X_i, X_j]$. Usually, only the non-vanishing brackets are mentioned. It can be examined whether or not an algebra with given structure constants defines a Lie algebra, by verifying the equations (1.1) for all possible values. In appendix A.1, algorithm `checkJacobi(C)` in Code A.2 checks whether or not `C` represents a Lie algebra. The input is an $(n \times n \times n)$-matrix `C` where $C(i, j, k)$ stands for the structure constant c_{ij}^k. For the convenience, only the non-zero entries $C(i, j, k)$ are implemented when $i < j$.

Definition 1.2.4 (Appearances of a basis vector)**.** *Let $\mathfrak{g}$ be an n-dimensional Lie algebra with basis $\mathcal{B} = \{X_1, \ldots, X_n\}$. A basis vector X_k is said to appear m times if there exist exactly m different pairs $\{i, j\}$ so that $c_{ij}^k = -c_{ji}^k \neq 0$.*

Of course, this definition depends on the choice of the basis.

Example 1.2.5. *Let $\mathfrak{g}$ be the four-dimensional Lie algebra over a field K with char$(K) \neq 0$ and basis $\mathcal{B} = \{X_1, X_2, X_3, X_4\}$ and non-vanishing Lie brackets*

$$[X_1, X_2] = X_2; \quad [X_1, X_3] = X_2 + X_3; \quad [X_1, X_4] = 2X_4 \quad \text{and} \quad [X_2, X_3] = X_4.$$

It is not so hard to show that the structure constants satisfy (1.1) for all $1 \leq i, j, k, l \leq 4$. For this Lie algebra, X_1 does not appear, X_2 and X_4 appear twice and X_3 appears once.

An important example of a Lie algebra is the following, motivated by the example of the first section.

Example 1.2.6. *Let V be a finite-dimensional vector space over a field K. Denote the dimension of V by $\dim(V) = n$. The set $\mathbf{gl}(V)$ denotes all linear maps from V to V. This is a Lie algebra for which the Lie bracket is defined by*

$$[X, Y] = X \circ Y - Y \circ X \qquad \text{for all } X, Y \in \mathbf{gl}(V).$$

It is easy to verify that the conditions on the Lie bracket are satisfied. Every linear map from V to V can be represented by an $(n \times n)$-matrix. Therefore, the vector space $\mathbf{gl}(n, K)$ of all $(n \times n)$-matrices also defines a Lie algebra, where the Lie bracket is given by

$$[A, B] = AB - BA \quad \text{for all } A, B \in \mathbf{gl}(n, K).$$

Here, AB denotes the usual matrix multiplication. This example will turn up a lot in this thesis.

The next definition is that of a Lie subalgebra, a vector subspace for which the Lie bracket is conserved for all elements.

Definition 1.2.7 (Lie subalgebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.*

It is clear that a Lie subalgebra is also a Lie algebra.

Example 1.2.8. *Let K be a field and consider $\mathfrak{g} = \{A \in K^{2\times 2} \mid trace(A) = 0\}$. Let*

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$$

be two arbitrary elements of $\mathfrak{g}$, hence $a, b, c, d, e, f \in K$. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} d & e \\ f & -d \end{pmatrix} - \begin{pmatrix} d & e \\ f & -d \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} &= \begin{pmatrix} ad+bf & ae-bd \\ cd-af & ce+ad \end{pmatrix} - \begin{pmatrix} ad+ce & bd-ae \\ af-cd & bf+ad \end{pmatrix} \\ &= \begin{pmatrix} bf-ce & 2(ae-bd) \\ 2(cd-af) & ce-bf \end{pmatrix} \end{aligned}$$

has zero trace, which means that $\mathfrak{g}$ is a Lie subalgebra of $\mathbf{gl}(2, K)$.

A basis for $\mathfrak{g}$ is given by

$$X_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the Lie brackets are defined by $[X_1, X_2] = X_3, [X_1, X_3] = -2X_1$ and $[X_2, X_3] = 2X_2$.

The next construction is that of an ideal, a special type of a subalgebra.

Definition 1.2.9 (Ideal)**.** *An ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and all $Y \in \mathfrak{h}$.*

Contrary to the definition of an ideal for rings, there is no distinction between right and left ideals due to skew-symmetry of the Lie bracket. An important example of an ideal is the centre of a Lie algebra.

Definition 1.2.10 (Centre)**.** *The centre $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined as*

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

This is indeed an ideal of $\mathfrak{g}$. Next, the product of two ideals is introduced.

Definition 1.2.11 (Product of ideals)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field K with ideals I and J. The product of I and J is given by*

$$[I, J] = span\{[X, Y] \in \mathfrak{g} \mid X \in I \text{ and } Y \in J\}.$$

Let $h \in [I, J]$, so there exist $n \in \mathbb{N}$, $a_i \in K, x_i \in I$ and $y_i \in J$ (for all $1 \le i \le n$) such that $h = \sum_{i=1}^{n} a_i [x_i, y_i]$. Then,

$$[g, h] = [g, \sum_{i=1}^{n} a_i [x_i, y_i]] = \sum_{i=1}^{n} a_i [g, [x_i, y_i]]$$

for all $g \in \mathfrak{g}$. By definition of the Jacobi identity,

$$[g, [x, y]] = [x, [g, y]] + [[g, x], y]$$

holds for all $g \in \mathfrak{g}, x \in I$ and $y \in J$. Since $[g, y] \in J$ and $[g, x] \in I$, it follows that

$$[x, [g, y]] \in [I, J] \quad \text{and} \quad [[g, x], y] \in [I, J].$$

Hence, $[g, h] \in [I, J]$ for every $h \in [I, J]$ and $g \in \mathfrak{g}$, which means that $[I, J]$ is an ideal of $\mathfrak{g}$. Note that in some cases, the set of all commutators $\{[X, Y] \in \mathfrak{g} \mid X \in I \text{ and } Y \in J\}$ is not an ideal.

Example 1.2.12. *Define* $\mathfrak{g}$ *as the set of all matrices of the form*

$$(f(x), g(y), h(x,y)) := \begin{pmatrix} 0 & f(x) & h(x,y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix},$$

where $f(x) \in \mathbb{R}[x]$*,* $g(y) \in \mathbb{R}[y]$ *and* $h(x,y) \in \mathbb{R}[x,y]$*. Then,* $\mathfrak{g}$ *is an (infinite-dimensional) algebra over* $\mathbb{R}$ *with the usual commutator bracket.*

Consider three arbitrary elements of $\mathfrak{g}$, namely $A_1 := (f_1(x), g_1(y), h_1(x,y))$, $A_2 := (f_2(x), g_2(y), h_2(x,y))$ and $A_3 := (f_3(x), g_3(y), h_3(x,y))$. By construction,

$$[(f_1(x), g_1(y), h_1(x,y)), (f_2(x), g_2(y), h_2(x,y))] = (0, 0, f_1(x)g_2(y) - f_2(x)g_1(y)).$$

Since $[A_1, [A_2, A_3]] = 0$ holds, the Jacobi identity is satisfied and $\mathfrak{g}$ is thus a Lie algebra. Further, $[\mathfrak{g}, \mathfrak{g}]$ consists of all matrices of the form

$$(0, 0, h(x,y)) = \begin{pmatrix} 0 & 0 & h(x,y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $h(x,y) \in \mathbb{R}[x,y]$. Denote $\mathfrak{g}_c$ for the set of all commutators. When $h(x,y) := x^2 + xy + y^2$, it is easy to see that $(0, 0, h(x,y)) \in [\mathfrak{g}, \mathfrak{g}]$, but $(0, 0, h(x,y)) \notin \mathfrak{g}_c$, although $(0, 0, x^2)$, $(0, 0, xy)$ and $(0, 0, y^2)$ belong to $\mathfrak{g}_c$. Hence, $\mathfrak{g}_c$ is not a vector space and thus not an ideal of $\mathfrak{g}$.

For $I = J = \mathfrak{g}$, the construction $[\mathfrak{g}, \mathfrak{g}]$ has an own name.

Definition 1.2.13 (Derived algebra)**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* K*. The derived algebra of* $\mathfrak{g}$ *is given by* $[\mathfrak{g}, \mathfrak{g}]$*.*

Since a Lie algebra is a vector space, it makes sense to consider the quotient vector space. Let $\mathfrak{g}$ be a Lie algebra with ideal I, then the cosets $X + I = \{X + Y \mid Y \in I\}$ (with $X \in \mathfrak{g}$) form the quotient vector space

$$\mathfrak{g}/I = \{X + I \mid X \in \mathfrak{g}\}.$$

It is possible to turn this into a Lie algebra, where the Lie bracket is defined as

$$[X + I, Y + I] := [X, Y] + I \quad \text{for all } X, Y \in \mathfrak{g}.$$

The value of $[X, Y] + I$ only depends on the cosets, not on the particular representatives. Let $X_1 + I = X_2 + I$ and $Y_1 + I = Y_2 + I$. By bilinearity of the Lie bracket in $\mathfrak{g}$,

$$\begin{aligned} [X_2, Y_2] &= [X_1 + (X_2 - X_1), Y_1 + (Y_2 - Y_1)] \\ &= [X_1, Y_1] + [X_2 - X_1, Y_1] + [X_1, Y_2 - Y_1] + [X_2 - X_1, Y_2 - Y_1]. \end{aligned}$$

The last three summands belong to I, so that $[X_2, Y_2] + I = [X_1, Y_1] + I$. Hence, the Lie bracket is well-defined. It is easy to verify that the Lie bracket on the quotient vector space is bilinear and fulfills the Jacobi identity.

Like other algebraic structures, the notion of a homomorphism makes sense for Lie algebras too.

Definition 1.2.14 (Homomorphism). *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the same field K. A linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism if*

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$$

holds for all $X, Y \in \mathfrak{g}_1$. A bijective homomorphism is called an isomorphism.

The bracket on the left side is taken in $\mathfrak{g}_1$ and the bracket on the right side in $\mathfrak{g}_2$.

The following example of a Lie algebra homomorphism will also show up in the next chapter.

Example 1.2.15. *Let $\mathfrak{g}$ be a Lie algebra over a field K. The adjoint homomorphism $ad : \mathfrak{g} \to \mathbf{gl}(\mathfrak{g})$ is defined as*

$$(ad(X)) : \mathfrak{g} \to \mathfrak{g} : Y \mapsto [X,Y] \quad \textit{for all } X, Y \in \mathfrak{g}.$$

Since the Lie bracket is bilinear, $\mathrm{ad}(X)$ is linear and belongs hence to $\mathbf{gl}(\mathfrak{g})$. Moreover, ad is linear for the same reason. Let $Z \in \mathfrak{g}$ be arbitrary. The Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

is equivalent with

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]].$$

For all $X, Y \in \mathfrak{g}$, this means that

$$\begin{aligned} \mathrm{ad}([X,Y]) &= \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X) \\ &= [\mathrm{ad}(X), \mathrm{ad}(Y)], \end{aligned}$$

where the first bracket is taken in $\mathfrak{g}$ and the second bracket in $\mathbf{gl}(\mathfrak{g})$. Hence, the adjoint map is indeed a Lie algebra homomorphism.

The next subsection is devoted to the introduction of some notions. Those concepts have an important role in Chapter 3 of this thesis, where the almost-inner derivations of special classes of Lie algebras will be computed.

1.2.2 Special constructions of Lie algebras

In the next chapter, almost-inner derivations are defined. This concept is geometrically important, especially for nilpotent Lie algebras. This kind of Lie algebras is introduced in this section. First, the notion of a direct sum of Lie algebras is explained.

Definition 1.2.16 (Direct sum of Lie algebras). *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over the same field K. The direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the vector space direct sum with $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$.*

It is clear that both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$ in this case.

Before the definition of a nilpotent Lie algebra can be given, some other notions have to be elaborated. The first concept is a generalisation of the derived algebra.

Definition 1.2.17 (Derived series). *Let $\mathfrak{g}$ be a Lie algebra over a field K. The derived series of $\mathfrak{g}$ is the series with terms*

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \qquad \textit{and} \qquad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \quad \textit{for } k \geq 2.$$

As for groups, the notion of solvability can also be introduced for Lie algebras.

Definition 1.2.18 (Solvable Lie algebra)**.** *A non-zero Lie algebra* $\mathfrak{g}$ *over a field* K *is solvable when* $\mathfrak{g}^{(k)} = 0$ *for some* $k \geq 1$*.*

A Lie algebra for which $[\mathfrak{g}, \mathfrak{g}] = 0$ is called 'abelian'. A non-abelian Lie algebra $\mathfrak{g}$ is 'metabelian' when $\mathfrak{g}^{(2)} = 0$. This concept will be important in Section 3.2 of the next chapter.

Example 1.2.19. *The four-dimensional Lie algebra* $\mathfrak{g}$ *with basis* $\mathcal{B} = \{X_1, X_2, X_3, X_4\}$ *and non-vanishing Lie brackets as in Example 1.2.5 is three-step solvable, since*

$$\mathfrak{g}^{(1)} = < X_2, X_3, X_4 >; \qquad \mathfrak{g}^{(2)} = < X_4 > \quad \textit{and} \quad \mathfrak{g}^{(k)} = 0 \ \textit{for all} \ k \geq 3.$$

Another notion is that of a semisimple Lie algebra.

Definition 1.2.20 (Semisimple Lie algebra)**.** *Let* $\mathfrak{g}$ *be a non-zero finite-dimensional Lie algebra over a field* K*. Then* $\mathfrak{g}$ *is semisimple if it has no non-zero solvable ideals.*

Note that when a Lie algebra $\mathfrak{g}$ is solvable, it cannot be semisimple, since $\mathfrak{g}$ itself is an ideal of $\mathfrak{g}$. The following Lie algebra is the standard example of a semisimple Lie algebra.

Example 1.2.21. *Let* $\mathfrak{g}$ *be the three-dimensional Lie algebra over a field* K *with char*$(K) \neq 2$ *and basis* $\mathcal{B} = \{X_1, X_2, X_3\}$ *and non-vanishing Lie brackets*

$$[X_1, X_2] = X_3; \qquad [X_1, X_3] = -2X_1 \quad \textit{and} \quad [X_2, X_3] = 2X_2.$$

Then $\mathfrak{g}$ *is semisimple.*

This Lie algebra even does not have proper ideals. Note that this is the same Lie algebra as in Example 1.2.8.

A notion related to solvability is nilpotency. First, the lower central series is introduced.

Definition 1.2.22 (Lower central series)**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* K*. The lower central series of* $\mathfrak{g}$ *is the series with terms*

$$\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \qquad \textit{and} \qquad \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}] \quad \textit{for } k \geq 2.$$

A Lie algebra is called nilpotent when there exists a natural number k such that every Lie bracket with more than k elements vanishes.

Definition 1.2.23 (Nilpotent Lie algebras)**.** *A non-zero Lie algebra* $\mathfrak{g}$ *over a field* K *is nilpotent when* $\mathfrak{g}^k = 0$ *for some* $k \geq 1$*.*

Most of the classes studied in Chapter 3 consist of nilpotent Lie algebras. It is easy to see that $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$ for all $k \in \mathbb{N}$. Hence, any nilpotent Lie algebra is also solvable. The converse is not true: the two-dimensional non-abelian Lie algebra (given by $[X_1, X_2] = X_1$) is solvable, but not nilpotent.

Definition 1.2.24 (Nilindex)**.** *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra. The nilindex of* $\mathfrak{g}$ *is the integer* $r \in \mathbb{N}_{\geq 1}$ *such that* $\mathfrak{g}^r = 0$ *and* $\mathfrak{g}^{r-1} \neq 0$*.*

Normally, the word 'nilindex' is not used for an abelian Lie algebra, since this is a stronger property than nilpotency.

Example 1.2.25. *Let $\mathfrak{g}$ be the infinite-dimensional Lie algebra over $\mathbb{R}$ as in Example 1.2.12. Then $\mathfrak{g}$ is nilpotent with nilindex two, since $[A_1,[A_2,A_3]] = 0$ for all $A_1, A_2, A_3 \in \mathfrak{g}$.*

Let $\mathfrak{g}$ be an n-dimensional Lie algebra over a field K with basis $\mathcal{B} = \{X_1,\dots,X_n\}$. When $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{g}^k = \mathfrak{g}$ for all $k \in \mathbb{N}$. Suppose that $\dim([\mathfrak{g},\mathfrak{g}]) = n-1$, say $[\mathfrak{g},\mathfrak{g}] = \operatorname{span}\{X_1,\dots,X_{n-1}\}$. Then,

$$\mathfrak{g}^2 = [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = [\mathfrak{g},\operatorname{span}\{X_1,\dots,X_{n-1}\}] = [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}^1$$

holds, which is impossible for nilpotent Lie algebras. Hence, this implies that for every nilpotent n-dimensional Lie algebra $\mathfrak{g}$, the inequality $\dim([\mathfrak{g},\mathfrak{g}]) \le n-2$ is true. Moreover,

$$\dim(\mathfrak{g}^k) \le n-k-1$$

holds for all $1 \le k \le n-1$. Therefore, the upper bound of the nilindex is equal to $n-1$. Lie algebras of dimension n with nilindex $n-1$ will be studied in Section 3.2.

Chapter 2

Almost-inner derivations

In this chapter, almost-inner derivations of Lie algebras are studied. Almost-inner derivations were first introduced in [12] in the study of spectral geometry. First, the definitions of an almost-inner derivation and of related concepts are given. The second section is devoted to the geometric importance of the concept. Further, some first observations and properties are stated and proven. There is a procedure to calculate $\mathrm{AID}(\mathfrak{g})$. This working method will be useful in the next chapter, were the almost-inner derivations will be computed for special classes of Lie algebras.

2.1 Definitions

This section explains some important concepts concerning almost-inner derivations. Before the definition can be given, there are other notions which have to be introduced. The first one is that of a derivation.

Definition 2.1.1 (Derivation)**.** *Let A be an algebra over a field K. A derivation of A is a K-linear map $D : A \to A$ such that*

$$D(XY) = XD(Y) + D(X)Y$$

holds for all $X, Y \in A$.

This identity is called 'Leibniz' rule'. The most familiar example is the following.

Example 2.1.2. *Let $A = C^\infty(\mathbb{R})$ be the vector space of all smooth functions $\mathbb{R} \to \mathbb{R}$. For $f, g \in A$, the product fg is given by $(fg)(x) = f(x)g(x)$. Equipped with this bilinear map, A is an algebra. The usual derivative, given by $D(f) = f'$ for all $f \in A$ is a derivation of A. Indeed, it follows from the product rule that*

$$D(fg) = (fg)' = f'g + fg' = (Df)g + fD(g)$$

holds for all $f, g \in A$.

It is clear that the set of all derivations of an algebra A forms a vector space, which is denoted by $\mathrm{Der}(A)$. Hence, it makes sense to define the dimension of the derivations as the dimension of the vector space. Recall that $\mathbf{gl}(A)$ is a Lie algebra with Lie bracket given by

$$[f, g] := f \circ g - g \circ f \quad \text{for all } f, g \in \mathbf{gl}(A).$$

By using the definition of a derivation and the fact that $\mathrm{Der}(A) \subset \mathbf{gl}(A)$, the following equations hold for every $D, E \in \mathrm{Der}(A)$ and for every $X, Y \in A$:

$$\begin{aligned}
[D,E](XY) &= D\big(E(XY)\big) - E\big(D(XY)\big) \\
&= D\big(XE(Y) + E(X)Y\big) - E\big(XD(Y) + D(X)Y\big) \\
&= D\big(XE(Y)\big) + D\big(E(X)Y\big) - E\big(XD(Y)\big) - E\big(D(X)Y\big) \\
&= XD\big(E(Y)\big) + D(X)E(Y) + E(X)D(Y) + D\big(E(X)\big)Y \\
&\quad - XE\big(D(Y)\big) - E(X)D(Y) - D(X)E(Y) - E\big(D(X)\big)Y \\
&= XD\big(E(Y)\big) + D\big(E(X)\big)Y - XE\big(D(Y)\big) - E\big(D(X)\big)Y \\
&= X[D,E](Y) + [D,E](X)Y.
\end{aligned}$$

This means that $[D, E]$ is a derivation too and $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathbf{gl}(A)$.

Let $\mathfrak{g}$ be a Lie algebra. Then a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a derivation, when

$$\varphi([X,Y]) = [X, \varphi(Y)] + [\varphi(X), Y] \quad \text{for all } X, Y \in \mathfrak{g}.$$

Let $\mathfrak{g}$ be a n-dimensional Lie algebra with basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ and structure constants c_{ij}^k for all $1 \leq i, j, k \leq n$. Since a derivation is a linear map, it admits a matrix representation. Denote with $D = (d_{ij})$ the corresponding matrix of the derivation φ, so $D(X_i) = \sum_{j=1}^{n} d_{ij} X_j$.

Remark 2.1.3. *In the literature, there is also another way to represent the linear transformation as a matrix: the coordinate of the j-th basis vector is written down in the j-th column of the matrix. In fact, this yields the transpose of the matrix representation which will be used in this thesis.*

By bilinearity of the Lie bracket, it suffices to check the above equation for the basis vectors. Let X_i and X_j be two arbitrary basis vectors. Then,

$$D([X_i, X_j]) = D\Big(\sum_{l=1}^{n} c_{ij}^l X_l\Big) = \sum_{l=1}^{n} c_{ij}^l D(X_l) = \sum_{l=1}^{n}\sum_{k=1}^{n} c_{ij}^l d_{lk} X_k \tag{2.1}$$

holds by bilinearity of the Lie bracket. Analogously, following equations are satisfied:

$$\begin{aligned}
[D(X_i), X_j] + [X_i, D(X_j)] &= \Big[\sum_{l=1}^{n} d_{il} X_l, X_j\Big] + \Big[X_i, \sum_{l=1}^{n} d_{jl} X_l\Big] \\
&= \sum_{l=1}^{n} d_{il}[X_l, X_j] + \sum_{l=1}^{n} d_{jl}[X_i, X_l] \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n} d_{il} c_{lj}^k X_k + \sum_{l=1}^{n}\sum_{k=1}^{n} d_{jl} c_{il}^k X_k \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n} (d_{il} c_{lj}^k + d_{jl} c_{il}^k) X_k. \qquad (2.2)
\end{aligned}$$

Since D is a derivation,

$$D([X_i, X_j]) = [D(X_i), X_j] + [X_i, D(X_j)]$$

is true for all $1 \leq i, j \leq n$. This means, by combining the equations (2.1) and (2.2), that

$$\sum_{l=1}^{n} c_{ij}^{l} d_{lk} = \sum_{l=1}^{n} (d_{il} c_{lj}^{k} + d_{jl} c_{il}^{k}) \tag{2.3}$$

has to be satisfied for all $1 \leq i, j, k \leq n$. This gives relations on the different matrix entries of the derivation. An example of such a computation is postponed to Proposition 3.2.5.

A derivation is completely characterised by the above equation. In appendix A.2, algorithm `derivations(C)` in Code A.3 computes the equations (2.3) for an arbitrary derivation of a given Lie algebra. As input, the $(n \times n \times n)$-matrix `C` contains the structure constants of $\mathfrak{g}$ as before. The output is an $(n^2 \times n^2)$-system, where every row stands for one relation on the matrix entries. The information for d_{ij} is represented by column $(i-1)n+j$ in the system. The obtained relations on the matrix entries can be visualised with the algorithm `makeBasisDerivations(C)`. It gives as output the matrix representation of an arbitrary derivation for the Lie algebra $\mathfrak{g}$, represented by the $(n \times n \times n)$-matrix `C`. Moreover, also the dimension of $\mathrm{Der}(\mathfrak{g})$ is printed. However, since this requires a long computing time, it is only useful for Lie algebras when n is small (at most around fifteen).

An important class of derivations consists of the inner derivations. Let $\mathfrak{g}$ be a Lie algebra. For every $X \in \mathfrak{g}$, the image of the adjoint homomorphism $\mathrm{ad}(X)$ is called an inner derivation.

Definition 2.1.4 (Inner derivation)**.** *Let $\mathfrak{g}$ be a Lie algebra and $X \in \mathfrak{g}$. The map*

$$ad(X) : \mathfrak{g} \to \mathfrak{g} : Y \mapsto [X, Y]$$

is called an inner derivation of $\mathfrak{g}$.

By the Jacobi identity,

$$\begin{aligned} \mathrm{ad}(X)([Y,Z]) &= [X,[Y,Z]] \\ &= [[X,Y],Z] + [Y,[X,Z]] \\ &= [\mathrm{ad}(X)(Y),Z] + [Y,\mathrm{ad}(X)(Z)] \end{aligned}$$

holds for all $Y, Z \in \mathfrak{g}$. Hence, every inner derivation is indeed a derivation. The symbol $\mathrm{Inn}(\mathfrak{g})$ stands for the set of all inner derivations of the Lie algebra $\mathfrak{g}$.

Let $\mathrm{ad}(X)$ be an arbitrary inner derivation of $\mathfrak{g}$, so $X \in \mathfrak{g}$. Further, let $D \in \mathrm{Der}(\mathfrak{g})$ be an arbitrary derivation of $\mathfrak{g}$ and let $Y \in \mathfrak{g}$. Then,

$$[D, \mathrm{ad}(X)](Y) = D([X,Y]) - [X, D(Y)] = [D(X), Y] = \mathrm{ad}(D(X))(Y) \tag{2.4}$$

follows. In the first step, the Lie bracket in $\mathrm{Der}(\mathfrak{g})$ is worked out, together with the definition of the adjoint map. The second equation holds since D is a derivation. This computation shows that $\mathrm{Inn}(\mathfrak{g})$ is even an ideal of $\mathrm{Der}(\mathfrak{g})$. By bilinearity, $\mathrm{Inn}(\mathfrak{g})$ can be generated by the maps $\mathrm{ad}(X_i) : \mathfrak{g} \to \mathfrak{g}$, where X_i is a basis vector, so $1 \leq i \leq n$.

As every derivation, an inner derivation can be represented by a matrix. Let now $\mathfrak{g}$ be an n-dimensional Lie algebra over K with basis $\mathcal{B} = \{X_1, \ldots, X_n\}$. Choose $k \in \{1, \ldots, n\}$ and denote with $H = (h_{ij})$ the corresponding matrix of the inner derivation $\mathrm{ad}(X_k) : \mathfrak{g} \to \mathfrak{g}$, thus

$$\mathrm{ad}(X_k)(X_i) = \sum_{j=1}^{n} h_{ij} X_j.$$

Moreover, $[X_k, X_i] = \sum_{j=1}^{n} c_{ki}^j X_j$ holds. Therefore, the entries of H are given by

$$h_{ij} = c_{ki}^j = -c_{ik}^j.$$

Consider an arbitrary $X = \sum_{k=1}^{n} a_k X_k \in \mathfrak{g}$, where $a_i \in K$ for all $1 \leq i \leq n$ and let H be the matrix representation of $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$. By bilinearity of the Lie bracket, the entries of $H = (h_{ij})$ are given by

$$h_{ij} = \sum_{k=1}^{n} -a_k c_{ik}^j.$$

For all $X \in \mathfrak{g}$ the map $\mathrm{ad}(X)$ is the zero-map if and only if $X \in Z(\mathfrak{g})$, which explains the equation

$$\mathrm{Inn}(\mathfrak{g}) \cong \frac{\mathfrak{g}}{Z(\mathfrak{g})}.$$

Hence, $\dim(\mathrm{Inn}(\mathfrak{g}))$ is easy to compute.

The following definition is the main topic of this thesis.

Definition 2.1.5 (Almost-inner derivation). *Let $\mathfrak{g}$ be a Lie algebra. A derivation φ is almost-inner if $\varphi(X) \in [X, \mathfrak{g}]$ for all $X \in \mathfrak{g}$.*

The definition means that there exists a map $B : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi(X) = [X, B(X)]$ for all $X \in \mathfrak{g}$. Hence, an inner derivation is almost-inner for which the map B has to be constant. The set of all almost-inner derivations of a Lie algebra $\mathfrak{g}$ is denoted by $\mathrm{AID}(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra with $\varphi_1, \varphi_2 \in \mathrm{AID}(\mathfrak{g})$. Choose $X \in \mathfrak{g}$ arbitrarily. Since φ_1 and φ_2 are almost-inner derivations, there exist X_1 and X_2 in $\mathfrak{g}$ such that $\varphi_1(X) = [X, X_1]$ respectively $\varphi_2(X) = [X, X_2]$. It is clear that $\mathrm{AID}(\mathfrak{g}) \subset \mathbf{gl}(\mathfrak{g})$. According to Example 1.2.6, the Lie bracket of φ_1 and φ_2 is given by

$$\begin{aligned}
[\varphi_1, \varphi_2](X) &= \varphi_1\varphi_2(X) - \varphi_2\varphi_1(X) \\
&= \varphi_1([X, X_2]) - \varphi_2([X, X_1]) \\
&= [\varphi_1(X), X_2] + [X, \varphi_1(X_2)] - [\varphi_2(X), X_1] - [X, \varphi_2(X_1)] \\
&= [[X, X_1], X_2] + [X, \varphi_1(X_2)] - [[X, X_2], X_1] - [X, \varphi_2(X_1)] \\
&= [[X, X_1], X_2] - [[X, X_2], X_1] + [X, \varphi_1(X_2) - \varphi_2(X_1)] \\
&= [X, [X_1, X_2]] + [X, \varphi_1(X_2) - \varphi_2(X_1)].
\end{aligned}$$

The third equality holds because φ_1 and φ_2 are derivations. In the last step, the Jacobi identity is used. Hence,

$$[\varphi_1, \varphi_2](X) = [X, [X_1, X_2] + \varphi_1(X_2) - \varphi_2(X_1)] \in [X, \mathfrak{g}]$$

holds, which means that $[\varphi_1, \varphi_2] \in \mathrm{AID}(\mathfrak{g})$ is an almost-inner derivation. Therefore, $\mathrm{AID}(\mathfrak{g})$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$. In Section 2.3, this definition is analysed more thoroughly.

A concept related to the notion of an almost-inner derivation, is a central almost-inner derivation.

Definition 2.1.6 (Central almost-inner derivation). *Let $\mathfrak{g}$ be a Lie algebra. An almost-inner derivation φ is central almost-inner if there exists an $X \in \mathfrak{g}$ such that $\varphi(Y) - ad(X)(Y) \in Z(\mathfrak{g})$ for all $Y \in \mathfrak{g}$.*

The set of all central almost-inner derivations of a Lie algebra $\mathfrak{g}$ is designated by CAID($\mathfrak{g}$). Next lemma shows that CAID($\mathfrak{g}$) is an ideal of AID($\mathfrak{g}$) for every Lie algebra $\mathfrak{g}$.

Lemma 2.1.7. *Let $\mathfrak{g}$ be a Lie algebra. Then CAID($\mathfrak{g}$) is an ideal of AID($\mathfrak{g}$).*

Proof. Let $\phi \in \mathrm{CAID}(\mathfrak{g})$ and $\varphi \in \mathrm{AID}(\mathfrak{g})$ be arbitrary. By the observations above, AID($\mathfrak{g}$) is a Lie algebra, hence $[\phi, \varphi] \in \mathrm{AID}(\mathfrak{g})$. The aim is to show that $[\phi, \varphi]$ is even a central almost-inner derivation of $\mathfrak{g}$. Since ϕ is central almost-inner, there exists $X \in \mathfrak{g}$ such that $\phi' := \phi - \mathrm{ad}(X)$ and $\phi'(\mathfrak{g}) \subseteq Z(\mathfrak{g})$. Let $Y \in \mathfrak{g}$ be arbitrary. It follows from (2.4) that

$$\begin{aligned}
[\varphi, \phi](Y) - \mathrm{ad}(\varphi(X))(Y) &= [\varphi, \phi](Y) - [\varphi, \mathrm{ad}(X)](Y) \\
&= [\varphi, \phi'](Y) \\
&= \varphi\phi'(Y) - \phi'\varphi(Y).
\end{aligned}$$

Define $\tilde{Y} := \phi'(Y)$, then there exists $Y' \in \mathfrak{g}$ such that $\varphi(\tilde{Y}) = [\tilde{Y}, Y']$. Since $\tilde{Y} \in Z(\mathfrak{g})$, it follows that $\varphi(\tilde{Y}) = 0$. Hence,

$$[\varphi, \phi](Y) - \mathrm{ad}(\varphi(X))(Y) \in Z(\mathfrak{g}),$$

holds, which means that $[\varphi, \phi] \in \mathrm{CAID}(\mathfrak{g})$ is a central almost-inner derivation of $\mathfrak{g}$. This completes the proof. □

By definition, every inner derivation is also central almost-inner. Hence, it is clear that the following holds for every Lie algebra $\mathfrak{g}$:

$$\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g}).$$

It is natural to investigate how all of these concepts are related.

In some cases, the question can be solved immediately. For semisimple complex Lie algebras, all four concepts are equal, so $\mathrm{Inn}(\mathfrak{g}) = \mathrm{CAID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

Proposition 2.1.8. *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, then the only derivations are the inner derivatons.*

Proof. This fact is proven in for example [8, page 85]. □

This means that all almost-inner derivations are trivial for finite-dimensional complex semisimple Lie algebras.

When the centre of a Lie algebra is trivial, all central almost-inner derivations are inner ones.

Lemma 2.1.9. *Let $\mathfrak{g}$ be a Lie algebra with $Z(\mathfrak{g}) = 0$. Then Inn($\mathfrak{g}$) = CAID($\mathfrak{g}$) holds.*

Proof. The inclusion $\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g})$ always holds. Suppose that $\varphi \in \mathrm{CAID}(\mathfrak{g})$ is an arbitrary central almost-inner derivation. By definition, there exists an element $X \in \mathfrak{g}$ so that $\varphi(Y) - \mathrm{ad}(X)(Y) \in Z(\mathfrak{g})$ for all $Y \in \mathfrak{g}$. Hence, $\varphi = \mathrm{ad}(X)$ is an inner derivation, which concludes the proof. □

For a two-step nilpotent Lie algebra, all almost-inner derivations are also central almost-inner.

Lemma 2.1.10. *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra. Then $CAID(\mathfrak{g}) = AID(\mathfrak{g})$ holds.*

Proof. The inclusion $\mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g})$ holds by definition. Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an arbitrary almost-inner derivation. Then $\varphi(X) \in [X, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$ holds since $\mathfrak{g}$ is two-step nilpotent. By definition, φ is a central almost-inner derivation. □

In Section 2.3, more theory is developed which makes it possible to compute $\mathrm{AID}(\mathfrak{g})$ for a given Lie algebra $\mathfrak{g}$. First, a geometric motivation is given. This illustrates why it is interesting to study almost-inner derivations.

2.2 Geometric motivation

The notion of an almost-inner derivation appeared in 1984. In their paper (see [12]), Gordon and Wilson provide an example which gives more insight in a question in spectral geometry. The first part of this section is devoted to this problem. Further, nilmanifolds and different types of group automorphisms are introduced to understand the result of Gordon and Wilson. In the last subsection, the connection with almost-inner derivations is elaborated.

2.2.1 Isospectral manifolds

In spectral geometry, relations between a certain kind of manifolds and spectra of differential operators are studied. The main example is the Laplace-Beltrami operator, a generalisation of the Laplace operator which can be used for general closed Riemannian manifolds.

Definition 2.2.1 (Riemannian manifold)**.** *A Riemannian manifold (M, g) is a real smooth manifold M with inner product g_p on T_pM for all $p \in M$ such that*

$$p \mapsto g_p(X(p), Y(p))$$

is a smooth function for all vectorfields X and Y on M.

The Laplace-Beltrami operator is defined as the divergence of the gradient and denoted with Δ or with ∇^2. One of the fundamental problems in spectral geometry is to determine to what extent the eigenvalues of the operator determine the geometry of a given manifold. First, some notions are introduced.

Definition 2.2.2 (Spectrum)**.** *Let (M, g) be a closed Riemannian manifold where the associated Laplace-Beltrami operator Δ acts on functions. The spectrum $spec(M, g)$ of (M, g) is the set of eigenvalues of Δ.*

When two Riemannian manifolds have the same spectrum, they are called 'isospectral'.

Definition 2.2.3 (Isospectral)**.** *Two closed Riemannian manifolds (M, g) and (M', g') are isospectral when $spec(M, g) = spec(M', g')$.*

For Riemannian manifolds, the right notion of an isomorphism is an isometry.

Definition 2.2.4 (Isometry)**.** *Let $f : M \to N$ be a diffeomorphism between two Riemannian manifolds (M, g) and (M', g'). Then, f is an isometry if*

$$g_p(X, Y) = g'_{f(p)}(\,\mathrm{d}f_p(X),\, \mathrm{d}f_p(Y))$$

holds for all $p \in M$ and for all $X, Y \in T_pM$.

One of the central questions in spectral geometry was 'are isospectral manifolds necessarily isometric?' This can be interpreted as follows: 'Is it possible to determine the whole geometry of the manifold by only looking at the eigenvalues?' It turns out that the answer is negative. In 1964, a counterexample was given by Milnor (see [14]), who constructed two isospectral and non-isometric flat tori of dimension 16.

In the following years, there have been many new examples, such as hyperbolic manifolds and Heisenberg manifolds. In 1984, Gordon and Wilson were the first to construct continuous families of isospectral manifolds which are non-isometric. In their paper (see [12]), they used the notions of a nilmanifold and an almost-inner automorphism. This is explained more in the following subsections.

2.2.2 Nilmanifolds

In this subsection, nilmanifolds are introduced. First, some other notions have to be explained.

Definition 2.2.5 (Left group action)**.** *Let G be a group and X a set. A left group action φ of G on X is a function*

$$\varphi : G \times X \to X : (g, x) \mapsto \varphi(g, x) = g \cdot x$$

such that for all $x \in X$ and for all $g, h \in G$, the equations

$$e \cdot x \;=\; x \quad \text{and} \quad gh \cdot x = g \cdot (h \cdot x)$$

are satisfied.

Analogously, a right group action φ of G on X is a function

$$\varphi : X \times G \to X : (x, g) \mapsto \varphi(x, g) = x \cdot g$$

such that for all $x \in X$ and for all $g, h \in G$, the equations

$$x \cdot e \;=\; x \quad \text{and} \quad x \cdot gh = (x \cdot g) \cdot h$$

hold. Due to the relation $(gh)^{-1} = h^{-1}g^{-1}$, a right action can be modified to a left action by composing the action with the inverse group operation. Mostly, the set X is in fact a topological space or even a Lie group. When X is a Lie group, the action is required to be continuous.

Definition 2.2.6 (Orbit)**.** *Consider a group G acting on a set X by left multiplication. The orbit of an element $x \in X$ is given by*

$$G \cdot x \;=\; \{g \cdot x \mid g \in G\}.$$

A similar definition exists for right actions. The quotient of a left action (or the orbit space) $G\backslash X$ is the set of all orbits of X under G. The quotient of a right action is denoted with X/G. The quotient forms a partition of X, where the equivalence relation is given by

$$x \sim y \qquad \Leftrightarrow \qquad \text{there exists } g \in G \text{ with } g \cdot x = y.$$

This means that two elements $x, y \in X$ are equivalent if and only if their orbits coincide. If a group action has only one orbit, it is called 'transitive'. This means that, for all points $x, y \in X$, there is a group element $g \in G$ with $y = g \cdot x$.

Definition 2.2.7 (Isotropy subgroup)**.** *Consider a group G acting on a set X by left multiplication. The isotropy subgroup G_x of an element $x \in X$ is given by*

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

It is clear that this defines a subgroup of G, for all points $x \in X$. A similar definition holds for right group actions. Notice that all points in the same orbit have conjugated isotropy subgroups. Indeed, consider $x, y \in X$ with $G \cdot x = G \cdot y$. This means that there exists $g \in G$ with $y = g \cdot x$. Suppose further that $h \in G_y$. Then, $h \cdot y = y$ is equivalent to $hg \cdot x = g \cdot x$ and $g^{-1}hg \cdot x = x$, which means that $h \in gG_xg^{-1}$. The other inclusion is shown similarly.

Definition 2.2.8 (Homogeneous space)**.** *Let G be a Lie group and X be a topological space. Then X is called homogeneous for G if there exists a transitive and continuous group action of G on X.*

Consider the map $l^x : G \to X : g \mapsto g \cdot x$, which has $G \cdot x$ as image. The equation $g \cdot x = h \cdot x$ is equivalent with $g^{-1}h \cdot x = x$, which means that $g^{-1}h \in G_x$ and thus $h \in gG_x$. This implies that $hG_x = gG_x$. Hence, there is a bijection between the space of left cosets G/G_x and the orbit $G \cdot x$. Therefore, for a transitive group action, the sets X and G/G_x are bijective for all $x \in X$. For right actions, similar results can be obtained.

All ingredients are introduced to understand what a nilmanifold is. The notion was first used by Anatoly Mal'cev in 1951. (see [13]).

Definition 2.2.9 (Nilmanifold)**.** *A nilmanifold is a quotient space $H\backslash G$ of a nilpotent Lie group G and a closed subgroup H of G.*

Equivalently, a nilmanifold can be seen as a homogeneous space for which the left transitive group action comes from a nilpotent Lie group.

From all nilmanifolds, the compact ones are of most interest. As Mal'cev showed, there exists a characterisation of compact nilmanifolds. To understand the method of working, there are some other notions which have to be explained first.

Definition 2.2.10 (Simply connected)**.** *Let X be a path connected topological space X. Then X is called simply connected if for any continuous map $f : S^1 \to X$, there exists a continuous map $H : S^1 \times [0,1] \to X$ and a point $x_0 \in X$ such that for all $z \in S^1$, the equations*

$$H(z,0) = f(z) \qquad \textit{and} \qquad H(z,1) = x_0$$

are satisfied.

Intuitively, this definition means that an arbitrary closed continuous curve in X is homotopic to a constant curve.

Another notion is that of a discrete subgroup of a given group.

Definition 2.2.11 (Discrete subgroup)**.** *A subgroup H of a topological group G is called discrete if the subspace topology of H in G is the discrete topology. This means that there exists an open cover of H such that every open subset contains exactly one element of H.*

Consider $\mathbb{R}$ with the standard topology. It is clear that the integers $\mathbb{Z}$ form a discrete subgroup. This is not the case for the rational numbers, since $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 2.2.12 (Uniform group action)**.** *A left (respectively right) action of a group G on a topological space X is uniform (also called cocompact) if the quotient space $G\backslash X$ (respectively X/G) is a compact space.*

Let N be a simply connected nilpotent Lie group and Γ be a discrete subgroup. If the subgroup Γ acts uniformly (via left multiplication) on N, then the quotient manifold $\Gamma\backslash N$ will be a compact nilmanifold. A theorem due to Mal'cev (see [13]) shows that every compact nilmanifold can be formed like that.

Next example shows that this is in fact a generalisation of a torus.

Example 2.2.13. *Let G be an abelian Lie group, which is of course nilpotent. For example, one can take the group of real numbers under addition, and the discrete cocompact subgroup consisting of the integers. The resulting nilmanifold (with $k \in \mathbb{N}$) is the generalised torus*

$$T^k = \frac{\mathbb{R}^k}{\mathbb{Z}^k}.$$

For $k = 1$, this is the circle and for $k = 2$, this is a torus. In the literature, a nilmanifold often is supposed to be compact. Also in this thesis, this assumption will be made. Therefore, sometimes the above property is used as the definition of a (compact) nilmanifold.

To obtain a Riemannian structure on an nilmanifold, choose a left-invariant metric on G. This metric in inherited by $\Gamma\backslash G$. In this way, a so called 'Riemannian nilmanifold' is acquired.

2.2.3 Automorphisms

The concepts for Lie algebras defined in Section 2.1 are closely related to some special types of group automorphisms. In this subsection, those notions are introduced and the geometric interpretation is provided. Especially, almost-inner automorphisms turn up in the study of spectral geometry of nilmanifolds. Further, the relation with almost-inner derivations is explained.

A group automorphism is a group isomorphism from a group to itself. The set of all automorphisms of a given group is denoted with $\mathrm{Aut}(G)$ and forms a group. Special subgroups are given in the following definitions.

Definition 2.2.14 (Inner automorphism)**.** *Let G be a Lie group. The inner automorphism $I_g : G \to G$ of G for $g \in G$ is given by $I_g(x) = gxg^{-1}$.*

The notation $\mathrm{Inn}(G)$ stands for the set of all inner automorphisms of the group G. It is clear that an inner automorphism is indeed an automorphism. Moreover, $\mathrm{Inn}(G)$ is even a normal subgroup of $\mathrm{Aut}(G)$. Consider now the group homomorphism

$$G \to \mathrm{Inn}(G) : g \mapsto (I_g : G \to G).$$

Since the kernel of this morphism is equal to the center $Z(G)$ of G, it follows from the isomorphism theorems that $G/Z(G) \cong \mathrm{Inn}(G)$. As explained in the previous section, a similar statement holds for $\mathrm{Inn}(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. Let G be a connected and simply connected Lie group. One can show that $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$ have $\mathrm{Inn}(\mathfrak{g})$ respectively $\mathrm{Der}(\mathfrak{g})$ as Lie algebra. Therefore, from now on, a Lie group G is always assumed to be connected and simply connected. An inner automorphism is a special case of an almost-inner automorphism.

Definition 2.2.15 (Almost-inner automorphism)**.** *Let G be a Lie group. An automorphism φ of G is almost-inner if and only if for all $x \in G$, there exists $g \in G$ such that $\varphi(x) = gxg^{-1}$.*

This notion was introduced differently in [12], but it is in fact equivalent to this one (see [10]). The set of all almost-inner automorphisms of G is denoted with $\mathrm{AIA}(G)$. Hence, an almost-inner automorphism is a generalisation of an inner one, where $g \in G$ can depend on $x \in G$. In [12], it is proven that $\mathrm{AIA}(G)$ is a Lie subgroup of $\mathrm{Aut}(G)$. The following theorem is due to Gordon and Wilson and is stated without proof.

Theorem 2.2.16 (Gordon and Wilson, see [12])**.** *Let $(\Gamma\backslash G, g)$ be a compact Riemannian nilmanifold and let $\varphi \in AIA(G)$. Then $(\varphi(\Gamma)\backslash G, g)$ is isospectral to $(\Gamma\backslash G, g)$. Moreover, $(\Gamma\backslash G, g)$ and $(\varphi(\Gamma)\backslash G, g)$ are non-isometric when $\varphi \notin Inn(G)$.*

This theorem was first stated and proven for the equivalent notion of an almost-inner automorphism, but it also holds for this definition (see [7], [9] and [10]). To calculate the almost-inner automorphisms, it is often easier to consider the almost-inner derivations, the equivalent notion on the Lie algebra level. The relation between the two concepts is expressed in the following proposition.

Proposition 2.2.17 (DeTurck and Gordon, see [7])**.** *Let G be a connected and simply connected nilpotent Lie group with nilindex r. Denote the Lie algebra of G with $\mathfrak{g}$. Then $AIA(G)$ is a simply-connected nilpotent Lie group with nilindex $\leq r-1$ and with Lie algebra $AID(\mathfrak{g})$.*

This kind of derivations has not much been studied and only in the case of very specific Lie algebras (see [6], [11], [12], and [15]). The aim of my thesis is to study this concept algebraically. In the next section, the definition of an almost-inner derivation is elaborated further.

2.3 First considerations

In this section, the definition of an almost-inner derivation is elaborated further. Moreover, some basic properties are stated and proven. This theory will be very useful in the next chapter, where almost-inner derivations of some specific classes are computed.

2.3.1 Conditions on the parameters of an almost-inner derivation

This subsection is devoted to derive which conditions a general map have to satisfy to be an almost-inner derivation. Let $\mathfrak{g}$ be a Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots X_n\}$ and structure constants c_{ij}^k (where $1 \leq i, j, k \leq n$), thus

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \quad \text{for all } 1 \leq i, j, k \leq n.$$

Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an almost-inner derivation. An almost-inner derivation is a linear map and can therefore be represented by a matrix. Let $D = (d_{ij})$ be the matrix representation of φ with respect to $\mathcal{B}$. Since every almost-inner derivation is a derivation, the conditions for a derivation

$$\sum_{l=1}^{n} c_{ij}^l d_{lk} = \sum_{l=1}^{n} (d_{il} c_{lj}^k + d_{jl} c_{il}^k)$$

has to be fulfilled too for all $1 \leq i, j, k \leq n$. Moreover, there are other conditions imposed by the definition of an almost-inner derivation. Indeed, there have to exist a_{ij} (with $1 \leq i, j \leq n$) so that

$$\varphi(X_i) = [X_i, \sum_{j=1}^{n} a_{ij} X_j] = \sum_{j=1}^{n} a_{ij} [X_i, X_j] = \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{n} c_{ij}^k X_k. \tag{2.5}$$

These values a_{ij} (with $1 \leq i, j \leq n$) are referred to as the 'parameters' of φ with respect to the basis $\mathcal{B}$. The parameter a_{ij} is said to 'belong to' the basis vector X_j. Of course, if the structure constants c_{ij}^k vanish for all $1 \leq k \leq n$, the value of the parameter a_{ij} does not matter. This motivates the following definition.

Definition 2.3.1 (Visible and invisible parameter)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$. Let $\varphi \in AID(\mathfrak{g})$ be an almost-inner derivation with parameters a_{ij} with respect to $\mathcal{B}$ (for $1 \leq i, j \leq n$). A parameter a_{ij} is visible if there exists a $1 \leq k \leq n$ so that $c_{ij}^k \neq 0$. A parameter a_{ij} is invisible if it is not visible.*

By bilinearity of a derivation, the last equation implies that for an arbitrary $X = \sum_{i=1}^{n} x_i X_i \in \mathfrak{g}$ (where $x_i \in K$ for all $1 \leq i \leq n$), the image of X under φ is given by

$$\varphi(X) = \varphi(\sum_{i=1}^{n} x_i X_i) = \sum_{i=1}^{n} x_i \varphi(X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} x_i a_{ij} c_{ij}^k X_k.$$

Besides, there have to exist $c_j \in K$ for $1 \leq j \leq n$ so that

$$\varphi(X) = [X, \sum_{j=1}^{n} c_j X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i c_j [X_i, X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} x_i c_j c_{ij}^k X_k.$$

Hence, there are two ways to write $\varphi(X)$. Combining the coefficients of the right hand side of these expressions, this gives a system of linear equations

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i c_j c_{ij}^k = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} c_{ij}^k \quad \text{for all } 1 \leq k \leq n \tag{2.6}$$

with unknowns $c_i \in K$ for all $1 \le i \le n$. Equivalently,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i(a_{ij} - c_j)c_{ij}^k = 0 \tag{2.7}$$

has to be satisfied for all $1 \le k \le n$ and for all $x_i \in K$ with $1 \le i \le n$. The purpose is to find conditions on the parameters a_{ij} (with $1 \le i, j \le n$) such that there exist c_j (with $1 \le i \le n$) for which the system of equations (2.7) has a solution for all possible values of x_i (with $1 \le i \le n$). Note that the unknowns c_j depend on the choices of x_j. Those conditions put in most cases some relations on the parameters. An arbitrary almost-inner derivation $\varphi : \mathfrak{g} \to \mathfrak{g}$ can be written as

$$\varphi : \mathfrak{g} \to \mathfrak{g} : X \mapsto D \cdot X,$$

where $D = (d_{ij})$ is the matrix representation of φ. Since

$$\varphi(X_i) = \sum_{j=1}^{n} d_{ij}X_j = \sum_{k=1}^{n} a_{ik} \sum_{j=1}^{n} c_{ik}^j X_j$$

holds by equation (2.5), the entries of D are given by

$$d_{ij} = \sum_{k=1}^{n} a_{ik}c_{ik}^j. \tag{2.8}$$

The next example clarifies these observations.

Example 2.3.2. *Let $\mathfrak{g}$ be the Lie algebra over $\mathbb{R}$ with basis $\mathcal{B} = \{X_1, \dots, X_8\}$ and with non-vanishing Lie brackets*

$$\begin{aligned} [X_1, X_3] &= X_6; & [X_1, X_4] &= X_7; & [X_1, X_5] &= X_8; \\ [X_2, X_3] &= X_8; & [X_2, X_4] &= X_6; & [X_2, X_5] &= X_7. \end{aligned}$$

Then, $\dim(AID(\mathfrak{g})) = 7$ *holds.*

It is clear that $\mathfrak{g}$ is a two-step nilpotent Lie algebra. Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ an arbitrary almost-inner derivation. By definition, there exist $a_{ij} \in \mathbb{R}$ with $1 \le i, j \le 8$ such that

$$\varphi : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & 0 & 0 & a_{24} & a_{25} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & -a_{31} & 0 & -a_{32} \\ 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{41} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{52} & -a_{51} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix}.$$

As already mentioned in Remark 2.1.3, the matrix representations in this thesis are the transposes of the representations sometimes used in the literature. By checking the conditions (2.3) with the computer algorithms of appendix A.2, it is easy to see that a general derivation is given by

$$\varphi : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix} \mapsto \begin{pmatrix} b_1 & 0 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ 0 & b_1 & b_3 & b_4 & b_2 & b_8 & b_9 & b_{10} \\ 0 & 0 & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ 0 & 0 & b_{13} & b_{11} & b_{12} & b_{17} & b_{18} & b_{19} \\ 0 & 0 & b_{12} & b_{13} & b_{11} & b_{20} & b_{21} & b_{22} \\ 0 & 0 & 0 & 0 & 0 & b_1 + b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & 0 & 0 & b_{13} & b_1 + b_{11} & b_{12} \\ 0 & 0 & 0 & 0 & 0 & b_{12} & b_{13} & b_1 + b_{11} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix},$$

where all $b_i \in \mathbb{R}$ (for all $1 \le i \le 22$), which means that $\dim(\mathrm{Der}(\mathfrak{g})) = 22$. There are no relations on the parameters due to the conditions for a derivation.

Since X_k does not appear for $1 \le k \le 5$, the structure constants c_{ij}^k vanish for all $1 \le k \le 5$ and all $1 \le i, j \le 8$. Hence, the first five rows of the system of (2.6) for this Lie algebra only contain zeros. The other equations are given by

$$\begin{aligned} -x_3c_1 - x_4c_2 + x_1c_3 + x_2c_4 &= x_1a_{13} + x_2a_{24} - x_3a_{31} - x_4a_{42} \\ -x_4c_1 - x_5c_2 + x_1c_4 + x_2c_5 &= x_1a_{14} + x_2a_{25} - x_4a_{41} - x_5a_{52} \\ -x_5c_1 - x_3c_2 + x_2c_3 + x_1c_5 &= x_1a_{15} + x_2a_{23} - x_3a_{32} - x_5a_{51}, \end{aligned}$$

where the first row represents the conditions for $k = 6$; the second and third row are the stipulations for $k = 7$ respectively $k = 8$. This can be written down as a matrix

$$\begin{pmatrix} -x_3 & -x_4 & x_1 & x_2 & 0 & x_1a_{13} + x_2a_{24} - x_3a_{31} - x_4a_{42} \\ -x_4 & -x_5 & 0 & x_1 & x_2 & x_1a_{14} + x_2a_{25} - x_4a_{41} - x_5a_{52} \\ -x_5 & -x_3 & x_2 & 0 & x_1 & x_1a_{15} + x_2a_{23} - x_3a_{32} - x_5a_{51} \end{pmatrix}, \tag{2.9}$$

in which the j-th column stands for c_j (with $1 \le j \le 5$); the last column is the right hand side. This system of equations must have a solution for all possible values of $x_i \in \mathbb{R}$ (with $1 \le i \le 5$). The calculation goes in different steps. During the next computations, columns with only zeros are omitted.

- Suppose that $x_1 = x_2 = x_3 = 0$ and $x_4 \neq 0 \neq x_5$.
  Then
  $$\begin{pmatrix} 0 & x_4 & x_4a_{42} \\ x_4 & x_5 & x_4a_{41} + x_5a_{52} \\ x_5 & 0 & x_5a_{51} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & a_{51} \\ 0 & 1 & a_{42} \\ 0 & 0 & x_4(a_{41} - a_{51}) + x_5(a_{52} - a_{42}) \end{pmatrix}$$
  implies that
  $$x_4(a_{41} - a_{51}) + x_5(a_{52} - a_{42}) = 0.$$
  By choosing $(x_4, x_5) = (1, 1)$ and $(x_4, x_5) = (1, -1)$, the equations
  $$\begin{aligned} a_{41} - a_{51} &= -(a_{52} - a_{42}) \\ a_{41} - a_{51} &= a_{52} - a_{42} \end{aligned}$$
  have to be satisfied. Hence, $a_{41} = a_{51}$ and $a_{52} = a_{42}$.

- Consider $x_1 = x_2 = x_4 = 0$ and $x_3 \neq 0 \neq x_5$.
  Analogously, it follows from

$$\begin{pmatrix} x_3 & 0 & x_3a_{31} \\ 0 & x_5 & x_5a_{52} \\ x_5 & x_3 & x_3a_{32} + x_5a_{51} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & a_{31} \\ 0 & 1 & a_{52} \\ 0 & 0 & x_3(a_{32} - a_{52}) + x_5(a_{51} - a_{31}) \end{pmatrix}$$

  that $a_{32} = a_{52}$ and $a_{51} = a_{31}$.

- Let $x_3 = x_4 = x_5 = 0$ and $x_1 \neq 0 \neq x_2$. The system then becomes

$$\begin{aligned} &\begin{pmatrix} x_1 & x_2 & 0 & x_1a_{13} + x_2a_{24} \\ 0 & x_1 & x_2 & x_1a_{14} + x_2a_{25} \\ x_2 & 0 & x_1 & x_1a_{15} + x_2a_{23} \end{pmatrix} \\ \to &\begin{pmatrix} x_1 & x_2 & 0 & x_1a_{13} + x_2a_{24} \\ 0 & x_1 & x_2 & x_1a_{14} + x_2a_{25} \\ 0 & -x_2^2 & x_1^2 & x_1^2a_{15} + x_1x_2(a_{23} - a_{13}) - x_2^2a_{24} \end{pmatrix} \\ \to &\begin{pmatrix} x_1^2 & 0 & -x_2^2 & x_1^2a_{13} + x_1x_2(a_{24} - a_{14}) - x_2^2a_{25} \\ 0 & x_1 & x_2 & x_1a_{14} + x_2a_{25} \\ 0 & 0 & x_1^3 + x_2^3 & x_1^3a_{15} + x_1^2x_2(a_{23} - a_{13}) + x_1x_2^2(a_{14} - a_{24}) + x_2^3a_{25} \end{pmatrix}. \end{aligned}$$

  This system of equations has a solution if and only if

$$x_1^3a_{15} + x_1^2x_2(a_{23} - a_{13}) + x_1x_2^2(a_{14} - a_{24}) + x_2^3a_{25} = 0$$

  holds whenever $x_1^3 = -x_2^3$ and $x_1 \neq 0 \neq x_2$. Working over $\mathbb{R}$, this gives one extra condition

$$a_{15} - a_{23} + a_{13} + a_{14} - a_{24} - a_{25} = 0.$$

For this example, those are the only cases that have to be treated. In the system of equations (2.9), all visible parameters belonging to X_1 and X_2 are multiplied with x_3, x_4 or x_5. Hence, to study the behaviour of those parameters, it can be assumed that $x_1 = x_2 = 0$. It is clear from the first two cases that $a_{31} = a_{41} = a_{51}$ and $a_{32} = a_{42} = a_{52}$. There will be no new conditions for the case $x_1 = x_2 = x_5 = 0$ and $x_3 \neq 0 \neq x_4$. Analogously, to study the behaviour of the visible parameters belonging to X_3, X_4 and X_5, it can be assumed that $x_3 = x_4 = x_5 = 0$. This is worked out in the third case. Note that there are no extra conditions when exactly one of X_1 and X_2 are equal to zero. An arbitrary almost-inner derivation φ looks like

$$\varphi : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & 0 & 0 & a_{24} & a_{25} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_2 \\ 0 & 0 & 0 & 0 & 0 & b_2 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_2 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix},$$

where $a_{13}, a_{14}, a_{15}, a_{24}, a_{25}, b_1$ and b_2 all belong to the field $\mathbb{R}$ and where

$$a_{23} \;=\; a_{13} + a_{14} - a_{24} + a_{15} - a_{25}.$$

Let $X = \sum_{i=1}^{8} a_i X_i \in \mathfrak{g}$ be arbitrary, where $a_i \in \mathbb{R}$ for all $1 \leq i \leq 8$. In matrix representation, $\operatorname{ad}(X) = \sum_{i=1}^{8} a_i \operatorname{ad}(X_i)$ is given by

$$\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -a_3 & -a_4 & -a_5 \\ 0 & 0 & 0 & 0 & 0 & -a_4 & -a_5 & -a_3 \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix}.$$

Hence, φ can be written as

$$\varphi = b_1 \operatorname{ad}(X_1) + b_2 \operatorname{ad}(X_2) - a_{13} \operatorname{ad}(X_3) - a_{14} \operatorname{ad}(X_4) - a_{15} \operatorname{ad}(X_5) + (a_{24} - a_{14})\psi_1 + (a_{25} - a_{15})\psi_2.$$

The map $\psi_1 : \mathfrak{g} \to \mathfrak{g}$, given by

$$\psi_1 : \mathfrak{g} \to \mathfrak{g} : \begin{cases} X_2 \mapsto X_6 - X_8 \\ X_i \mapsto 0 \quad \text{for all } 1 \leq i \leq 8 \text{ with } i \neq 2 \end{cases}$$

is indeed an almost-inner derivation. Let $X = \sum_{i=1}^{8} x_i X_i$ with $x_i \in \mathbb{R}$ (for $1 \leq i \leq 8$), then

$$\psi_1(X) \;=\; x_2(X_6 - X_8) \;=\; \frac{x_2}{(x_1^2 - x_1 x_2 + x_2^2)} [X, (x_1 - x_2)X_3 + x_2 X_4 - x_1 X_5].$$

Further, the derivation

$$\psi_2 : \mathfrak{g} \to \mathfrak{g} : \begin{cases} X_2 \mapsto X_7 - X_8 \\ X_i \mapsto 0 \quad \text{for all } 1 \leq i \leq 8 \text{ with } i \neq 2 \end{cases}$$

is almost-inner. Consider $X = \sum_{i=1}^{8} x_i X_i$ with $x_i \in \mathbb{R}$ (for $1 \leq i \leq 8$). Then

$$\psi_2(X) \;=\; x_2(X_7 - X_8) \;=\; \frac{x_2}{(x_1^2 - x_1 x_2 + x_2^2)} [X, -x_2 X_3 + x_1 X_4 + (x_2 - x_1)X_5]$$

holds. It is clear that no linear combination of ψ_1 and ψ_2 belongs to $\operatorname{Inn}(\mathfrak{g})$. As a conclusion, this shows that $\dim(\operatorname{AID}(\mathfrak{g})) = 7$.

As the last example illustrates, there is in most cases a lot of computation needed to calculate $\operatorname{AID}(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. A few steps always come back. Let $\mathfrak{g}$ be a Lie algebra over a field K and let $\varphi \in \operatorname{AID}(\mathfrak{g})$ be an arbitrary almost-inner derivation.

1. Choose a convenient basis $\mathcal{B}$ and determine the Lie brackets;

2. Set up the matrix representation of φ with parameters with respect to $\mathcal{B}$;
3. Derive relations on the parameters by conditions for
   - a derivation
   - an almost-inner derivation;
4. Determine the linearly independent almost-inner derivations and check.

The first step was very easy in the example, because this was given in advance. However, for some classes of Lie algebras, it is very hard to construct the non-vanishing Lie brackets (this is for example the case in Section 3.4). The second step already gives an upper bound for the dimension. In the third step, the relations between the parameters are checked, by combining different conditions. When all relations are revealed, it is possible to express the dimension of the space of almost-inner derivations. If $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$, it is interesting to specify how an almost-inner derivation (which is not inner) looks like. In some cases, it is possible to obtain a result without going through all the steps. However, this procedure will be very useful to compute $\mathrm{AID}(\mathfrak{g})$ for the Lie algebras of the classes of Chapter 3.

In many cases, the third step is the hardest one. The conditions due to the definition of a derivation can be computed using the algorithms of A.2. For an almost-inner derivation, also the system of equations (2.7) has to be satisfied. This system has to hold for all possible values of the given field. Especially for fields with infinitely many elements, this is difficult to verify. Therefore, it is important to have some properties concerning almost-inner derivations. This will be elaborated in the next subsection.

2.3.2 Properties to compute the almost-inner derivations

In this subsection, some first results are obtained, which ease the computation of the space of almost-inner derivations for a given Lie algebra.

Let $\mathfrak{g}$ be an n-dimensional Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$ and structure constants c_{ij}^k (where $1 \leq i, j, k \leq n$). Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an almost-inner derivation of $\mathfrak{g}$. Then $\varphi(X_i) \in [X_i, \mathfrak{g}]$ holds by definition for all $1 \leq i \leq n$. Since φ is a linear map, it is clear that

$$\dim(\mathrm{AID}(\mathfrak{g})) \leq \sum_{i=1}^{n} \dim([X_i, \mathfrak{g}])$$

holds. This gives an upper bound for $\dim(\mathrm{AID}(\mathfrak{g}))$. Motivated by this equation, the dimension of a basis vector is introduced.

Definition 2.3.3 (Dimension of a basis vector)**.** *Let $\mathfrak{g}$ be a Lie algebra with basis $\mathcal{B} = \{X_1, \dots, X_n\}$. The dimension of the basis vector X_i is defined as*

$$d_i := \dim([X_i, \mathfrak{g}]).$$

Every inner derivation is also almost-inner, hence

$$\dim(\mathrm{Inn}(\mathfrak{g})) \leq \dim(\mathrm{AID}(\mathfrak{g})) \leq \sum_{i=1}^{n} d_i. \tag{2.10}$$

As explained in the last section, the conditions due to the definition of an almost-inner derivation can force some visible parameters to be equal. When all visible parameters belonging to the same basis vector has to be the same, the vector is called fixed.

Definition 2.3.4 (Fixed basis vector)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$. Let $\varphi \in AID(\mathfrak{g})$ be an almost-inner derivation with parameters a_{ij} with respect to $\mathcal{B}$ for $1 \le i, j \le n$. A basis vector X_j is fixed if there exists a value $a_j \in K$ so that $a_j = a_{ij}$ for all visible parameters belonging to X_j. Then a_j is called the 'fixed value' for X_j.*

The hard part in this theoretical description is to find which basis vectors are fixed. However, in some cases, the conditions are trivially satisfied, as is stated in the following important remark. It will be useful in Chapter 3.

Remark 2.3.5. *Let $\mathfrak{g}$ be a Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$. Let $\varphi \in AID(\mathfrak{g})$ be an almost-inner derivation with parameters a_{ij} with respect to $\mathcal{B}$ for all $1 \le i, j \le n$. Suppose that X_j is a basis vector with at most one visible parameter. Then X_j is fixed by definition.*

This is for example the case for all basis vectors in the centre. Whether or not a basis vector is fixed, depends on the choice of a basis. However, when all basis vectors are fixed, there is a nice relation with a general property of the Lie algebra, namely that every almost-inner derivation is an inner derivation. This fact is stated in the next lemma.

Lemma 2.3.6. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$. Then the equation*

$$Inn(\mathfrak{g}) = CAID(\mathfrak{g}) = AID(\mathfrak{g})$$

holds if and only if every basis vector X_i is fixed ($1 \le i \le n$).

Proof. Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an arbitrary almost-inner derivation with matrix representation $D = (d_{ij})$. Suppose first that $\mathrm{Inn}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g})$. Then φ is an inner derivation. Hence, there exist values $a_k \in K$ (with $1 \le k \le n$) such that $\varphi = \sum_{i=k}^{n} a_k \mathrm{ad}(X_k)$. In matrix representation, this implies that

$$d_{ij} = \sum_{k=1}^{n} -a_k c_{ik}^{j}$$

holds. It follows from equation (2.8) that every basis vector X_k is fixed with fixed value $-a_k$ (where $1 \le k \le n$).

Conversely, if every basis vector X_k is fixed with fixed value $a_k \in K$ (for $1 \le k \le n$), the matrix representation of φ is given by

$$d_{ij} = \sum_{k=1}^{n} a_k c_{ik}^{j}.$$

Hence, φ is a linear combination of the inner derivations $\mathrm{ad}(X_k)$ with coefficients $-a_k$ (for $1 \le k \le n$). Since $\varphi \in \mathrm{AID}(\mathfrak{g})$ was arbitrary, this completes the proof. □

Let $\mathfrak{g}$ be an n-dimensional Lie algebra over a field K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$. To prove that a basis vector X_i is fixed, it suffices to show that $a_{ji} = a_{li}$ for all visible parameters belonging to X_i. Next two lemmas are very technical and not very practical at first sight. They give sufficient conditions for some visible parameters belonging to the same basis vector to be equal, based on the equations (2.7). By applying those lemmas several times, it is possible to prove that some basis vectors are fixed. This approach will be used frequently in the next chapter.

Lemma 2.3.7. *Let $\mathfrak{g}$ be a Lie algebra over K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$ and structure constants c_{ij}^k. Let $\varphi \in AID(\mathfrak{g})$ be an almost-inner derivation with parameters a_{ij} (with $1 \le i, j, k \le n$). Let $(i, j, k, l) \in \{1, \dots, n\}^4$ such that the following conditions are satisfied:*

- *$c_{ij}^k \neq 0$ and $c_{il}^k \neq 0$;*
- *$c_{pj}^k = c_{pl}^k = 0$ for all $1 \le p \le n$ and $p \neq i$.*

Then, $a_{ji} = a_{li}$ follows.

Proof. There is nothing to show when $j = l$, so suppose that $j \neq l$. By equation (2.7), the condition

$$\sum_{q=1}^{n} \sum_{p=1}^{n} x_q (a_{qp} - c_p) c_{qp}^r = 0$$

has to be satisfied for all $1 \le r \le n$ and for all $x_q \in K$ with $1 \le q \le n$. Let $x_j \neq 0$ and $x_l \neq 0$ and choose $x_q = 0$ when $q \neq j, l$. The equation for the basis vector X_k becomes

$$\sum_{p=1}^{n} x_j (a_{jp} - c_p) c_{jp}^k + \sum_{p=1}^{n} x_l (a_{lp} - c_p) c_{lp}^k = 0.$$

The values c_p (with $1 \le p \le n$) depend on the chosen values of x_q (with $1 \le q \le n$). By assumption on the structure constants, this means that

$$x_j (a_{ji} - c_i) c_{ji}^k + x_l (a_{li} - c_i) c_{li}^k = 0$$

holds for all $x_j \neq 0 \neq x_l$, which is equivalent to

$$x_j a_{ji} c_{ji}^k + x_l a_{li} c_{li}^k = (x_j c_{ji}^k + x_l c_{li}^k) c_i.$$

Choose now $x_j = -x_l \frac{c_{li}^k}{c_{ji}^k}$, then it follows that

$$x_l c_{li}^k (-a_{ji} + a_{li}) = 0.$$

Since $x_l \neq 0$ and $c_{ji}^k \neq 0$, this implies that $a_{ji} = a_{li}$. □

Many Lie algebras satisfy the conditions of the lemma. In particular, this is true when the basis vector X_k appears exactly twice, namely for the pairs $\{X_i, X_j\}$ and $\{X_i, X_l\}$. In this case, it follows that $a_{ji} = a_{li}$. The same result is obtained when the conditions are slightly changed.

Lemma 2.3.8. *Let $\mathfrak{g}$ be a Lie algebra over K with basis $\mathcal{B} = \{X_1, \dots, X_n\}$ and structure constants c_{ij}^k. Let $\varphi \in AID(\mathfrak{g})$ be an almost-inner derivation with parameters a_{ij} (where $1 \le i, j, k \le n$). Let $(i, j, k, l, m) \in \{1, \dots, n\}^5$ with $k \neq m$ such that the following conditions are satisfied:*

- $c_{ij}^k \neq 0$ *and* $c_{il}^m \neq 0$;
- $c_{pj}^k = c_{pl}^m = 0$ *for all* $1 \leq p \leq n$ *with* $p \neq i$;
- $c_{pl}^k = c_{pj}^m = 0$ *for all* $1 \leq p \leq n$.

Then, $a_{ji} = a_{li}$ *follows.*

Proof. For $j = l$, the result immediately follows, so suppose that $j \neq l$. Again, the condition

$$\sum_{q=1}^{n}\sum_{p=1}^{n} x_q(a_{qp} - c_p)c_{qp}^r = 0$$

has to be satisfied for all $1 \leq r \leq n$ and for all $x_q \in K$ with $1 \leq q \leq n$. Fix $x_j \neq 0$ and $x_l \neq 0$ and set $x_q = 0$ for all $q \neq j, l$. In particular, the equations for the basis vectors X_k and X_m become

$$\begin{aligned} \sum_{p=1}^{n} x_j(a_{jp} - c_p)c_{jp}^k + \sum_{p=1}^{n} x_l(a_{lp} - c_p)c_{lp}^k &= 0 \\ \sum_{p=1}^{n} x_j(a_{jp} - c_p)c_{jp}^m + \sum_{p=1}^{n} x_l(a_{lp} - c_p)c_{lp}^m &= 0. \end{aligned}$$

Note that the values c_p (with $1 \leq p \leq n$) depend on the chosen values of x_q (with $1 \leq q \leq n$). By assumption on the structure constants, this further reduces to

$$\begin{aligned} x_j(a_{ji} - c_i)c_{ji}^k &= 0 \\ x_l(a_{li} - c_i)c_{li}^m &= 0, \end{aligned}$$

from which the result easily follows, since $c_{ij}^k \neq 0 \neq c_{il}^m$ and $x_j \neq 0 \neq x_l$. □

Although the conditions of this lemma seem to be demanding, next chapter shows that they are satisfied for a lot of Lie algebras. In particular, suppose that the basis vector X_k appears once. Then all conditions on the structure constants c_{ij}^k are satisfied (where $1 \leq i, j \leq n$). When X_k and X_m both appear once, namely for the pairs $\{X_i, X_j\}$ respectively $\{X_i, X_l\}$, it follows from the lemma that $a_{ji} = a_{li}$. This situation will frequently occur for the Lie algebras studied in Chapter 3. Next lemma states that an almost-inner derivation of a Lie algebra induces an almost-inner derivation of the quotient of that Lie algebra by an ideal.

Lemma 2.3.9. *Let* $\mathfrak{g}$ *be a Lie algebra with ideal* $\mathfrak{h}$*. Let* $\varphi \in AID(\mathfrak{g})$ *be an almost-inner derivation of* $\mathfrak{g}$*. Then* $\varphi(\mathfrak{h}) \subseteq \mathfrak{h}$*. Moreover,* φ *induces an almost-inner derivation of* $\mathfrak{g}/\mathfrak{h}$*.*

Proof. Choose $X \in \mathfrak{h}$ arbitrarily. By definition of an almost-inner derivation, it follows that

$$\varphi(X) \in [X, \mathfrak{g}] \subseteq [\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}.$$

Since $X \in \mathfrak{h}$ was arbitrary, this proves that $\varphi(\mathfrak{h}) \subseteq \mathfrak{h}$. Define

$$\overline{\varphi} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} : X + \mathfrak{h} \mapsto \overline{\varphi}(X + \mathfrak{h}) := \varphi(X) + \mathfrak{h}.$$

This is a well-defined linear map due to the first statement. Choose $X+\mathfrak{h}, Y+\mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ arbitrarily. Then

$$\overline{\varphi}([X+\mathfrak{h}, Y+\mathfrak{h}]) = \overline{\varphi}([X,Y]+\mathfrak{h}) = \varphi([X,Y])+\mathfrak{h}$$

holds, as well as

$$\begin{aligned}[\overline{\varphi}(X+\mathfrak{h}), Y+\mathfrak{h}]+[X+\mathfrak{h}, \overline{\varphi}(Y+\mathfrak{h})] &= [\varphi(X)+\mathfrak{h}, Y+\mathfrak{h}]+[X+\mathfrak{h}, \varphi(Y)+\mathfrak{h}] \\ &= [\varphi(X), Y]+\mathfrak{h}+[X, \varphi(Y)]+\mathfrak{h}.\end{aligned}$$

Combining these equations and using that φ is a derivation of $\mathfrak{g}$, it follows that $\overline{\varphi}$ is a derivation of $\mathfrak{g}/\mathfrak{h}$. Let $X+\mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ be arbitrary. Since φ is an almost-inner derivation of $\mathfrak{g}$, there exists $Y \in \mathfrak{g}$ such that $\varphi(X) = [X,Y]$. Hence,

$$\overline{\varphi}(X+\mathfrak{h}) = \varphi(X)+\mathfrak{h} = [X,Y]+\mathfrak{h} = [X+\mathfrak{h}, Y+\mathfrak{h}],$$

which shows that $\overline{\varphi}$ is an almost-inner derivation of $\mathfrak{g}/\mathfrak{h}$. □

When a Lie algebra is the direct sum of two Lie algebras, the set of almost-inner derivations $\mathrm{AID}(\mathfrak{g})$ satisfy an easy rule. This is worked out in the following proposition.

Proposition 2.3.10. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ over the same field. Then*

$$AID(\mathfrak{g}) = AID(\mathfrak{g}_1) \oplus AID(\mathfrak{g}_2)$$

holds.

Proof. The proof goes in two steps. First, let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an almost-inner derivation and let $X \in \mathfrak{g}$ be arbitrary. Then $X = X_1 + X_2$, where $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$. Since $\varphi \in \mathrm{AID}(\mathfrak{g})$, there exists $Y = Y_1+Y_2 \in \mathfrak{g}$ (with $Y_1 \in \mathfrak{g}_1$ and $Y_2 \in \mathfrak{g}_2$) such that $\varphi(X) = [X,Y]$. By construction,

$$\begin{aligned}\varphi(X) &= [X,Y] \\ &= [X_1+X_2, Y_1+Y_2] \\ &= [X_1,Y_1]+[X_1,Y_2]+[X_2,Y_1]+[X_2,Y_2] \\ &= [X_1,Y_1]+[X_2,Y_2],\end{aligned}$$

where the last equality holds since $[\mathfrak{g}_1,\mathfrak{g}_2] = 0$, by definition of the direct sum. This means that $\varphi_1 := \varphi|_{\mathfrak{g}_1}$ is an almost-inner derivation of $\mathfrak{g}_1$. Analogously, $\varphi_2 := \varphi|_{\mathfrak{g}_2} \in \mathrm{AID}(\mathfrak{g}_2)$ is satisfied. Hence, φ can be written as $\varphi = \varphi_1 \oplus \varphi_2$, which is defined as

$$\varphi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_1 \oplus \mathfrak{g}_2 : X_1 + X_2 \mapsto (\varphi_1 \oplus \varphi_2)(X_1+X_2) = \varphi_1(X_1)+\varphi_2(X_2).$$

Moreover, let $X = X_1+X_2 \in \mathfrak{g}$ be arbitrary, with $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$. Let $\varphi_1 \in \mathrm{AID}(\mathfrak{g}_1)\oplus 0$ and $\varphi_2 \in 0 \oplus \mathrm{AID}(\mathfrak{g}_1)$ be arbitrary. Then

$$\begin{aligned}[\varphi_1,\varphi_2](X) &= [\varphi_1,\varphi_2](X_1+X_2) \\ &= [\varphi_1,\varphi_2](X_1)+[\varphi_1,\varphi_2](X_2) \\ &= \varphi_1\varphi_2(X_1)-\varphi_2\varphi_1(X_1)+\varphi_1\varphi_2(X_2)-\varphi_2\varphi_1(X_2) \\ &= 0\end{aligned}$$

follows. This means that $[\mathrm{AID}(\mathfrak{g}_1), \mathrm{AID}(\mathfrak{g}_2)] = 0$ holds and completes the proof of the first inclusion.

Conversely, let $\varphi_1 + \varphi_2 \in \mathrm{AID}(\mathfrak{g}_1) \oplus \mathrm{AID}(\mathfrak{g}_1)$ be arbitrary, where $\varphi_1 \in \mathrm{AID}(\mathfrak{g}_1) \oplus 0$ and $\varphi_2 \in 0 \oplus \mathrm{AID}(\mathfrak{g}_2)$. Define the map $(\varphi_1 + \varphi_2)$ as

$$(\varphi_1 + \varphi_2) : \mathfrak{g} \to \mathfrak{g} : X = X_1 + X_2 \mapsto \varphi_1(X_1) + \varphi_2(X_2),$$

where $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$. By definition of an almost-inner derivation of $\mathfrak{g}$, there exist $Y_1 \in \mathfrak{g}_1$ and $Y_2 \in \mathfrak{g}_2$ such that $\varphi_1(X_1) = [X_1, Y_1]$ and $\varphi_2(X_2) = [X_2, Y_2]$. Note that these brackets are in $\mathfrak{g}$. Since $[X_1, Y_2] = 0 = [X_2, Y_1]$, it follows that

$$\begin{aligned}
(\varphi_1 + \varphi_2)(X) &:= \varphi_1(X_1) + \varphi_2(X_2) \\
&= [X_1, Y_1] + [X_2, Y_2] \\
&= [X_1, Y_1] + [X_2, Y_2] + [X_1, Y_2] + [X_2, Y_1] \\
&= [X_1 + X_2, Y_1 + Y_2].
\end{aligned}$$

Hence, the map $\varphi_1 + \varphi_2$ defines an almost-inner derivation of $\mathfrak{g}$. □

In particular, this proposition is useful for Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is abelian. In this case, $\mathrm{AID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g}_1)$ holds. An example of this situation is postponed to the next chapter.

Chapter 3

Different classes of Lie algebras

The procedure to compute the almost-inner derivations of a Lie algebra will be used for different classes of Lie algebras. In particular, it is studied for which of these classes of Lie algebras there exist non-inner almost-inner derivations. The first section concerns (complex) Lie algebras of dimension $n \leq 4$. For these Lie algebras, all almost-inner derivations are inner. In Section 3.2, filiform Lie algebras are studied. For such a Lie algebra $\mathfrak{g}$ which is also metabelian, the dimension of $\mathrm{AID}(\mathfrak{g})$ is at most one more than the dimension of $\mathrm{Inn}(\mathfrak{g})$. Further, two-step nilpotent Lie algebras can be defined by a graph; this is worked out in the following section. The fourth section is about free nilpotent Lie algebras; results for this class are only obtained when the nilindex is two or three. Finally, (strictly) uppertriangular matrices are treated. For the Lie algebras studied in the last three classes, the only almost-inner derivations are the inner ones. Except for the first and last class, all types are examples of nilpotent Lie algebras. This is since the geometric motivation only holds for those Lie algebras.

3.1 Low-dimensional Lie algebras

In this section, all non-isomorphic complex Lie algebras of dimension $n \leq 4$ are listed. It is clear that a one-dimensional Lie algebra can not have non-zero brackets. When the dimension is equal to $n = 2$, next proposition even holds for an arbitrary field. This is a well-known result, see for example [8, page 20].

Proposition 3.1.1. *Let K be an arbitrary field and let $\mathfrak{g}$ be a non-abelian two-dimensional Lie algebra over K. Then there exists a basis $\mathcal{B} = \{X_1, X_2\}$ such that the Lie bracket is given by $[X_1, X_2] = X_1$. Hence, for every field K, there is, up to isomorphism, a unique non-abelian two-dimensional Lie algebra over K.*

Proof. Let $\mathfrak{g}$ be a non-abelian two-dimensional Lie algebra over the field K. Let $\mathcal{B} = \{X, Y\}$ be a basis of $\mathfrak{g}$. Since $\mathfrak{g}$ is non-abelian, $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{[X, Y]\}$ is one-dimensional. Let $X_1 \in [\mathfrak{g}, \mathfrak{g}]$ be non-zero and extend to a basis $\{X_1, \tilde{Y}\}$ of the vector space $\mathfrak{g}$. Then, $[X_1, \tilde{Y}] \in [\mathfrak{g}, \mathfrak{g}]$ is non-zero, since $[X_1, \tilde{Y}]$ spans $[\mathfrak{g}, \mathfrak{g}]$. This means that $[X_1, \tilde{Y}] = kX_1$ for $k \in K^*$. For $X_2 := k^{-1}\tilde{Y}$, the Lie bracket of $\mathfrak{g}$ is given by $[X_1, X_2] = X_1$, where $\{X_1, X_2\}$ is a basis of $\mathfrak{g}$. It is easy to check that this indeed defines a Lie algebra. □

The unique non-abelian two dimensional Lie algebra over K is denoted by $\mathfrak{r}_2(K)$ or $\mathrm{aff}(K)$. For Lie algebras of dimension $n \geq 3$, it is difficult to classify the Lie algebras over

Table 3.1: Overview of the three-dimensional complex Lie algebras

$\mathfrak{g}$	Non-zero Lie brackets
$\mathbb{C}^3$	-
$\mathfrak{n}_3(\mathbb{C})$	$[X_1, X_2] = X_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[X_1, X_2] = X_1$
$\mathfrak{r}_3(\mathbb{C})$	$[X_1, X_2] = X_2$; $[X_1, X_3] = X_2 + X_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C})$	$[X_1, X_2] = X_2$; $[X_1, X_3] = \lambda X_3$ where $\lambda \in \mathbb{C}^*$
$\mathfrak{sl}_2(\mathbb{C})$	$[X_1, X_2] = X_3$; $[X_1, X_3] = -2X_1$; $[X_2, X_3] = 2X_2$

an arbitrary field. However, for complex Lie algebras of dimension $n \leq 4$, the result is known.

Proposition 3.1.2. *Every complex three-dimensional Lie algebra is isomorphic to at least one of the Lie algebras in Table 3.1.*

Proof. A proof of this fact can be found in [8]. □

Moreover, it is also shown in the proof that $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$ if and only if $\lambda = \mu$ or $\lambda\mu = 1$ for $\lambda, \mu \in \mathbb{C}$. All other Lie algebras in the table are non-isomorphic. The Lie algebra $\mathfrak{n}_3(\mathbb{C})$ is an example of a standard graded filiform Lie algebra and a strictly uppertriangular matrix over $\mathbb{C}$. Those classes of Lie algebras will be studied later in this chapter (in 3.2 respectively 3.5.1). For complex Lie algebras of dimension $n = 4$, there also exists a classification.

Proposition 3.1.3. *Every complex four-dimensional Lie algebra is isomorphic to at least one of the Lie algebras in Table 3.2.*

Proof. This fact is stated in [3]. □

One can show that $\mathfrak{g}_{10}(\alpha_1) \cong \mathfrak{g}_{10}(\alpha_2)$ if and only if $\alpha_1 = \alpha_2$ or $\alpha_1\alpha_2 = 1$. Moreover, when $\alpha, \beta \neq 0$, the relation $\mathfrak{g}_9(\alpha, \beta) \cong \mathfrak{g}_9(\alpha', \beta')$ holds exactly when (α', β') is one of the following:

$$(\alpha, \beta), \quad (\beta, \alpha), \quad \left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right), \quad \left(\frac{\beta}{\alpha}, \frac{1}{\alpha}\right), \quad \left(\frac{1}{\beta}, \frac{\alpha}{\beta}\right) \text{ and } \left(\frac{\alpha}{\beta}, \frac{1}{\beta}\right).$$

Further, also following relations hold:

$$\begin{aligned} \mathfrak{g}_7(0) &\cong \mathfrak{r}_{3,1}(\mathbb{C}) \oplus \mathbb{C}; \\ \mathfrak{g}_9(\alpha, 0) &\cong \mathfrak{r}_{3,\alpha}(\mathbb{C}) \oplus \mathbb{C} \quad \text{with } \alpha \neq 0, 1; \\ \mathfrak{g}_9(0, 1) &\cong \mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}. \end{aligned}$$

All other Lie algebras in the table are non-isomorphic.

Next propositions show that there are no non-inner almost-inner derivations for a complex Lie algebra $\mathfrak{g}$ when $\mathfrak{g}$ has dimension $n \leq 4$. For this, the classification of the non-isomorphic complex Lie algebras is used.

Proposition 3.1.4. *Let $\mathfrak{g}$ be a complex Lie algebra of dimension $n \leq 3$. Then all almost-inner derivations are inner.*

Table 3.2: Overview of the four-dimensional complex Lie algebras, where $\alpha, \beta \in \mathbb{C}$

$\mathfrak{g}$	Non-zero Lie brackets
$\mathfrak{g}_0 = \mathbb{C}^4$	-
$\mathfrak{g}_1 = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$	$[X_1, X_2] = X_3$
$\mathfrak{g}_2 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$	$[X_1, X_2] = X_1$
$\mathfrak{g}_3 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[X_1, X_2] = X_1$; $[X_3, X_4] = X_3$
$\mathfrak{g}_4 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[X_1, X_2] = X_3$; $[X_1, X_3] = -2X_1$; $[X_2, X_3] = 2X_2$
$\mathfrak{g}_5 = \mathfrak{n}_4(\mathbb{C})$	$[X_1, X_2] = X_3$; $[X_1, X_3] = X_4$
$\mathfrak{g}_6$	$[X_1, X_2] = X_2$; $[X_1, X_3] = X_3$; $[X_1, X_4] = X_4$
$\mathfrak{g}_7(\alpha)$	$[X_1, X_2] = X_2$; $[X_1, X_3] = X_3$; $[X_1, X_4] = X_3 + \alpha X_4$
$\mathfrak{g}_8$	$[X_1, X_2] = X_2$; $[X_1, X_3] = X_3$; $[X_1, X_4] = 2X_4$; $[X_2, X_3] = X_4$
$\mathfrak{g}_9(\alpha, \beta)$	$[X_1, X_2] = X_2$; $[X_1, X_3] = X_2 + \alpha X_3$; $[X_1, X_4] = X_3 + \beta X_4$
$\mathfrak{g}_{10}(\alpha)$	$[X_1, X_2] = X_2$; $[X_1, X_3] = X_2 + \alpha X_3$; $[X_1, X_4] = (\alpha + 1)X_4$; $[X_2, X_3] = X_4$

Proof. The proof goes by calculation of the almost-inner derivations for all Lie algebras of the classification of Table 3.1. By Lemma 2.3.6, it suffices to show that every basis vector is fixed for an arbitrary almost-inner derivation.

- Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an arbitrary almost-inner derivation of $\mathfrak{g}$, where $\mathfrak{g}$ is one of the Lie algebras $\mathbb{C}$, $\mathbb{C}^2$, $\mathfrak{r}_2(\mathbb{C})$, $\mathbb{C}^3$, $\mathfrak{n}_3(\mathbb{C})$ or $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$. Then every basis vector is fixed due to Remark 2.3.5.

- For the Lie algebra $\mathfrak{r}_3(\mathbb{C})$ with almost-inner derivation $\varphi \in \mathrm{AID}(\mathfrak{r}_3(\mathbb{C}))$, Remark 2.3.5 implies that X_2 and X_3 are fixed. The basis vector X_1 is fixed by Lemma 2.3.7 with $(i, j, k, l) = (1, 2, 2, 3)$.

- Let $\varphi \in \mathrm{AID}(\mathfrak{r}_{3,\lambda}(\mathbb{C}))$ be an arbitrary almost-inner derivation of $\mathfrak{r}_{3,\lambda}(\mathbb{C})$. Then, the basis vectors X_2 and X_3 are fixed by Remark 2.3.5. Lemma 2.3.8 with $(i, j, k, l, m) = (1, 2, 2, 3, 3)$ shows that X_1 is fixed too.

- As showed in Example 1.2.21, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is semisimple. The result now immediately follows from Proposition 2.1.8. □

The same result holds for all complex four-dimensional Lie algebras. For this, the classification of Table 3.2 is used.

Proposition 3.1.5. *Let $\mathfrak{g}$ be a complex Lie algebra of dimension $n = 4$. Then all almost-inner derivations are inner.*

Proof. The proof goes by calculation of the almost-inner derivations for all Lie algebras of the classification of Table 3.2. By Lemma 2.3.6, it suffices to show that every basis vector is fixed for an arbitrary almost-inner derivation φ. In the proof, all parameters with respect to φ are denoted with a_{ij} (where $1 \leq i, j \leq n$).

- The first five Lie algebras of the list all are direct sums of lower-dimensional Lie algebras for which all almost-inner derivations are inner derivations. By Proposition 2.3.10, the result immediately follows.

- Let $\varphi \in \mathrm{AID}(\mathfrak{n}_4(\mathbb{C}))$ be an arbitrary almost-inner derivation of $\mathfrak{n}_4(\mathbb{C})$. By Remark 2.3.5, the basis vectors X_2, X_3 and X_4 are fixed. Lemma 2.3.8 with $(i,j,k,l,m) = (1,2,3,3,4)$ shows that X_1 is fixed too, since a_{21} and a_{31} are the only visible parameters belonging to X_1.

- Consider the Lie algebra $\mathfrak{g}_6$ with arbitrary almost-inner derivation φ. Then, the basis vectors X_2, X_3 and X_4 are fixed by Remark 2.3.5. There are three visible parameters belonging to X_1. Lemma 2.3.8 with $(i,j,k,l,m) = (1,2,2,3,3)$ shows that $a_{21} = a_{31}$. By the same lemma with $(i,j,k,l,m) = (1,2,2,4,4)$, it follows that $a_{21} = a_{41}$. Hence, the basis vector X_1 is fixed too, which completes the proof.

- For the Lie algebras $\mathfrak{g}_7(\alpha)$ with arbitrary $\varphi \in \mathrm{AID}(\mathfrak{g}_7(\alpha))$, Remark 2.3.5 implies that the basis vectors X_2, X_3 and X_4 are fixed. By Lemma 2.3.8 with $(i,j,k,l,m) = (1,2,2,3,3)$, it follows that $a_{21} = a_{31}$. Moreover, $a_{31} = a_{41}$ holds by Lemma 2.3.7 with $(i,j,k,l) = (1,3,3,4)$. These two equations show that X_1 is fixed, since there are three visible parameters belonging to X_1.

- Let $\varphi \in \mathrm{AID}(\mathfrak{g}_8)$ be an almost-inner derivation of the Lie algebra $\mathfrak{g}_8$. The basis vector X_4 is fixed by Remark 2.3.5. To show that the other basis vectors are fixed, the procedure of Subsection 2.3.1 is used. By definition, there exist $a_{ij} \in \mathbb{C}$ with $1 \leq i,j \leq 4$ such that

$$\varphi : \mathfrak{g}_8 \to \mathfrak{g}_8 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_{12} & a_{13} & 2a_{14} \\ 0 & -a_{21} & 0 & a_{23} \\ 0 & 0 & -a_{31} & -a_{32} \\ 0 & 0 & 0 & -2a_{41} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}.$$

By checking the equations (2.3) for a derivation, it is clear that φ has to be of the form

$$\varphi : \mathfrak{g}_8 \to \mathfrak{g}_8 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_2 \\ 0 & b_6 & b_7 & -b_1 \\ 0 & 0 & 0 & b_4 + b_7 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix},$$

where $b_i \in \mathbb{C}$ for all $1 \leq i \leq 7$, hence $\dim(\mathrm{Der}(\mathfrak{g}_8)) = 7$. This calculation can be done by hand, but also with the aid of the computer algorithms of appendix A.2. By combining the two matrices, it turns out that $a_{12} = a_{32} = b_1$ and $a_{13} = a_{23} = b_2$, which shows that both X_2 and X_3 are fixed, since those are the only visible parameters belonging to the corresponding basis vectors. Moreover, from Lemma 2.3.8 with $(i,j,k,l,m) = (1,2,2,3,3)$, it follows that $a_{21} = a_{31}$ and therefore also $b_4 + b_7 = -2a_{21} = -2a_{41}$. Hence, X_1 is fixed too.

- Let φ be an almost-inner derivation of $\mathfrak{g}_9(\alpha,\beta)$. By Remark 2.3.5, the basis vectors X_2, X_3 and X_4 are fixed. From Lemma 2.3.7 with $(i,j,k,l) = (1,2,2,3)$, it follows that $a_{21} = a_{31}$. Lemma 2.3.8 with $(i,j,k,l,m) = (1,2,2,4,3)$ shows that $a_{21} = a_{41}$. Since X_1 has three visible parameters, it is fixed too.

- Consider the Lie algebras $\mathfrak{g}_{10}(\alpha)$ with arbitrary almost-inner derivation φ. It is clear that $\dim(\mathrm{Inn}(\mathfrak{g}_{10}(\alpha))) = 4$ when $\alpha \neq -1$ and $\dim(\mathrm{Inn}(\mathfrak{g}_{10}(\alpha))) = 3$ otherwise. The purpose is to show that the same dimensions hold for $\mathrm{AID}(\mathfrak{g}_{10}(\alpha))$. By definition of an almost-inner derivation, there exist $a_{ij} \in \mathbb{C}$ with $1 \le i, j \le 4$ such that

$$\varphi : \mathfrak{g}_{10}(\alpha) \to \mathfrak{g}_{10}(\alpha) : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_{12}+a_{13} & \alpha a_{13} & (\alpha+1)a_{14} \\ 0 & -a_{21} & 0 & a_{23} \\ 0 & -a_{31} & -\alpha a_{31} & -a_{32} \\ 0 & 0 & 0 & -(\alpha+1)a_{41} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}.$$

By checking the equations (2.3) for a derivation, one sees that φ has to be of the form

$$\varphi : \mathfrak{g}_{10}(\alpha) \to \mathfrak{g}_{10}(\alpha) : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & b_1 - b_2 & \alpha b_1 & b_3 \\ 0 & b_4 & 0 & b_1 \\ 0 & b_5 & b_4 + (\alpha-1)b_5 & b_2 \\ 0 & 0 & 0 & 2b_4 + (\alpha-1)b_5 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix},$$

where $b_i \in \mathbb{C}$ for all $1 \le i \le 5$, hence $\dim(\mathrm{Der}(\mathfrak{g}_{10}(\alpha))) = 5$. Again, this calculation can be done by hand or with the algorithms of appendix A.2. An almost-inner derivation is a derivation. Hence, for all possible values of α, the equation $b_4 = b_5$ is satisfied, which means that $a_{21} = a_{31} = a_{41}$. Further, $a_{12} + a_{13} = a_{23} + a_{32}$ holds. Therefore, φ can be written as

$$\varphi : \mathfrak{g}_{10}(\alpha) \to \mathfrak{g}_{10}(\alpha) : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_{23}+a_{32} & \alpha a_{13} & (\alpha+1)a_{14} \\ 0 & -a_{21} & 0 & a_{23} \\ 0 & -a_{21} & -\alpha a_{21} & -a_{32} \\ 0 & 0 & 0 & -(\alpha+1)a_{21} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}.$$

This implies that $\dim(\mathrm{AID}(\mathfrak{g}_{10}(\alpha))) = 4$ when $\alpha \neq -1$ and $\dim(\mathrm{AID}(\mathfrak{g}_{10}(\alpha))) = 3$ otherwise. □

The statement of the last two propositions does not hold for Lie algebras of a higher dimension. In the next section, there are examples of five-dimensional Lie algebras with non-inner almost-inner derivations.

3.2 Filiform Lie algebras

This section is devoted to filiform Lie algebras: nilpotent Lie algebras which have the maximal possible nilindex. For this reason, those Lie algebras are regarded as the 'less' nilpotent ones. For a filiform Lie algebra $\mathfrak{g}$ which is moreover metabelian, there exist general results concerning $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{AID}(\mathfrak{g})$. To be able to prove these propositions, a theorem due to Bratzlavsky is needed. First, the definition of a filiform Lie algebra is introduced.

Definition 3.2.1 (Filiform Lie algebra). *A Lie algebra $\mathfrak{g}$ of dimension n is filiform if $\mathfrak{g}$ is nilpotent with nilindex $n-1$.*

By definition, $\dim(\mathfrak{g}^k) = n - k - 1$ for $1 \le k \le n-1$. This explains the name 'filiform', which means threadlike.

Example 3.2.2. *Let $\mathfrak{g}_n$ be the n-dimensional Lie algebra with basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ and Lie brackets defined by*

$$[X_1, X_i] = X_{i+1} \quad \textit{for all } 2 \leq i \leq n-1.$$

This Lie algebra is called the standard graded filiform Lie algebra of dimension n. Of course, this only makes sense when $n \geq 3$.

The complex standard graded filiform Lie algebras of dimension three and four, $\mathfrak{n}_3(\mathbb{C})$ respectively $\mathfrak{n}_4(\mathbb{C})$, already appeared in 3.1. This is the standard example of a filiform Lie algebra. Moreover, it is also an example of a metabelian filiform Lie algebra. For those Lie algebras, there exists a classification result.

Proposition 3.2.3. *Let $\mathfrak{g}$ be a metabelian filiform Lie algebra over a field K with dimension $n \geq 3$. Then there exists a basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ such that the Lie brackets are given by*

- $[X_1, X_i] = X_{i+1}$ *for all* $2 \leq i \leq n-1$;
- $[X_2, X_i] = \sum_{k=i+2}^{n} c_{k-i-1} X_k$ *for all* $3 \leq i \leq n-2$,

where $c_j \in K$ for all $1 \leq j \leq n-4$.

Proof. A proof of this fact was first given by Bratzlavsky and can be found in [2]. □

Note that the second set of equations only defines non-zero brackets if $n \geq 5$. Hence, if the dimension is at most 4, all metabelian filiform Lie algebras are standard graded. Further, it is clear that the metabelian filiform Lie algebra is standard graded if and only if $c_i = 0$ for all $1 \leq i \leq n-4$. The basis in the proposition is called the 'graded basis' of the Lie algebra. With the aid of this basis, it is possible to prove a general result concerning the almost-inner derivations of metabelian filiform Lie algebras. Last section showed that complex Lie algebras of dimension $n \leq 4$ do not permit non-inner almost-inner derivations. This property also holds for the standard graded filiform Lie algebras.

Proposition 3.2.4. *Let $\mathfrak{g}$ be a standard graded filiform Lie algebra over a field K. Then all almost-inner derivations are inner.*

Proof. Denote n for the dimension of $\mathfrak{g}$. Since $\mathfrak{g}$ is standard graded filiform, there exists a basis $\{X_1, \ldots, X_n\}$ such that the non-vanishing Lie brackets of $\mathfrak{g}$ are given by $[X_1, X_i] = X_{i+1}$ for all $i \in \{2, \ldots, n-1\}$. Let φ be an arbitrary almost-inner derivation of $\mathfrak{g}$. For all $2 \leq i \leq n$, the basis vector X_i is fixed by Remark 2.3.5. Due to Lemma 2.3.6, it suffices to show that X_1 is fixed too. Consider $2 \leq i \neq j \leq n-1$ arbitrary, then $c_{1i}^{i+1} = 1 = c_{1j}^{j+1}$. Moreover,

$$c_{pi}^{i+1} = 0 = c_{pj}^{j+1} \qquad \text{and} \qquad c_{qj}^{i+1} = 0 = c_{qi}^{j+1}$$

hold for all $1 < p \leq n$ and for every $1 \leq q \leq n$, since X_{i+1} and X_{j+1} appear once. Hence, by Lemma 2.3.8 with $(i, j, k, l, m) = (1, i, i+1, j, j+1)$, this means that $a_{i1} = a_{j1}$. Since i and j were chosen arbitrarily, this shows that X_1 is fixed, which completes the proof. □

Standard graded filiform Lie algebras are a special case of metabelian filiform Lie algebras. For general metabelian filiform Lie algebras, another approach is needed. Before the result can be stated, more theory is needed about the derivations of this class of nilpotent Lie algebras. Let $\mathfrak{g}$ be an n-dimensional metabelian filiform Lie algebra. By Proposition 3.2.3, there exists a 'graded' basis $\mathcal{B} = \{X_1, \dots, X_n\}$. For this basis, the matrix representation of an arbitrary derivation can be computed.

Proposition 3.2.5. *Let $\mathfrak{g}$ be an n-dimensional metabelian filiform Lie algebra (where $n \geq 5$) over a field K with graded basis $\mathcal{B} = \{X_1, \dots, X_n\}$. Let $\varphi : \mathfrak{g} \to \mathfrak{g}$ be a linear map with matrix $D = (d_{ij})$. Then, φ is a derivation of $\mathfrak{g}$ if and only if the following equations hold:*

- $d_{ij} = 0$ *for all* $1 \leq j < i \leq n$;
- $d_{ii} = d_{22} + (i-2)d_{11}$ *for all* $3 \leq i \leq n$;
- $d_{i(i+1)} = d_{23} + (i-3)c_1 d_{12}$ *for all* $3 \leq i \leq n-1$;
- $d_{ij} = d_{2(j-i+2)} - \sum_{k=3}^{j-i+1} c_{j-i+2-k} d_{1k} + (i-3)c_{j-i} d_{12}$ *for all* $3 \leq i < j-1 \leq n-1$,

together with the extra equations

- $c_1(2d_{11} - d_{22}) = 0$ *when* $n \geq 5$;
- $\sum_{l=1}^{j-2} (l+1)c_l c_{j-l-1} d_{12} + c_{j-1}(jd_{11} - d_{22}) = 0$ *for all* $3 \leq j \leq n-3$ *when* $n \geq 6$.

Proof. Consider first that $\mathfrak{g}$ is a metabelian filiform Lie algebra and φ is a linear map with matrix $D = (d_{ij})$. The proof of this lemma is very technical, since it consists of verifying the conditions for a map to be a derivation. For all $1 \leq i, j \leq n$, the requirement for $[X_i, X_j]$ is given by

$$\varphi([X_i, X_j]) = [\varphi(X_i), X_j] + [X_i, \varphi(X_j)],$$

where $1 \leq k \leq n$. Due to the specific definition of the Lie brackets, this leads to several different cases. For the moment, only the first four statements will be proven.

- For the Lie bracket $[X_1, X_2]$, the condition is

$$\begin{aligned}
\sum_{j=1}^{n} d_{3j} X_j &= d_{11}X_3 - \sum_{j=3}^{n-2} d_{1j} \sum_{k=j+2}^{n} c_{k-j-1} X_k + \sum_{j=2}^{n-1} d_{2j} X_{j+1} \\
&= d_{11}X_3 - \sum_{k=3}^{n-2} d_{1k} \sum_{j=k+2}^{n} c_{j-k-1} X_j + \sum_{j=3}^{n} d_{2(j-1)} X_j \\
&= d_{11}X_3 - \sum_{j=5}^{n} d_{1k} \sum_{k=3}^{j-2} c_{j-k-1} X_j + \sum_{j=3}^{n} d_{2(j-1)} X_j.
\end{aligned}$$

The second and last equality are obtained by renaming respectively replacing the summands. This means that the following equations have to be satisfied:

$$
\begin{aligned}
d_{31} &= d_{32} = 0 \\
d_{33} &= d_{11} + d_{22} \\
d_{34} &= d_{23} \\
d_{3j} &= d_{2(j-1)} - \sum_{k=3}^{j-2} c_{j-k-1} d_{1k} \quad \text{for all } 5 \le j \le n.
\end{aligned}
$$

Those are exactly the expressions for d_{3j} that have to be fulfilled in the four first statements of the proposition, where $1 \le j \le n$.

- The conditions due to $[X_1, X_i]$ where $3 \le i \le n-2$ are given by

$$\sum_{j=1}^{n} d_{(i+1)j} X_j = d_{11} X_{i+1} + d_{12} \sum_{j=i+2}^{n} c_{j-i-1} X_j + \sum_{j=2}^{n-1} d_{ij} X_{j+1}.$$

This means that for all $4 \le i \le n-1$, the equations

$$\sum_{j=1}^{n} d_{ij} X_j = d_{11} X_i + d_{12} \sum_{j=i+1}^{n} c_{j-i} X_j + \sum_{j=3}^{n} d_{(i-1)(j-1)} X_j.$$

hold. Hence,

$$
\begin{aligned}
d_{i1} &= d_{i2} = 0 \\
d_{ij} &= d_{(i-1)(j-1)} \quad \text{for all } 3 \le j < i \\
d_{ii} &= d_{11} + d_{(i-1)(i-1)} \\
d_{ij} &= d_{(i-1)(j-1)} + c_{j-i} d_{12} \quad \text{for all } j > i
\end{aligned}
$$

have to be fulfilled, where $4 \le i \le n-1$. Inductively, this gives the equations

$$
\begin{aligned}
d_{ij} &= d_{(i-j+2),2} \quad \text{for all } 3 \le j < i \\
d_{ii} &= (i-3) d_{11} + d_{33} \\
d_{i(i+1)} &= d_{34} + (i-3) c_1 d_{12} \quad \text{for all } j > i \\
d_{ij} &= d_{3(j-i+3)} + (i-3) c_{j-i} d_{12} \quad \text{for all } j > i.
\end{aligned}
$$

All previous equations together give the desired expressions for d_{ij} in the first four requirements of the proposition, where $3 \le i \le n-1$ and $1 \le j \le n$.

- The condition for $[X_1, X_{n-1}]$ is

$$\sum_{j=1}^{n} d_{nj} X_j = d_{11} X_n + \sum_{j=2}^{n-1} d_{(n-1)j} X_{j+1} = d_{11} X_n + \sum_{j=3}^{n} d_{(n-1)(j-1)} X_j.$$

This corresponds to the equations

$$
\begin{aligned}
d_{n1} &= d_{n2} = 0 \\
d_{nj} &= d_{(n-1)(j-1)} \quad \text{for all } 3 \le j \le n-1 \\
d_{nn} &= d_{11} + d_{(n-1)(n-1)}.
\end{aligned}
$$

Hence, the first two statements of the proposition also hold for $i = n$.

- The Lie bracket $[X_1, X_n]$ has as stipulation that $0 = \sum_{j=2}^{n-1} d_{nj} X_{j+1}$ has to be satisfied, which means that $d_{nj} = 0$ for all $2 \leq j \leq n-1$.
- The requirements for the brackets $[X_2, X_i]$ with $3 \leq i \leq n-2$ are postponed to the end of this proof. The Lie bracket $[X_2, X_{n-1}]$ has as condition

$$0 = d_{21} X_n - d_{(n-1)1} X_3 + \sum_{j=3}^{n-2} d_{(n-1)j} \sum_{k=j+2}^{n} c_{k-j-1} X_k.$$

By previous observations, $d_{(n-1)j} = 0$ for all $1 \leq j \leq n-2$. Hence, this shows that $d_{21} = 0$.

This finishes the proof of the first four requirements of the proposition.

- For $[X_2, X_n]$, the condition is given by

$$0 = -d_{n1} X_3 + \sum_{j=3}^{n-2} d_{nj} \sum_{k=j+2}^{n} c_{k-j-1} X_k$$

This does not define new relations, because $d_{nj} = 0$ for all $1 \leq j \leq n-1$.

- The conditions due to $[X_i, X_j]$ where $3 \leq i < j \leq n-2$ are

$$0 = d_{i1} X_{j+1} + d_{i2} \sum_{k=j+2}^{n} c_{k-j-1} X_k - d_{j1} X_{i+1} - d_{j2} \sum_{k=i+2}^{n} c_{k-i-1} X_k.$$

Since $d_{i1} = d_{i2} = 0$ for all $3 \leq i \leq n$, this does not give new conditions.

- The brackets $[X_i, X_{n-1}]$ where $3 \leq i \leq n-2$ give the equations

$$0 = d_{i1} X_n - d_{(n-1)1} X_{i+1} - d_{(n-1)2} \sum_{j=i+2}^{n} c_{j-i-1} X_j.$$

By the same reasoning as before, there are no new relations.

- For $[X_i, X_n]$ where $3 \leq i \leq n-2$, the equations

$$0 = -d_{n1} X_{i+1} - d_{n2} \sum_{j=i+2}^{n} c_{j-i-1} X_j.$$

have to be satisfied. Hence, there are no new conditions.

- The stipulation due to $[X_{n-1}, X_n]$ is $0 = -d_{n1} X_n$.

There is only one case left. It suffices to show that this leads to the fifth and sixth statement. The conditions due to $[X_2, X_i]$ where $3 \leq i \leq n-2$ are given by

$$\sum_{k=i+2}^{n} c_{k-i-1} \sum_{j=1}^{n} d_{kj} X_j = d_{21} X_{i+1} + d_{22} \sum_{k=i+2}^{n} c_{k-i-1} X_k - d_{i1} X_3 + \sum_{j=3}^{n-2} d_{ij} \sum_{k=j+2}^{n} c_{k-j-1} X_k.$$

Since $d_{ij} = 0$ for all $1 \le i < j \le n$, the left side can be written as

$$\sum_{k=i+2}^{n} \sum_{j=k}^{n} c_{k-i-1} d_{kj} X_j = \sum_{j=i+2}^{n} \sum_{k=j}^{n} c_{j-i-1} d_{jk} X_k = \sum_{k=i+2}^{n} \sum_{j=i+2}^{k} c_{j-i-1} d_{jk} X_k,$$

where the roles of j and k and the summands are changed in the first respectively second equality. By using that $d_{ij} = 0$ for all $1 \le i < j \le n$, the right side is equal to

$$d_{22} \sum_{k=i+2}^{n} c_{k-i-1} X_k + \sum_{j=i}^{n-2} d_{ij} \sum_{k=j+2}^{n} c_{k-j-1} X_k = d_{22} \sum_{k=i+2}^{n} c_{k-i-1} X_k + \sum_{k=i+2}^{n} d_{ij} \sum_{j=i}^{k-2} c_{k-j-1} X_k.$$

In the last equality, the summands were changed. Combining the left and the right side, this gives the equation

$$\sum_{k=i+2}^{n} \sum_{j=i+2}^{k} c_{j-i-1} d_{jk} X_k = d_{22} \sum_{k=i+2}^{n} c_{k-i-1} X_k + \sum_{k=i+2}^{n} \sum_{j=i}^{k-2} d_{ij} c_{k-j-1} X_k.$$

Hence, for all $i + 2 \le k \le n$, there is the condition

$$\sum_{j=i+2}^{k} c_{j-i-1} d_{jk} X_k = d_{22} c_{k-i-1} X_k + \sum_{j=i}^{k-2} d_{ij} c_{k-j-1} X_k,$$

or equivalently,

$$\sum_{l=1}^{k-i-1} c_l d_{(l+i+1)k} = d_{22} c_{k-i-1} + \sum_{l=1}^{k-i-1} d_{i(l+i-1)} c_{k-l-i} = d_{22} c_{k-i-1} + \sum_{l=1}^{k-i-1} c_l d_{i(k-l-1)}.$$

The first equality is obtained by renaming the summands, the second by summing in a different order. This means that

$$\sum_{l=1}^{k-i-1} c_l \left(d_{(l+i+1)k} - d_{i(k-l-1)} \right) - d_{22} c_{k-i-1} = 0 \tag{3.1}$$

holds for all $i + 2 \le k \le n$, where $i \in \{3, \ldots, n-2\}$. For $k - i = 2$, this gives the equation $c_1(d_{55} - d_{33}) - c_1 d_{22} = 0$, which can be written as

$$c_1(2d_{11} - d_{22}) = 0.$$

This is the fifth statement of the proposition and the only case when $n = 5$. When $i + 3 \le k \le n$ and $3 \le i \le n - 3$, equation (3.1) is equivalent to

$$\sum_{l=1}^{k-i-2} c_l \left(d_{(l+i+1)k} - d_{i(k-l-1)} \right) + c_{k-i-1}(d_{kk} - d_{ii} - d_{22}) = 0.$$

When $l = k - i - 2$, the calculations from before imply that

$$d_{(l+i+1)k} - d_{i(k-l-1)} = \left(d_{23} + (l+i-2) c_1 d_{12} \right) - \left(d_{23} + (i-3) c_1 d_{12} \right) = (l+1) c_{k-l-i-1} d_{12}.$$

For $l < k - i - 2$, the equations

$$d_{(l+i+1)k} = d_{2(k-l-i+1)} - \sum_{m=3}^{k-l-i} c_{k-l-i+1-m} d_{1m} + (l+i-2) c_{k-l-i-1} d_{12}$$

$$d_{i(k-l-1)} = d_{2(k-l-i+1)} - \sum_{m=3}^{k-l-i} c_{k-l-i+1-m} d_{1m} + (i-3) c_{k-l-i-1} d_{12}$$

follow from the fourth equation, so the same result holds. By the above observations,

$$d_{kk} - d_{ii} - d_{22} \;=\; (k-i)d_{11} - d_{22}$$

is satisfied. Hence, this gives the equation

$$\sum_{l=1}^{k-i-2} (l+1)c_l c_{k-l-i-1} d_{12} + c_{k-i-1}\big((k-i)d_{11} - d_{22}\big) \;=\; 0$$

for all $i+3 \leq k \leq n$ and all $3 \leq i \leq n-3$, or equivalently,

$$\sum_{l=1}^{j-2}(l+1)c_l c_{j-l-1} d_{12} + c_{j-1}\big(jd_{11} - d_{22}\big) \;=\; 0,$$

which holds for all $3 \leq j \leq n-3$ (so only if $n \geq 6$). This completes the proof for the fifth and the sixth statement of the proposition.

Conversely, let $\varphi : \mathfrak{g} \to \mathfrak{g}$ be a linear map of $\mathfrak{g}$ for which the matrix entries satisfy the above equations. It is clear that φ is a derivation of $\mathfrak{g}$ in this case. □

As is stated in the previous proposition, d_{1k} and d_{2k} for $3 \leq k \leq n$ can be chosen arbitrarily in K, which shows that $\dim(\mathrm{Der}(\mathfrak{g})) \geq 2n-4$ for an n-dimensional metabelian filiform Lie algebra $\mathfrak{g}$ with $n \geq 5$. Moreover, the matrix entry d_{ij} (with $3 \leq i \leq n$ and $1 \leq j \leq n$) can be written as linear combination

$$d_{ij} \;=\; \sum_{k=1}^{n} a_k d_{1k} + b_k d_{2k},$$

where $a_k, b_k \in K$ (for $1 \leq k \leq n$) are specified in the proposition. Hence, there is no choice for those values. The relation between d_{11}, d_{12} and d_{22} is written down in the last two equations of the proposition, so this gives zero to three extra degrees of freedom. As a conclusion,

$$2n-4 \;\leq\; \dim(\mathrm{Der}(\mathfrak{g})) \;\leq\; 2n-1$$

holds, where $\mathfrak{g}$ is an n-dimensional metabelian filiform Lie algebra of dimension $n \geq 5$. Note that there are only two metabelian filiform Lie algebras which are not covered by the proposition, namely the standard graded filiform Lie algebras $\mathfrak{g}_3$ and $\mathfrak{g}_4$ of dimension $n = 3$ respectively $n = 4$. By the algorithms of appendix A.2, it is easy to verify that a general derivation φ of $\mathfrak{g}_3$ is given by

$$\varphi : \mathfrak{g}_3 \to \mathfrak{g}_3 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \mapsto \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ 0 & 0 & b_1 + b_5 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

where $b_i \in K$ for $1 \leq i \leq 6$. A general derivation φ of $\mathfrak{g}_4$ is of the form

$$\varphi : \mathfrak{g}_4 \to \mathfrak{g}_4 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \mapsto \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & b_1 + b_5 & b_6 \\ 0 & 0 & 0 & 2b_1 + b_5 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix},$$

with $b_i \in K$ for $1 \leq i \leq 7$. Hence $\dim(\mathrm{Der}(\mathfrak{g}_3)) = 6$ and $\dim(\mathrm{Der}(\mathfrak{g}_4)) = 7$ hold.

For metabelian filiform Lie algebras, there exists a global result concerning the almost-inner derivations. For standard graded filiform ones, this was proven in Proposition 3.2.4. The other metabelian filiform Lie algebras are treated in the next statement.

Proposition 3.2.6. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra over a field K and suppose that $\mathfrak{g}$ is metabelian filiform and not standard graded. Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a graded basis for $\mathfrak{g}$. Then*

$$CAID(\mathfrak{g}) = AID(\mathfrak{g}) = Inn(\mathfrak{g}) \oplus < E_{n,2} >,$$

where $E_{n,2}$ denotes the derivation

$$E_{n,2} : \mathfrak{g} \to \mathfrak{g} : \begin{cases} X_2 & \mapsto & X_n \\ X_i & \mapsto & 0 \quad \textit{for all } i \in \{1, 3, \ldots, n\}. \end{cases}$$

Proof. Let $\mathfrak{g}$ be a metabelian filiform Lie algebra. Then there exists a graded basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ as in Proposition 3.2.3. Since $\mathfrak{g}$ is not standard graded, $n \geq 5$. Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an almost-inner derivation with matrix representation $D = (d_{ij})$. By definition of an almost-inner derivation and of the particular Lie brackets, there exist $a_{ij} \in K$ for $1 \leq i, j \leq n$ such that

$$\begin{aligned}
\varphi(X_1) &= [X_1, \sum_{j=1}^{n} a_{1j} X_j] = \sum_{j=2}^{n-1} a_{1j} X_{j+1} = \sum_{k=3}^{n} a_{1(k-1)} X_k, \\
\varphi(X_2) &= [X_2, \sum_{j=1}^{n} a_{2j} X_j] = -a_{21} X_3 + \sum_{j=3}^{n-2} \sum_{k=j+2}^{n} a_{2j} c_{k-j-1} X_k \\
&= -a_{21} X_3 + \sum_{k=5}^{n} \sum_{j=3}^{k-2} a_{2j} c_{k-j-1} X_k, \\
\varphi(X_i) &= [X_i, \sum_{j=1}^{n} a_{ij} X_j] = -a_{i1} X_{i+1} - \sum_{k=i+2}^{n} a_{i2} c_{k-i-1} X_k \qquad \text{for all } 3 \leq i \leq n-2, \\
\varphi(X_{n-1}) &= [X_{n-1}, \sum_{j=1}^{n} a_{(n-1)j} X_j] = -a_{(n-1)1} X_n, \\
\text{and} \quad \varphi(X_n) &= [X_n, \sum_{j=1}^{n} a_{nj} X_j] = 0.
\end{aligned}$$

From equation (2.8), it follows that $d_{ij} = \sum_{k=1}^{n} a_{ik} c_{ik}^{j}$ for all $1 \leq i, j \leq n$. Hence, it is clear that

$$\begin{aligned}
d_{i(i+1)} &= -a_{i1} \qquad \text{for all } 2 \leq i \leq n-1 \\
d_{12} &= d_{24} = 0.
\end{aligned}$$

Since D has to fulfill the conditions for a derivation too, Proposition 3.2.5 and the above observations imply moreover that

$$d_{i(i+1)} = d_{23} + (i-3) c_1 d_{12} = d_{23} \qquad \text{for all } 3 \leq i \leq n-1.$$

This means that $d_{23} = -a_{i1} = -a_{21}$ for all $3 \leq i \leq n-1$ and hence X_1 is fixed.

Furthermore, let p be the smallest value $p \in \{1, \ldots, n-4\}$ such that $c_p \neq 0$. This exists, since $\mathfrak{g}$ is not standard graded. Then, the equation

$$d_{i(i+p+1)} = d_{2(p+3)} - \sum_{k=3}^{p+2} c_{p+3-k} d_{1k} + (i-3) c_{p+1} d_{12} = d_{2(p+3)} - c_p d_{13}$$

is satisfied for all $3 \le i \le n-p-1$. If $p = 1$, then $d_{2(p+3)} = d_{24} = 0$. For $p > 1$, the same result follows, since

$$d_{2(p+3)} = \sum_{j=3}^{p+1} a_{2j} c_{p+2-j} = 0$$

holds by definition of an almost-inner derivation. This means that

$$d_{i(i+p+1)} = -c_p d_{13} = -c_p a_{i2} \qquad \text{for all } 3 \le i \le n-p-1,$$

where the definition of the almost-inner derivation is used in the second equality. Hence, $a_{i2} = d_{13} = a_{12}$ for all $3 \le i \le n-p-1$, thus X_2 is fixed.

The next part consists of showing that X_l is fixed for all $3 \le l \le n-3$. By Proposition 3.2.5,

$$d_{in} = d_{2(n-i+2)} - \sum_{k=3}^{n-i+1} c_{n-i+2-k} d_{1k} + (i-3) c_{n-i} d_{12} = d_{2(n-i+2)} - \sum_{k=3}^{n-i+1} c_{n-i+2-k} d_{1k}$$

is satisfied for all $3 \le i \le n-2$. Hence,

$$d_{in} = d_{2(n-i+2)} - c_{n-i-1} d_{13} - \sum_{k=4}^{n-i+1} c_{n-i+2-k} d_{1k}$$

holds for all $3 \le i \le n-3$. Besides, for $3 \le i \le n-2$, also the equation $d_{in} = -a_{i2} c_{n-i-1}$ is fulfilled. Together, this implies that

$$d_{2(n-i+2)} = \sum_{k=4}^{n-i+1} c_{n-i+2-k} d_{1k}$$

for all $3 \le i \le n-3$, or equivalently

$$d_{2j} = \sum_{k=4}^{j-1} c_{j-k} d_{1k} = \sum_{k=4}^{j-1} c_{j-k} a_{1(k-1)} \qquad \text{for all } 5 \le j \le n-1.$$

In the second equation, the equivalent expression for the matrix entry d_{1k} is used. Moreover, by definition of an almost-inner derivation,

$$d_{2j} = \sum_{k=3}^{j-2} c_{j-k-1} a_{2k} = \sum_{k=4}^{j-1} c_{j-k} a_{2(k-1)}$$

is satisfied for all $5 \le j \le n$. It follows that $a_{2(k-1)} = a_{1(k-1)}$ for all $4 \le k \le n-2$. Thus,

$$a_{2l} = a_{1l} \qquad \text{for all } 3 \le l \le n-3$$

holds, which means that X_l is fixed for all $3 \le l \le n-3$.

Notice further that X_{n-1} is fixed by Remark 2.3.5 since there is only one visible parameter for X_{n-1}. Moreover, X_n is fixed because $X_n \in Z(\mathfrak{g})$. The basis vector X_{n-2} has two visible parameters, which gives rise to at most one extra dimension. Further, $E_{n,2}$ is indeed an almost-inner derivation. Let $X = \sum_{i=1}^{n} x_i X_i \in \mathfrak{g}$ be arbitrary (where $x_i \in K$ for all $1 \le i \le n$). If $x_1 \ne 0$, then

$$E_{n,2}(X) = x_2 X_n = [x_1 X_1 + \cdots + x_n X_n, \frac{x_2}{x_1} X_{n-1}]$$

is satisfied. If $x_1 = 0$, then

$$E_{n,2}(X) = x_2 X_n = [x_2X_2 + \cdots + x_nX_n, \frac{1}{c_p}X_{n-p-1}]$$

holds, where p is the smallest number such that $c_p \neq 0$. This exists since $\mathfrak{g}$ is not standard graded. This consideration concludes the proof. □

The advantage in the metabelian case is the existence of a 'graded basis'. In this way, the Lie brackets can be expressed very elegantly and the whole class can be treated simultaneously. For other classes of filiform Lie algebras, all Lie algebras have to be studied individually, which means that there is no global result. Next example shows that there are three-step solvable filiform Lie algebras for which $\mathrm{CAID}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g})$.

Example 3.2.7. *Let $\mathfrak{g}$ be the six-dimensional Lie algebra over a field K with basis $\mathcal{B} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ and with non-vanishing Lie brackets*

$$\begin{array}{llll} [X_1,X_2] = X_3; & [X_1,X_3] = X_4; & [X_1,X_4] = X_5; & [X_1,X_5] = X_6; \\ [X_2,X_3] = X_5; & [X_2,X_4] = X_6; & [X_2,X_5] = X_6; & [X_3,X_4] = -X_6. \end{array}$$

Then, $\mathfrak{g}$ is three-step solvable filiform with $\dim(\mathit{CAID}(\mathfrak{g})) = 5$ *and* $\dim(\mathit{AID}(\mathfrak{g})) = 6$.

The conditions on solvability and nilpotency are not hard to verify. Let $\varphi \in \mathrm{AID}(\mathfrak{g})$ be an arbitrary almost-inner derivation. By definition, there exist $a_{ij} \in K$ with $1 \leq i,j \leq 6$ such that

$$\varphi : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & -a_{21} & 0 & a_{23} & a_{24}+a_{25} \\ 0 & 0 & 0 & -a_{31} & -a_{32} & -a_{34} \\ 0 & 0 & 0 & 0 & -a_{41} & a_{43}-a_{42} \\ 0 & 0 & 0 & 0 & 0 & -(a_{51}+a_{52}) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix}.$$

By checking the conditions (2.3) with the computer algorithms of appendix A.2, it is easy to see that a general derivation is given by

$$\varphi : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} \mapsto \begin{pmatrix} b_1 & b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 2b_1 & b_6 & b_1 & b_7 & b_8 \\ 0 & 0 & 3b_1 & b_6 & b_1-b_2 & -b_3-b_4+b_7 \\ 0 & 0 & 0 & 4b_1 & b_1+b_6 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & 5b_1 & 2b_1+b_6-b_2 \\ 0 & 0 & 0 & 0 & 0 & 7b_1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix},$$

where all $b_i \in K$ (for all $1 \leq i \leq 8$), which means that $\dim(\mathrm{Der}(\mathfrak{g})) = 8$. Since an almost-inner derivation is a derivation, it turns out that $a_{21} = a_{31} = a_{41} = -b_6$ and $a_{12} = a_{32} = b_2$. Moreover, $-a_{34} = -b_3 - b_4 + b_7 = -a_{13} - a_{14} + a_{23}$ holds, as well as $a_{43} - a_{42} = -b_2 + b_3 = -b_2 + a_{13}$ and $-(a_{51} + a_{52}) = b_6 - b_2$. By definition of an almost-inner derivation, the equations

$$\sum_{i=1}^{6}\sum_{j=1}^{6} x_i a_{ij} c_{ij}^k = \sum_{i=1}^{6}\sum_{j=1}^{6} x_i c_j c_{ij}^k = 0$$

have to be fulfilled for all $1 \le k \le 6$ and for all $x_i \in K$ with $1 \le i \le 6$. For $k = 6$, this means that

$$\begin{aligned} &x_1 a_{15} + x_2(a_{24} + a_{25}) - x_3 a_{34} + x_4(a_{43} - a_{42}) - x_5(a_{51} + a_{52}) \\ &\qquad = -x_5 c_1 - (x_4 + x_5)c_2 + x_4 c_3 + (x_2 - x_3)c_4 + (x_1 + x_2)c_5 \end{aligned}$$

has to hold for $x_i \in K$ with $1 \le i \le 6$. For $(x_1, x_2, x_3, x_4, x_5) = (-1, 1, 1, 0, 0)$, this means that $-a_{15} + a_{24} + a_{25} - a_{34} = 0$, or equivalently

$$a_{24} + a_{25} \;=\; a_{15} + a_{34} \;=\; a_{15} + a_{13} + a_{14} - a_{23}.$$

Combining the above observations, this means that the almost-inner derivation φ is of the form

$$\varphi : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & b_2 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & b_6 & 0 & a_{23} & a_{15} + a_{13} + a_{14} - a_{23} \\ 0 & 0 & 0 & b_6 & -b_2 & -a_{13} - a_{14} + a_{23} \\ 0 & 0 & 0 & 0 & b_6 & -b_2 + a_{13} \\ 0 & 0 & 0 & 0 & 0 & b_6 - b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix}.$$

Let $X = \sum_{i=1}^{6} a_i X_i \in \mathfrak{g}$ be arbitrary, where $a_i \in K$ for all $1 \le i \le 6$. In matrix representation, $\mathrm{ad}(X) = \sum_{i=1}^{6} a_i \mathrm{ad}(X_i)$ is given by

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g} : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & -a_2 & -a_3 & -a_4 & -a_5 \\ 0 & 0 & a_1 & 0 & -a_3 & -a_4 - a_5 \\ 0 & 0 & 0 & a_1 & a_2 & a_4 \\ 0 & 0 & 0 & 0 & a_1 & a_2 - a_3 \\ 0 & 0 & 0 & 0 & 0 & a_1 + a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix},$$

which shows that $\dim(\mathrm{Inn}(\mathfrak{g})) = 5$. Hence, φ can be written as

$$\varphi \;=\; b_6\, \mathrm{ad}(X_1) - b_2\, \mathrm{ad}(X_2) - a_{13}\, \mathrm{ad}(X_3) - a_{14}\, \mathrm{ad}(X_4) - a_{15}\, \mathrm{ad}(X_5) + (a_{23} - a_{13})\, \psi,$$

where $\psi : \mathfrak{g} \to \mathfrak{g}$ is given by

$$\psi : \mathfrak{g} \to \mathfrak{g} : \begin{cases} X_2 & \mapsto \quad X_5 - X_6 \\ X_3 & \mapsto \quad X_6 \\ X_i & \mapsto \quad 0 \quad \text{for all } \; i \in \{1, 4, 5, 6\}. \end{cases}$$

The map ψ is indeed an almost-inner derivation. Let $X = \sum_{i=1}^{6} x_i X_i$ with $x_i \in K$ (for $1 \le i \le 6$). If $x_1 \neq 0$, then

$$\psi(X) \;=\; x_2 X_5 + (x_3 - x_2) X_6 \;=\; [X, \frac{x_2}{x_1} X_4 + \frac{x_3 - x_2}{x_1} X_5].$$

Further,

$$\psi(X) = x_2X_5 + (x_3 - x_2)X_6 = [\sum_{i=2}^{6} x_iX_i, X_3 - X_4 - \frac{x_4}{x_2}X_5]$$

is satisfied if $x_1 = 0$ and $x_2 \neq 0$. For $x_1 = x_2 = 0$,

$$\psi(X) = x_3X_6 = [\sum_{i=3}^{6} x_iX_i, -X_4]$$

holds. It is easy to see that ψ is not central almost-inner. For this Lie algebra,

$$\mathrm{Inn}(\mathfrak{g}) = \mathrm{CAID}(\mathfrak{g}) \subsetneq \mathrm{AID}(\mathfrak{g}) \subsetneq \mathrm{Der}(\mathfrak{g})$$

is fulfilled. Hence, the result for metabelian filiform Lie algebras can not be generalised to general filiform ones.

3.3 Two-step nilpotent Lie algebras determined by graphs

There is a strong connection between finite simple graphs and some two-step nilpotent Lie algebras. Let $G(V, E)$ be a finite simple graph with vertices $V = \{X_1, \dots, X_n\}$ and edges E. If there is an edge between X_i and X_j (with $i < j$), it is denoted with Y_{ij}. Let X be the vector space with basis the elements of V and let Y be the vector space with basis the edges of E. The vector space $\mathfrak{g} = X \oplus Y$ can be viewed as a Lie algebra, where the brackets are given by

$$\begin{aligned} [X_i, X_j] &= \begin{cases} Y_{ij} & \text{if } i < j \text{ and } Y_{ij} \in E; \\ -Y_{ji} & \text{if } i > j \text{ and } Y_{ji} \in E; \\ 0 & \text{if there is no edge between } X_i \text{ and } X_j; \end{cases} \\ [X_i, Y_{jk}] &= 0 \qquad \text{for all } X_i \in V \text{ and for all } Y_{jk} \in E; \\ [Y_{ij}, Y_{kl}] &= 0 \qquad \text{for all } Y_{ij}, Y_{kl} \in E. \end{aligned}$$

It is easy to see that by construction, this defines a nilpotent Lie algebra with nilindex two.

Conversely, an arbitrary nilpotent Lie algebra with nilindex two is determined by a graph when there exists a basis such that every basis vector appears at most once. It is not always clear at first sight if this condition is satisfied.

Example 3.3.1. *The two-step nilpotent Lie algebra $\mathfrak{g}$ over a field K with basis $\mathcal{B} = \{X_1, X_2, X_3, X_4\}$ and Lie brackets*

$$[X_1, X_2] = X_4 \quad \text{and} \quad [X_1, X_3] = X_4$$

can be determined by a graph.

Consider the basis $\mathcal{B}' = \{X_1, X_2, X_3', X_4\}$, where $X_3' = X_3 - X_2$. It is not difficult to see that for this new basis, the only non-zero Lie bracket is $[X_1, X_2] = X_4$, since

$$[X_1, X_3'] = [X_1, X_3] - [X_1, X_2] = 0.$$

This corresponds to a graph with three vertices (X_1,X_2 and X_3') and one edge (X_4). However, not all two-step nilpotent Lie algebras can be formed in that way.

Example 3.3.2. *The two-step nilpotent Lie algebra $\mathfrak{g}$ over a field K with basis $\mathcal{B} = \{X_1, X_2, X_3, X_4, X_5\}$ and Lie brackets*

$$[X_1, X_2] = X_5 \quad \text{and} \quad [X_3, X_4] = X_5$$

can not be determined by a graph.

In this example, it is easy to see that for every basis, there exists a basis vector which appears twice.

For two-step nilpotent Lie algebras determined by graphs, there exists a global result about the almost-inner derivations.

Proposition 3.3.3. *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra determined by a graph. Then all almost-inner derivations of $\mathfrak{g}$ are inner.*

Proof. Let $G(V, E)$ be the simple graph corresponding to $\mathfrak{g}$. Then $\mathfrak{g}$ can be written as $\mathfrak{g} = X \oplus Y$, where X is the vector space with basis the elements of V and Y the vector space with basis the edges of E. Denote the dimension of $\mathfrak{g}$ with n. Let φ be an arbitrary almost-inner derivation of $\mathfrak{g}$ with parameters a_{ij}, where $1 \le i, j \le n$. It is enough to show that all basis vectors are fixed. The basis elements of Y belong to the centre of $\mathfrak{g}$ and are hence fixed. By Remark 2.3.5, it suffices to consider the basis vectors X_i of X with more than one visible parameter. Let $X_i \in X$ be an arbitrary basis element with at least two visible parameters. Choose j, k, l and m arbitrarily in $\{1, \dots, n\}$ so that $c_{ij}^k \neq 0 \neq c_{il}^m$ and $j \neq l$. This means that a_{ji} and a_{li} are visible parameters belonging to X_i. By construction, k and m have to be different, since every edge belongs to exactly two vertices. By the same reasoning, X_k and X_m appear once. Hence, the conditions for Lemma 2.3.8 are fulfilled for (i, j, k, l, m), which shows that $a_{ji} = a_{li}$. Since j, k, l and m were arbitrary, this means that X_i has to be fixed. Therefore, all basis vectors of $\mathfrak{g}$ are fixed. Lemma 2.3.6 concludes the proof. □

Note that the converse of this proposition does not hold. Indeed, for the Lie algebra from Example 3.3.2, all almost-inner derivations are inner. This immediately follows from equation (2.10), since $\dim(\mathrm{Inn}(\mathfrak{g})) = 4$ and

$$\sum_{i=1}^{5} d_i = 1 + 1 + 1 + 1 + 0 = 4,$$

where d_i is as in Definition 2.3.3.

The result of the previous proposition can not be generalised for two-step nilpotent Lie algebras.

Example 3.3.4. *Let $\mathfrak{g}$ be the six-dimensional Lie algebra over a field K with basis $\mathcal{B} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ and Lie brackets defined by*

$$[X_1, X_2] = X_5, \quad [X_1, X_3] = X_6 \quad \text{and} \quad [X_3, X_4] = X_5.$$

Then, $\mathfrak{g}$ is two-step nilpotent with

$$\dim\left(\frac{AID(\mathfrak{g})}{Inn(\mathfrak{g})}\right) = 6 - 4 = 2.$$

Analogously as in Example 3.3.2, the Lie algebra $\mathfrak{g}$ can not be determined by a graph. Remark first that six is an upper bound for the dimension of the almost-inner derivations, since $\sum_{i=1}^{n} d_i = 2+1+2+1+0+0 = 6$. The derivation $\varphi_1 : \mathfrak{g} \to \mathfrak{g}$ defined as

$$\varphi_1 : \mathfrak{g} \to \mathfrak{g} : \begin{cases} X_1 & \mapsto & X_6 \\ X_i & \mapsto & 0 \quad \text{for all } i \in \{2, \dots, 6\} \end{cases}$$

is almost-inner. Let $X = \sum_{i=1}^{6} x_i X_i$ with $x_i \in K$ (for $1 \leq i \leq 6$). The definition is automatically satisfied when $x_1 = 0$. If $x_1 \neq 0$, then

$$\varphi_1(X) = x_1 X_6 = x_4 X_5 + x_1 X_6 - x_4 X_5 = [X, \frac{x_4}{x_1} X_2 + X_3].$$

Further, the derivation

$$\varphi_2 : \mathfrak{g} \to \mathfrak{g} : \begin{cases} X_3 & \mapsto & X_6 \\ X_i & \mapsto & 0 \quad \text{for all } i \in \{1, 2, 4, 5, 6\} \end{cases}$$

is almost-inner. Consider $X = \sum_{i=1}^{6} x_i X_i$ with $x_i \in K$ (for $1 \leq i \leq 6$). When $x_3 = 0$, the definition is immediately satisfied. If $x_3 \neq 0$, then

$$\varphi_2(X) = x_3 X_6 = x_2 X_5 + x_3 X_6 - x_2 X_5 = [X, -X_1 - \frac{x_2}{x_3} X_4]$$

holds. It is easy to see that no linear combination of φ_1 and φ_2 belongs to $\mathrm{Inn}(\mathfrak{g})$.

Moreover, due to Proposition 2.3.10, it is possible to construct a Lie algebra $\mathfrak{g}$ for which

$$\dim \left(\frac{\mathrm{AID}(\mathfrak{g})}{\mathrm{Inn}(\mathfrak{g})} \right)$$

is arbitrary large. However, $\mathrm{CAID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g})$ will always hold when $\mathfrak{g}$ is two-step nilpotent Lie algebra, by Lemma 2.1.10.

3.4 Free nilpotent Lie algebras

Section 3.2 was about filiform Lie algebras, the so-called 'less' nilpotent Lie algebras. Then, a special case of the 'most' (non-abelian) nilpotent Lie algebras were treated. The next class consists of the free nilpotent Lie algebras, where all nilindices can occur. First, the notion of a free Lie algebra is explained and a suitable basis is worked out. Further, for free nilpotent Lie algebras with nilindex is equal to two or three, results concerning the almost-inner derivations are proven.

As is the case for groups, a Lie algebra can be free too.

Definition 3.4.1 (Free Lie algebra). *Let X be a set and $\mathfrak{g}$ a Lie algebra. Let $i : X \to \mathfrak{g}$ be a set map. The Lie algebra $\mathfrak{g}$ is free on X if for every Lie algebra $\tilde{\mathfrak{g}}$ with a set map $f : X \to \tilde{\mathfrak{g}}$, there is a unique Lie algebra morphism $\varphi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ with $f = \varphi \circ i$.*

One can show that for every set X, there is a unique free Lie algebra generated by X. This Lie algebra has $M := |X|$ generators and is denoted with $\mathfrak{g}_M$. By definition, the only relations for free Lie algebras are due to the skew-symmetry and the Jacobi-identity.

Definition 3.4.2 (Length of a generator)**.** *Let $\mathfrak{g}_M$ be a free Lie algebra with M generators. The length d of a generator X_i (with $1 \le i \le M$) is defined as $d(X_i) = 1$. The length of a Lie bracket in the generators is defined recursively as*

$$d([X,Y]) = d(X) + d(Y) \qquad \textit{where } X,Y \in \mathfrak{g}_M \quad \textit{and } X \neq 0 \neq Y.$$

Moreover, $d(0) = 0$ holds.

As an example, consider $\mathfrak{g}_5$, the free Lie algebra on 5 generators. Then $d(X_2) = 1$ and $d([X_3, X_4]) = 2$, which shows that $d([X_2,[X_3,X_4]]) = 3$. This implies that

$$d([X_3,X_5],[X_2,[X_3,X_4]]) = 5.$$

Note that $d([X_2,[[X_1,X_3],[X_1,X_3]]]) = 0$, since $[[X_1,X_3],[X_1,X_3]] = 0$. Let $X \in \mathfrak{g}_M$ and $X \neq 0$, then X can be written as Lie bracket in the generators. The length of X can more or less be seen as the number of generators which are used in this expression. The length of a Lie bracket is used in the definition of a Hall set.

Definition 3.4.3 (Hall set)**.** *Let $\mathfrak{g}_M$ be a free Lie algebra with M generators. Then a set $\mathcal{B}_M = \{X_1, X_2, \dots\}$ is called a Hall set for $\mathfrak{g}_M$ when it satisfies the following conditions:*

- *The first M elements are the generators $X_1, \dots X_M$;*
- *If i and j are positive integers so that $d(X_i) < d(X_j)$, then $i < j$ holds;*
- *Let $X_i, X_j \in \mathfrak{g}_M$, then $[X_i, X_j] \in \mathcal{B}_M$ if and only if both $X_i, X_j \in \mathcal{B}_M$, the inequality $i < j$ holds and either X_j is a generator, or $X_j = [X_l, X_m]$ for some $X_l, X_m \in \mathcal{B}_M$ with $l \le i$.*

Consider the free Lie algebra with three generators. Then $A := [X_1,[X_1,X_3]] \in \mathcal{B}_3$ and $B := [X_2,[X_1,X_3]] \in \mathcal{B}_3$, but $[X_1,[X_2,X_3]] \notin \mathcal{B}_3$. Since $A, B \in \mathcal{B}$, they can be written as $A := X_a$ respectively $B := X_b$, where $a, b \in \mathbb{N}$. However, there is no fixed rule to decide whether $a < b$ or $b < a$. Of course, the Hall set depends on the choice or ordering. Let $X_{m_1}, X_{m_2} \in \mathcal{B}_3$ with $m_1 \neq m_2$ and suppose that

$$X_{m_1} := [X_i, X_j] \quad \text{and} \quad X_{m_2} := [X_k, X_l],$$

where $X_i, X_j, X_k, X_l \in \mathcal{B}_3$. In this thesis, the convention is used that the relation

$$m_1 < m_2 \quad \text{if and only if} \quad (i < k) \text{ or } (i = k \text{ and } j < l)$$

holds. It turns out that the Hall set can be used as a basis for a free Lie algebra.

Theorem 3.4.4. *Let $\mathfrak{g}_M$ be a free Lie algebra with M generators. A Hall set for $\mathfrak{g}_M$ defines a basis for $\mathfrak{g}_M$.*

Proof. A proof of this fact is given in [5, chapter 7]. □

This basis has infinitely many elements. A related notion is that of a free nilpotent Lie algebra.

Definition 3.4.5 (Free nilpotent Lie algebra)**.** *The free nilpotent Lie algebra $\mathfrak{g}_{M,r}$ with M generators and with nilindex r is the quotient of the free Lie algebra with M generators by the ideal $\mathfrak{g}^{r+1}$.*

Hence, the Lie algebra $\mathfrak{g}_{M,r}$ is generated by all Lie brackets of length $\leq r$, since every Lie bracket existing of more than r elements vanishes. Consider $\mathfrak{g}_M$, the free Lie algebra on M generators. Define

$$\mathcal{B} := \{X \in \mathcal{B}_M \mid d(X) \leq r\}$$

as the basis vectors of the Hall basis for $\mathfrak{g}_M$ with length $d \leq r$. Then $\mathcal{B}$ is a basis for $\mathfrak{g}_{M,r}$. To ease the notation, each basis vector is denoted as X_i for a suitable i, and not as the Lie bracket of generators. The 'length of a basis vector X_i which is no generator' (so $i > M$) is then defined as the length of the corresponding Lie bracket.

The following example shows the construction and calculation of a Hall basis and computes the dimension of the free nilpotent Lie algebra $\mathfrak{g}_{2,4}$.

Example 3.4.6. *The Hall basis $\mathcal{B}$ of the free nilpotent Lie algebra $\mathfrak{g}_{2,4}$ with two generators and nilindex four contains eight basis vectors.*

Below, all non-vanishing Lie brackets of a given length are listed. It is clear that all Lie brackets of length $l \geq 4$ are equal to zero, since $\mathfrak{g}_{2,4}$ has nilindex four. When the bracket belongs to the Hall basis $\mathcal{B}$, it can be written as X_a, for a suitable $a \in \mathbb{N}$. To determine the value of a, the previous convention is used. Of course, the generators X_1 and X_2 have length one.

- The only non-vanishing Lie bracket of length two is $X_3 := [X_1, X_2]$.
- There are exactly two non-zero Lie brackets of length three, namely

  $$X_4 := [X_1, [X_1, X_2]] \qquad \text{and} \qquad X_5 := [X_2, [X_1, X_2]].$$

- According to Theorem 3.4.4, the Lie brackets of length four which belong to the Hall basis $\mathcal{B}$ are

  $$\begin{aligned} & X_6 := [X_1, [X_1, [X_1, X_2]]]; \qquad X_7 := [X_1, [X_2, [X_1, X_2]]] \\ \text{and} \quad & X_8 := [X_2, [X_2, [X_1, X_2]]]. \end{aligned}$$

The Lie bracket $[X_1, [X_2, [X_1, X_2]]]$ is non-zero, but does not belong to the Hall basis, since it can be written as linear combination of basis vectors. Indeed, it follows from the Jacobi identity

$$[X_1, [X_2, [X_1, X_2]]] + [X_2, [[X_1, X_2], X_1]] + [[X_1, X_2], [X_1, X_2]] = 0$$

that $[X_1, [X_2, [X_1, X_2]]] = [X_2, [X_1, [X_1, X_2]]]$. As a result, $\mathfrak{g}_{2,4}$ has Hall basis $\mathcal{B} = \{X_1, \ldots, X_8\}$ and non-vanishing Lie brackets given by

$$\begin{aligned} &[X_1, X_2] = X_3; \qquad [X_1, X_3] = X_4; \qquad [X_2, X_3] = X_5; \\ &[X_1, X_4] = X_6; \qquad [X_1, X_5] = X_7; \qquad [X_2, X_4] = X_7 \quad \text{and} \quad [X_2, X_5] = X_8. \end{aligned}$$

The dimension of a free nilpotent Lie algebra $\mathfrak{g}_{M,r}$ with M generators and nilindex r can be computed explicitly due to a theorem of Witt, without constructing the Hall basis. Therefore, some terminology has to be introduced first.

Definition 3.4.7 (Möbiusfunction)**.** *Let $d \in \mathbb{N}$ be a natural number with prime factorisation $d = \prod_{i=1}^{q} p_i^{n_i}$. The Möbiusfunction $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined as*

$$\mu(d) = \begin{cases} 1 & \text{when } d = 1; \\ (-1)^q & \text{if } n_i = 1 \text{ for all } 1 \le i \le q; \\ 0 & \text{otherwise.} \end{cases}$$

In this notation, all p_i are different prime numbers and $n_i > 0$ holds for all $1 \le i \le q$.

This function appears in Witt's theorem. The formula, written with the notations of this thesis, determines the number of basis vectors of a given length.

Theorem 3.4.8 (Witt, 1937)**.** *Let $\mathfrak{g}_{M,r}$ be a free nilpotent Lie algebra with M generators and nilindex r. For all $k \le r$, the number ψ_k of basis vectors of length k is given by*

$$\psi_k(M) = \frac{1}{k} \sum_{d|k} \mu(d)\, M^{k/d}.$$

Proof. A proof of this fact can be found in [16]. □

Note that

$$\psi_1(M) = M; \quad \psi_2(M) = \frac{M^2 - M}{2}; \quad \psi_3(M) = \frac{M^3 - M}{3} \quad \text{and} \quad \psi_4(M) = \frac{M^4 - M^2}{4}.$$

From Witt's theorem, the dimension of $\mathfrak{g}_{M,r}$ immediately follows.

Corollary 3.4.9. *Let $\mathfrak{g}_{M,r}$ be a free nilpotent Lie algebra with M generators and nilindex r. The dimension of $\mathfrak{g}_{M,r}$ is given by*

$$\dim(\mathfrak{g}_{M,r}) = \sum_{i=1}^{r} \psi_r(M).$$

In the next example, the dimension of $\mathfrak{g}_{2,4}$ is checked with Witt's formula.

Example 3.4.10. *The free nilpotent Lie algebra $\mathfrak{g}_{2,4}$ with two generators and nilindex four is eight-dimensional.*

It follows from Example 3.4.6 that

$$\dim(\mathfrak{g}_{2,4}) = \psi_1(2) + \psi_2(2) + \psi_3(2) + \psi_4(2) = 2 + 1 + 2 + 3 = 8,$$

where $\psi_i(2)$ stands for the number of basis vectors of dimension i (where $1 \le i \le 4$).

Although it is possible to define a Hall basis for all free nilpotent Lie algebras, it is difficult to give a description of the Lie brackets when the nilindex is large. When the nilindex is two or three, the basis vectors can easily be written down explicitly. This makes it possible to prove a general result concerning the almost-inner derivations in that case. In the next subsections, this is worked out in detail.

3.4.1 Free nilpotent Lie algebras with nilindex two

The free nilpotent Lie algebras $\mathfrak{g}_{M,2}$ with M generators and with nilindex $r = 2$ are not so hard to build. Since $\mathfrak{g}_{M,2}$ is two-step nilpotent, the Jacobi identity automatically is satisfied. According to Definition 3.4.3, all Lie brackets $[X_i, X_j]$ with $1 \le i < j \le M$ belong to the Hall basis. Moreover, those are the only new elements, because all Lie brackets with length more than two vanish. Hence, $\mathfrak{g}_{M,2}$ has dimension

$$M + \binom{M}{2} = M + \frac{M(M-1)}{2} = \frac{M(M+1)}{2}.$$

Note that this satisfies Witt's formula, since $\psi_1(M) = M$ and $\psi_2(M) = \frac{M(M-1)}{2}$. The Lie brackets are given by $[X_i, X_j] = X_m$ for a suitable $M+1 \le m \le \frac{M(M+1)}{2}$. This m can be computed exactly if the previous ordering of the basis vectors is used. Let $1 \le i < j \le M$ and consider $[X_i, X_j] = X_m$. With the preceding convention, m is equal to

$$\begin{aligned} m &= \sum_{l=0}^{i-1}(M-l) + (j-i) = iM - \sum_{l=1}^{i-1} l + (j-i) \\ &= iM - \frac{i(i-1)}{2} + (j-i) = iM - \frac{i(i+1)}{2} + j. \end{aligned}$$

The next example follows this convention.

Example 3.4.11. *The free nilpotent Lie algebra $\mathfrak{g}_{4,2}$ is ten-dimensional with Lie brackets*

$$\begin{aligned} &[X_1, X_2] = X_5; \quad [X_1, X_3] = X_6; \quad [X_1, X_4] = X_7; \\ &[X_2, X_3] = X_8; \quad [X_2, X_4] = X_9; \quad [X_3, X_4] = X_{10}. \end{aligned}$$

Consider the Lie algebras $\mathfrak{g}_{M,2}$. It is easy to see that, with the above ordering, the basis vectors $X_1, \ldots, X_M$ do not appear and all other basis vectors appear exactly once. It turns out that those Lie algebras $\mathfrak{g}_{M,2}$ are a special case of the Lie algebras constructed Section 3.3. Indeed, this Lie algebra corresponds to a complete graph, where $X_1, \ldots, X_m$ stand for the M vertices and $X_{M+1}, \ldots, X_n$ are the edges.

Proposition 3.4.12. *Let $\mathfrak{g}_{M,2}$ be a free nilpotent Lie algebra with M generators and with nilindex $r = 2$. Then all almost-inner derivations are inner.*

Proof. Since the Lie algebra $\mathfrak{g}_{M,2}$ corresponds to the complete graph with M generators, the result immediately follows from Proposition 3.3.3. □

Next subsection is devoted to free nilpotent Lie algebras with nilindex $r = 3$.

3.4.2 Free nilpotent Lie algebras with nilindex three

Let $\mathfrak{g}_{M,3}$ be a free nilpotent Lie algebra with M generators and with nilindex equal to $r = 3$. As in the previous case, for all $1 \le i < j \le M$, the basis vectors of length two can be written as

$$[X_i, X_j] = X_m \qquad \text{with} \qquad m = iM - \frac{i(i+1)}{2} + j.$$

Since $\mathfrak{g}_{M,3}$ has nilindex three, all brackets of the form $[X_i, X_j]$ are non-zero, where X_i is a generator and X_j a basis vector of length two. This gives M times $\binom{M}{2}$ new brackets which are non-zero. However, according to Definition 3.4.3, only the brackets of the form

$$[X_i, [X_j, X_k]] \quad \text{with } 1 \le j < k \le M \text{ and } 1 \le j \le i \le M.$$

are basis vectors. This can be explained as follows. The Jacobi identity

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0 \tag{3.2}$$

holds for all $1 \le i, j, k \le M$. When two or more values of i, j and k are the same, this equation is automatically satisfied by skew-symmetry. This means that $[X_i, [X_j, X_k]]$ defines a new basis vector if $i = j$ or $i = k$. Otherwise, when $1 \le i < j < k \le M$, equation (3.2) can be written as

$$[X_i, [X_j, X_k]] = [X_j, [X_i, X_k]] - [X_k, [X_i, X_j]] = 0.$$

Hence, only two of the brackets in the expression above can be basis vectors. The next example will clarify the previous considerations. The ordering on the basis vectors is as in the previous convention.

Example 3.4.13. *The free nilpotent Lie algebra* $\mathfrak{g}_{3,3}$ *has Lie brackets*

$$\begin{array}{lll}
[X_1, X_2] = X_4; & [X_1, X_3] = X_5; & [X_2, X_3] = X_6; \\
[X_1, X_4] = X_7; & [X_1, X_5] = X_8; & [X_1, X_6] = X_{10} - X_{12}; \\
[X_2, X_4] = X_9; & [X_2, X_5] = X_{10}; & [X_2, X_6] = X_{11}; \\
[X_3, X_4] = X_{12}; & [X_3, X_5] = X_{13}; & [X_3, X_6] = X_{14}.
\end{array}$$

The brackets in the first row already appear for $\mathfrak{g}_{M,2}$. The three different possibilities for the new brackets are treated below. All Lie brackets $[X_i, X_j]$ with $1 \le i \le M$ and $M + 1 \le j \le \frac{M(M+1)}{2}$ belong to one of the three cases.

- The Lie bracket $[X_1, X_4]$ defines a new basis vector, because $X_4 = [X_1, X_2]$.
- The Lie bracket $[X_2, X_5]$ can be written as $[X_2, [X_1, X_3]]$. Since $1 \le 2$, this defines a new basis vector.
- To see whether or not the Lie bracket $[X_1, X_6]$ defines a new basis vector, note that the basis vector X_6 can be written as $[X_2, X_3] = X_6$. According to equation (3.2), the Lie bracket $[X_1, [X_2, X_3]]$ is equal to the linear combination

  $$[X_1, [X_2, X_3]] = [X_2, [X_1, X_3]] - [X_3, [X_1, X_2]] = [X_2, X_5] - [X_3, X_4] = X_{10} - X_{12},$$

  since $2 > 1$.

As explained before, there are M times $\binom{M}{2}$ new non-zero brackets, from which $\binom{M}{3}$ elements are linear combinations of the others: all different combinations (i, j, k) with $1 \le i < j < k \le M$ define only two new basis vectors of $\mathfrak{g}_{M,3}$. Hence, the dimension of $\frac{\mathfrak{g}^3_{M,3}}{\mathfrak{g}^2_{M,3}}$ is given by

$$\begin{aligned}
M\binom{M}{2} - \binom{M}{3} &= \frac{M^2(M-1)}{2} - \frac{M(M-1)(M-2)}{3 \cdot 2} \\
&= \frac{M(M-1)}{2} \cdot \frac{3M - M + 2}{3} = \frac{M(M-1)(M+1)}{3}.
\end{aligned}$$

When $M = 2$, the dimension of $\frac{\mathfrak{g}^3_{2,3}}{\mathfrak{g}^2_{2,3}}$ is equal to $M\binom{M}{2} = 2 = \frac{M(M-1)(M+1)}{3}$. Hence, the formula for the dimension remains valid. Note that $\psi_3(M) = \frac{M(M^2-1)}{3}$, which is an application of Witt's formula. This implies that the dimension of $\mathfrak{g}_{M,3}$ is

$$\begin{aligned} M + \frac{M(M-1)}{2} + \frac{M(M-1)(M+1)}{3} &= M\left(\frac{6 + 3(M-1) + 2(M^2-1)}{6}\right) \\ &= M\left(\frac{2M^2 + 3M + 1}{6}\right). \end{aligned}$$

As an example, the dimension of $\mathfrak{g}_{3,3}$ is given by

$$M\left(\frac{2M^2 + 3M + 1}{6}\right) = 3\left(\frac{2 \cdot 3^2 + 3 \cdot 3 + 1}{6}\right) = 14.$$

Consider the Lie algebras $\mathfrak{g}_{M,3}$. With the above convention, the basis vectors $X_1, \dots, X_M$ do not appear. Further, the basis vectors of length two appear once. Let X_p a basis vector of length three, so $X_p = [X_j, X_s]$ where X_j is a generator and X_s can be written as Lie bracket of two generators, say $X_s = [X_i, X_k]$. When $1 \le i < j \ne k \le n$, then X_p appears twice, namely for $\{X_j, X_s\}$ and also for $\{X_i, X_t\}$, where

$$X_t = [X_j, X_k] \quad (\text{if } j < k) \qquad \text{or} \qquad X_t = [X_k, X_j] \quad (\text{if } k < j).$$

Otherwise, X_p appears only once.

Remark 3.4.14. *Let $\mathfrak{g}_{M,3}$ be a free nilpotent Lie algebra with M generators and nilindex three. Denote the Hall basis with $\mathcal{B} = \{X_1, \dots, X_n\}$. The observations from before show that if X_k appears twice, say for $\{X_i, X_j\}$ and $\{X_a, X_b\}$, then a, b, i and l are pairwise different.*

Next proposition shows that $\mathrm{AID}(\mathfrak{g}_{M,3}) = \mathrm{Inn}(\mathfrak{g}_{M,3})$ for all possible values of M.

Proposition 3.4.15. *Let $\mathfrak{g}_{M,3}$ be a free nilpotent Lie algebra with M generators and nilindex $r = 3$. Then all almost-inner derivations are inner.*

Proof. Denote n for the dimension of $\mathfrak{g}_{M,3}$. Let $\varphi \in \mathrm{AID}(\mathfrak{g}_{M,3})$ with parameters a_{ij} (where $1 \le i, j \le n$). Choose X_i arbitrarily (where $1 \le i \le n$). By Lemma 2.3.6, it suffices to show that X_i is fixed. This is satisfied when all visible parameters belonging to X_i are equal. Let $(j, k, l, m) \in \{1, \dots, n\}^4$ be arbitrary values such that $c_{ij}^k \ne 0$ and $c_{il}^m \ne 0$, so a_{ji} and a_{li} are visible parameters belonging to X_i. It follows from Remark 3.4.14 that $k \ne m$. It is enough to prove that $a_{ji} = a_{li}$. This is done in different steps.

- Consider first that the basis vector X_i is a generator. Choose $1 \le b \le M$ and $M + 1 \le r \le n$ so that

  $$c_{ib}^r \ne 0 \qquad \text{and} \qquad c_{bp}^k = 0 = c_{bp}^m \quad \text{for all } 1 \le p \le n$$

  holds. This is always possible when X_k and X_m appear just once. When the basis vector X_k appears twice, then X_k has length three and there is one other generator X_a such that $c_{as}^k \ne 0$, for a suitable s. Therefore, a is excluded in the choice for b. The same reasoning also holds when X_m appears twice. Hence, for $M > 3$, there are thus at least $M-3$ possible choices for b, note that b cannot be equal to i. When $M=2$, there are no basis vectors which appear twice; for $M=3$, there are only two basis vectors which appear twice. It is clear from Example 3.4.13 that those vectors cannot be both X_k and X_m. This means that the values $1 \le b \le M$ and $1 \le r \le n$ also can be found in these cases. Further, $c_{pj}^k = c_{pl}^m = 0$ for all $1 \le p \le n$ with $p \ne i$ holds by Remark 3.4.14. Moreover, the basis vector X_r appears once, since it has length two. Therefore,

$$\begin{aligned} c_{qb}^r &= 0 \quad \text{for all } 1 \le q \le n \text{ with } q \ne i \\ \text{and} \quad c_{pj}^r &= c_{pl}^r = 0 \quad \text{for all } 1 \le q \le n \end{aligned}$$

hold. Lemma 2.3.8 can be used with (i,j,k,b,r) respectively (i,l,m,b,r), which shows that $a_{ji} = a_{bi}$ respectively $a_{li} = a_{bi}$. Together, this implies that $a_{ji} = a_{li}$, which was to show.

- Suppose now that X_i is not a generator. If $X_i \notin Z(\mathfrak{g}_{M,3})$, it is automatically fixed. Otherwise, there exists a unique pair $1 \le a < b \le M$ so that $[X_a, X_b] = X_i$. Denote X_s and X_t for the basis vectors $X_s = [X_a, X_i]$ respectively $X_t = [X_b, X_i]$. By construction, X_s and X_t appear once. Hence, it follows from Lemma 2.3.8 with (i,a,s,b,t) that $a_{ai} = a_{bi}$. The rest of the proof goes in two different steps.
  - When $c_{ap}^k = 0$ for all $1 \le p \le n$, Lemma 2.3.8 can be used with (i,j,k,a,s), which shows that $a_{ji} = a_{ai}$. Otherwise, since X_k appears at most twice, the equation $c_{bp}^k = 0$ is satisfied for all $1 \le p \le n$. Hence, Lemma 2.3.8 with (i,j,k,b,t) implies that $a_{ji} = a_{bi}$.
  - If $c_{ap}^l = 0$ holds for all $1 \le p \le n$, it follows from Lemma 2.3.8 with (i,l,m,a,s) that $a_{li} = a_{ai}$. If this is not the case, $c_{bp}^m = 0$ holds for all $1 \le p \le n$, since the basis vector X_m appears at most twice. Lemma 2.3.8 with (i,l,m,b,t) then gives that $a_{li} = a_{bi}$.

  Since $a_{ai} = a_{bi}$, the above observations show that $a_{ji} = a_{li}$.

In both cases, $a_{ji} = a_{li}$ is satisfied. Since a_{ji} and a_{li} were two arbitrary visible parameters belonging to X_i, the basis vector X_i is fixed. This finishes the proof. □

The different cases in the proof of the proposition will be clarified with an example.

Example 3.4.16. *For the fourteen-dimensional free nilpotent Lie algebra $\mathfrak{g}_{3,3}$, it follows that Inn$(\mathfrak{g}_{3,3})$ = AID$(\mathfrak{g}_{3,3})$.*

The Lie brackets for this Lie algebra are listed in Example 3.4.13. Let $\varphi \in \mathrm{AID}(\mathfrak{g}_{3,3})$ be an almost-inner derivation with parameters a_{ij} (where $1 \le i,j \le 14$).

- Consider the inequalities $c_{24}^9 \ne 0$ and $c_{25}^{10} \ne 0$. The way to show that $a_{42} = a_{52}$ is written down in the first part of the proof, since X_2 is a generator. Consider the Lie brackets

$$[X_2, X_4] = [X_2, [X_1, X_2]] = X_9 \qquad \text{and} \qquad [X_2, X_5] = [X_2, [X_1, X_3]] = X_{10}.$$

  The basis vector X_9 appears once, but X_{10} appear twice. Indeed, the equation $c_{16}^{10} = 1$ holds. Thus, b cannot be equal to one or two, which means that $b = 3$ is the only possibility (and $r = 6$). Further, the basis vector $X_6 = [X_2, X_3]$ appears only once. Hence, Lemma 2.3.8 with $(i,j,k,l,m) = (2,4,9,3,6)$ shows that $a_{42} = a_{32}$. Moreover, it follows from the same lemma with $(i,j,k,l,m) = (2,5,10,3,6)$ that $a_{52} = a_{32}$. Those equations imply that $a_{52} = a_{42}$.

- Consider the inequalities $c_{61}^{10} \neq 0$ and $c_{62}^{11} \neq 0$. The corresponding visible parameters a_{16} and a_{26} belong to X_6, which can be written as $X_6 = [X_2, X_3]$. Further, the basis vectors $X_{11} = [X_2, X_6]$ and $X_{14} = [X_3, X_6]$ appear once, so $(a,b,s,t) = (2,3,11,14)$. Lemma 2.3.8 with $(6,2,11,3,14)$ shows that $a_{26} = a_{36}$. Further, $c_{2p}^{10} \neq 0$ holds for $p = 5$, but $c_{3p}^{10} = 0$ is satisfied for all $1 \leq p \leq 14$. Lemma 2.3.8 can be used with $(i,j,k,l,m) = (6,1,10,3,14)$ to show that $a_{16} = a_{36}$. Next, $c_{26}^{11} = 1$, but $c_{3p}^{11} = 0$ holds for all $1 \leq p \leq n$, since X_{11} appears only once. It follows from Lemma 2.3.8 with $(i,j,k,l,m) = (6,2,11,3,14)$ that $a_{26} = a_{36}$. Both observations together give the desired $a_{16} = a_{26}$.

All other cases are analogous. Of course, in the last example, it can already be derived that $a_{16} = a_{26}$ without the last step. However, the example is worked out completely to illustrate that the procedure is valid whether or not one or both of the values j and l are equal to a or b in the proof of the proposition.

Let $\mathfrak{g}_{M,r}$ be a free nilpotent Lie algebra with M generators and nilindex $r > 3$. It is still possible to build a Hall basis and to determine the number of basis vectors. However, it becomes difficult to describe the relations between the basis vectors and hence the Lie brackets, since there are a lot of different cases which have to be treated. Therefore, it is hard to compute $\mathrm{AID}(\mathfrak{g}_{M,r})$, since the first step in the procedure mentioned in Subsection 2.3.1 cannot be executed. Hence, there is no general result known concerning the almost-inner derivations of free nilpotent Lie algebras $\mathfrak{g}_{M,r}$ with M generators and nilindex $r > 3$.

3.5 Triangular matrices

This section is devoted to the study of the set of all strictly uppertriangular matrices and the set of all uppertriangular matrices. It will turn out that those two subsets of $\mathbf{gl}(n,K)$ are in fact Lie subalgebras. First, some notation is introduced. Fix an $n \in \mathbb{N}$ and a field K. Consider $\mathbf{gl}(n,K)$, the Lie algebra of all $(n \times n)$-matrices over the field K, introduced in Example 1.2.6. The matrix E_{ij} is defined as

$$(E_{ij})_{k,l} = \begin{cases} 1 & \text{if } (k,l) = (i,j); \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathcal{B}_n = \{E_{ij} \in K^{n \times n} \mid 1 \leq i,j \leq n\}$ forms a basis of $\mathbf{gl}(n,K)$. Further, the Lie brackets are given by

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

Next subsections describe the definition and the almost-inner derivations for the Lie algebras of (strictly) uppertriangular matrices. Of course, all properties obtained for these classes of Lie algebras also hold for the Lie algebras of all (strictly) lowertriangular matrices.

3.5.1 Strictly uppertriangular matrices

The notation $\mathfrak{n}_n(K) \subset \mathbf{gl}(n, K)$ stands for the set of all strictly uppertriangular $(n \times n)$-matrices over the field K. Throughout this section, it will be assumed that $n \geq 2$, since $\mathfrak{n}_1(K)$ only contains zero. A basis for the vector space $\mathfrak{n}_n(K)$ is given by

$$\mathcal{B}_n = \{E_{ij} \in K^{n\times n} \mid 1 \leq i < j \leq n\}.$$

Choose $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$ arbitrarily. Since

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$$

holds, the Lie bracket of two strictly uppertriangular matrices is again strictly uppertriangular. Hence, $\mathfrak{n}_n(K)$ is a Lie subalgebra of $\mathbf{gl}(n, K)$. From the basis of $\mathfrak{n}_n(K)$, it is clear that the dimension of this Lie algebra is

$$\dim(\mathfrak{n}_n(K)) = \frac{n(n-1)}{2}.$$

The Lie algebra $\mathfrak{n}_2(K) \cong K$ is abelian. When $n > 2$, the Lie algebra $\mathfrak{n}_n(K)$ is nilpotent with nilindex $n-1$.

Example 3.5.1. *The Lie algebra $\mathfrak{n}_3(K)$ over the field K is three-dimensional and has $\mathcal{B} = \{E_{12}, E_{23}, E_{13}\}$ as basis. The only non-zero Lie bracket is $[E_{12}, E_{23}] = E_{13}$. This is the standard graded filiform Lie algebra of dimension $n = 3$.*

In the next lemma, the dimension of the Lie algebra of inner derivations of $\mathfrak{n}_n(K)$ is determined.

Lemma 3.5.2. *Let $\mathfrak{n}_n(K)$ be the Lie algebra of all strictly uppertriangular $(n\times n)$-matrices over the field K. When $n > 1$, it follows that*

$$\dim(\mathit{Inn}(\mathfrak{n}_n(K))) = \dim(\mathfrak{n}_n(K)) - 1 = \frac{n^2 - n - 2}{2}.$$

Proof. The dimension of $\mathrm{Inn}(\mathfrak{n}_n(K))$ is equal to the dimension of

$$\mathrm{vct}\{\mathrm{ad}(E_{ij}) : \mathfrak{n}_n(K) \to \mathfrak{n}_n(K) \mid 1 \leq i < j \leq n\}.$$

Remark first that E_{1n} belongs to the centre of $\mathfrak{n}_n(K)$, which shows the result for $n = 2$. Further, when $n > 2$, all inner derivations $\mathrm{ad}(E_{ij}) : \mathfrak{n}_n(K) \to \mathfrak{n}_n(K)$ are linear independent when $1 \leq i < j \leq n$ and $(i, j) \neq (1, n)$. Indeed, for all $1 \leq j < k < n$, the only inner derivation of a basis vector which maps E_{kn} to (a non-zero multiple of) E_{jn} is $\mathrm{ad}(E_{jk})$. Further, $\mathrm{ad}(E_{jn})$ is the only adjoint map belonging to a basis vector which maps E_{1j} to (a non-zero multiple of) $-E_{1n}$, where $1 < j < n$. This completes the proof. □

Next proposition shows that the only almost-inner derivations for $\mathfrak{n}_n(K)$ are the inner ones. The proof is based on the fact that an almost-inner derivation also satisfies the conditions for an arbitrary derivation.

Proposition 3.5.3. *Let $\mathfrak{n}_n(K)$ be the Lie algebra of all strictly uppertriangular $(n \times n)$-matrices over the field K. Then all almost-inner derivations are inner derivations.*

Proof. An abelian Lie algebra does not admit inner derivation, which shows the result for $n = 2$. When $n = 3$, the result immediately follows from Proposition 3.2.4. Suppose now that $n > 3$. Let $\varphi \in \mathrm{AID}(\mathfrak{n}_n(K))$ be an almost-inner derivation of $\mathfrak{n}_n(K)$. Then there exist parameters a_{ij}^{kl} with $1 \le i, j, k, l \le n$ such that

$$\begin{aligned}
\varphi(E_{1i}) &= \sum_{t=i+1}^{n} a_{1i}^{it} E_{1t} \qquad \text{with} \quad 1 < i < n;\\
\varphi(E_{1n}) &= 0;\\
\varphi(E_{jk}) &= \sum_{t=1}^{j-1} -a_{jk}^{tj} E_{tk} + \sum_{t=k+1}^{n} a_{jk}^{kt} E_{jt} \qquad \text{with} \quad 1 < j < k < n;\\
\varphi(E_{jn}) &= \sum_{t=1}^{j-1} -a_{jn}^{tj} E_{tn} \qquad \text{with} \quad 1 < j < n.
\end{aligned}$$

To enlarge the readability, the second index of the parameters is now written as a superscript. By Lemma 2.3.6, it suffices to show that all basis vectors are fixed. This goes in different steps.

- Remark that E_{1n} belongs to the centre and is hence fixed by Remark 2.3.5.
- Choose E_{1i} arbitrarily with $1 < i < n$. The visible parameters belonging to E_{1i} are a_{ij}^{1i} (with $i < j \le n$). When $i = n - 1$, there is only one visible parameter, namely $a_{n-1,n}^{1,n-1}$. Hence, $E_{1,n-1}$ is fixed by Remark 2.3.5. Otherwise, let $i < j < n$ be arbitrary. By definition of the Lie brackets and of an arbitrary almost-inner derivation,

  $$\varphi([E_{ij}, E_{jn}]) = \varphi(E_{in}) = \sum_{t=1}^{i-1} -a_{in}^{ti} E_{tn}$$

  holds. Moreover, φ is a derivation and therefore,

  $$\begin{aligned}
  \varphi([E_{ij}, E_{jn}]) &= [\varphi(E_{ij}), E_{jn}] + [E_{ij}, \varphi(E_{jn})]\\
  &= [\sum_{t=1}^{i-1} -a_{ij}^{ti} E_{tj} + \sum_{t=j+1}^{n} a_{ij}^{jt} E_{it}, E_{jn}] + [E_{ij}, \sum_{t=1}^{j-1} -a_{jn}^{tj} E_{tn}]\\
  &= \sum_{t=1}^{i-1} -a_{ij}^{ti} E_{tn}
  \end{aligned}$$

  has to be satisfied. Combining both equations leads to the following result:

  $$a_{in}^{ti} = a_{ij}^{ti} \qquad \text{where} \quad 1 \le t < i < j < n.$$

  In particular, this shows that for $t = 1$, all visible parameters belonging to E_{1i} are the same, thus E_{1i} is fixed.
- Consider an arbitrary basis vector E_{jk}, where $1 < j < k < n$. The visible parameters belonging to E_{jk} are

  $$a_{ij}^{jk} \quad \text{and} \quad a_{kl}^{jk}, \qquad \text{with} \quad 1 \le i < j < k < l \le n,$$

  because the Lie brackets $[E_{ij}, E_{jk}]$ and $[E_{jk}, E_{kl}]$ are non-zero for all values $1 \le i < j < k < l \le n$. To show that E_{jk} is fixed, it suffices to prove that all those visible parameters are equal.

Since φ is a derivation,

$$
\begin{aligned}
0 &= \varphi([E_{1j}, E_{kn}]) \\
&= [\varphi(E_{1j}), E_{kn}] + [E_{1j}, \varphi(E_{kn})] \\
&= [\sum_{t=j+1}^{n} a_{1j}^{jt} E_{1t}, E_{kn}] + [E_{1j}, \sum_{t=1}^{k-1} -a_{kn}^{tk} E_{tn}] \\
&= a_{1j}^{jk} E_{1n} - a_{kn}^{jk} E_{1n}
\end{aligned}
$$

follows. This means that

$$a_{1j}^{jk} = a_{kn}^{jk} \qquad \text{where} \quad 1 < j < k < n \tag{3.3}$$

holds. When $j = 2$ and $k = n - 1$, those are the only two visible parameters.

Suppose now that $k < n - 1$. Choose an arbitrary $k < l < n$. It follows then that

$$
\begin{aligned}
0 &= \varphi([E_{1j}, E_{kl}]) \\
&= [\varphi(E_{1j}), E_{kl}] + [E_{1j}, \varphi(E_{kl})] \\
&= [\sum_{t=j+1}^{n} a_{1j}^{jt} E_{1t}, E_{kl}] + [E_{1j}, \sum_{t=1}^{k-1} -a_{kl}^{tk} E_{tl} + \sum_{t=l+1}^{n} a_{kl}^{lt} E_{kt}] \\
&= a_{1j}^{jk} E_{1l} - a_{kl}^{jk} E_{1l},
\end{aligned}
$$

since φ is a derivation. Therefore,

$$a_{1j}^{jk} = a_{kl}^{jk} \qquad \text{where} \quad 1 < j < k < l < n \tag{3.4}$$

holds. When $j = 2$ and $k < n - 1$, equations (3.3) and (3.4) show that all visible parameters belonging to E_{2k} are the same.

When $2 < j$, choose an arbitrary $1 < i < j$. By the same reasoning, it follows from the definition of a derivation that

$$
\begin{aligned}
0 &= \varphi([E_{ij}, E_{kn}]) \\
&= [\varphi(E_{ij}), E_{kn}] + [E_{ij}, \varphi(E_{kn})] \\
&= [\sum_{t=1}^{i-1} -a_{ij}^{ti} E_{tj} + \sum_{t=j+1}^{n} a_{ij}^{jt} E_{it}, E_{kn}] + [E_{ij}, \sum_{t=1}^{k-1} -a_{kn}^{tk} E_{tn}] \\
&= a_{ij}^{jk} E_{in} - a_{kn}^{jk} E_{in}.
\end{aligned}
$$

This means that

$$a_{ij}^{jk} = a_{kn}^{jk} \qquad \text{where} \quad 1 < i < j < k < n. \tag{3.5}$$

For $j > 2$ and $k = n-1$, the equations (3.3) and (3.5) imply that the visible parameters belonging to $E_{j,n-1}$ are equal.

The only case left is when $2 < j < k < n-1$. Choose now $1 < i < j$ and $k < l < n$. Since φ is an almost-inner derivation, the conditions for a derivation have to be satisfied.

Hence,

$$\begin{aligned} 0 &= \varphi([E_{ij}, E_{kl}]) \\ &= [\varphi(E_{ij}), E_{kl}] + [E_{ij}, \varphi(E_{kl})] \\ &= [\sum_{t=1}^{i-1} -a_{ij}^{ti} E_{tj} + \sum_{t=j+1}^{n} a_{ij}^{jt} E_{it}, E_{kl}] + [E_{ij}, \sum_{t=1}^{k-1} -a_{kl}^{tk} E_{tl} + \sum_{t=l+1}^{n} a_{kl}^{lt} E_{kt}] \\ &= a_{ij}^{jk} E_{il} - a_{kl}^{jk} E_{il} \end{aligned}$$

follows. This means that

$$a_{ij}^{jk} = a_{kl}^{jk} \qquad \text{where} \qquad 1 < i < j < k < l < n. \tag{3.6}$$

Combining the equations (3.4), (3.5) and (3.6), it is clear that $a_{ij}^{jk} = a_{kl}^{jk}$ holds, where $1 \leq i < j < k < l \leq n$. This means that all visible basis vectors belonging to E_{jk} are the same, thus E_{jk} is fixed.

- Let E_{jn} be an arbitrary basis vector with $1 < j < n$. The visible parameters belonging to E_{jn} are a_{ij}^{jn} (with $1 \leq i < j$). When $j = 2$, there is only one visible parameter, namely a_{12}^{2n}. Therefore, E_{2n} is fixed by Remark 2.3.5. Otherwise, choose $1 < i < j$ arbitrarily. The equation

  $$\varphi([E_{1i}, E_{ij}]) = \varphi(E_{1j}) = \sum_{t=j+1}^{n} a_{1j}^{jt} E_{1t}$$

  holds by definition of the Lie brackets. Further,

  $$\begin{aligned} \varphi([E_{1i}, E_{ij}]) &= [\varphi(E_{1i}), E_{ij}] + [E_{1i}, \varphi(E_{ij})] \\ &= [\sum_{t=i+1}^{n} a_{1i}^{it} E_{1t}, E_{ij}] + [E_{1i}, \sum_{t=1}^{i-1} -a_{ij}^{ti} E_{tj} + \sum_{t=j+1}^{n} a_{ij}^{jt} E_{it}] \\ &= \sum_{t=j+1}^{n} a_{ij}^{jt} E_{1t}. \end{aligned}$$

  is satisfied, since φ is a derivation. It follows from these equations that

  $$a_{1j}^{jt} = a_{ij}^{jt} \qquad \text{where} \qquad 1 < i < j < t \leq n.$$

  For $t = n$, this shows that all visible parameters belonging to E_{jn} are the same, thus E_{jn} is fixed.

This finishes the proof, since all basis vectors are fixed. □

For the Lie algebra of all uppertriangular matrices, the same result as before can be obtained. The approach is similar, but it requires more work since there are more non-zero brackets.

3.5.2 Uppertriangular matrices

This subsection is devoted to $\mathfrak{t}_n(K)$, the set of all uppertriangular $(n \times n)$-matrices over the field K. A basis for this vector space is given by

$$\mathcal{B}_n = \{E_{ij} \in K^{n\times n} \mid 1 \le i \le j \le n\}.$$

Let E_{ij} and E_{kl} be two uppertriangular matrices, so $1 \le i \le j \le n$ and $1 \le k \le l \le n$. Then, the Lie bracket

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$

is again uppertriangular. This means that $\mathfrak{t}_n(K)$ is a Lie subalgebra of $\mathbf{gl}(n, K)$ and the dimension is

$$\dim(\mathfrak{t}_n(K)) = \frac{n(n+1)}{2}.$$

Note that the Lie algebra $\mathfrak{n}_n(K)$ is an ideal of $\mathfrak{t}_n(K)$, since the Lie bracket of an uppertriangular matrix with a strictly uppertriangular matrix is again strictly uppertriangular. It is easy to see that $\mathfrak{t}_1(K) \cong K$, so this Lie algebra is abelian (and hence nilpotent and solvable). When $n > 1$, the Lie algebras $\mathfrak{t}_n(K)$ are not nilpotent, since for example $[E_{11}, E_{12}] = E_{12}$. Therefore, $E_{12} \in \mathfrak{t}_n(K)^k$ for all $k \ge 1$. However, the Lie algebras are still solvable for $n > 1$.

Example 3.5.4. *The Lie algebra $\mathfrak{t}_2(K)$ over the field K is three-dimensional and has $\mathcal{B} = \{E_{11}, E_{12}, E_{22}\}$ as basis. The non-zero Lie brackets are*

$$[E_{11}, E_{12}] = E_{12}; \quad \textit{and} \quad [E_{12}, E_{22}] = E_{12}.$$

Define $X_3 := E_{11} + E_{22}$. Then $\mathcal{B}' = \{E_{12}, E_{22}, X_3\}$ is also a basis for $\mathfrak{t}_2(K)$. The Lie brackets are now given by

$$[E_{12}, E_{22}] = E_{12}; \qquad [E_{12}, X_3] = -E_{12} + E_{12} = 0 \quad \text{and} \quad [E_{22}, X_3] = 0.$$

Let $\varphi \in \mathrm{AID}(\mathfrak{t}_2(K))$ be an arbitrary almost-inner derivation. For the basis $\mathcal{B}'$, all basis vectors are fixed due to Remark 2.3.5, since they have at most one visible parameter.

The following proposition describes the dimension of the inner derivations of $\mathfrak{t}_n(K)$.

Proposition 3.5.5. *Let $\mathfrak{t}_n(K)$ be the Lie algebra of all uppertriangular $(n \times n)$-matrices over the field K. It follows that*

$$\dim(\mathit{Inn}(\mathfrak{t}_n(K))) = \dim(\mathfrak{t}_n(K)) - 1 = \frac{n^2+n-2}{2}.$$

Proof. The dimension of $\mathrm{Inn}(\mathfrak{t}_n(K))$ is equal to the dimension of

$$\mathrm{vct}\{\mathrm{ad}(E_{ij}) : \mathfrak{t}_n(K) \to \mathfrak{t}_n(K) \mid 1 \le i \le j \le n\}.$$

Let $X = \sum_{i=1}^{n}\sum_{j=i}^{n} x_{ij}E_{ij}$ be an arbitrary element of $\mathfrak{t}_n(K)$, where $x_{ij} \in K$ for all $1 \le i \le j \le n$. By definition of an inner derivation,

$$\mathrm{ad}(E_{jk})(X) = \sum_{i=1}^{j} -x_{ij}E_{ik} + \sum_{i=k}^{n} x_{ki}E_{ji} \qquad \text{with } 1 \le i < j \le n$$

holds. It is easy to see that all those maps are linear independent. Indeed, $\mathrm{ad}(E_{jk})$ is the only adjoint map of a basis vector which maps E_{kk} to (a non-zero multiple of) E_{jk}. Further, the inner derivations of the basis vectors E_{ii} (with $1 \le i \le n$) are given by

$$\begin{aligned}
\mathrm{ad}(E_{11})(X) &= \sum_{i=2}^{n} x_{1i}E_{1i};\\
\mathrm{ad}(E_{jj})(X) &= \sum_{i=1}^{j-1} -x_{ij}E_{ij} + \sum_{i=j+1}^{n} x_{ji}E_{ji} \qquad \text{where} \qquad 1 < j < n;\\
\mathrm{ad}(E_{nn})(X) &= \sum_{i=1}^{n-1} -x_{in}E_{in}.
\end{aligned}$$

For these maps, it follows that

$$\begin{aligned}
\sum_{j=1}^{n} \mathrm{ad}(E_{jj})(X) &= \sum_{i=2}^{n} x_{1i}E_{1i} + \sum_{j=2}^{n-1}\sum_{i=1}^{j-1} -x_{ij}E_{ij} + \sum_{j=2}^{n-1}\sum_{i=j+1}^{n} x_{ji}E_{ji} + \sum_{i=1}^{n-1} -x_{in}E_{in}\\
&= \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} x_{ji}E_{ji} + \sum_{j=2}^{n}\sum_{i=1}^{j-1} -x_{ij}E_{ij}\\
&= \sum_{i=2}^{n}\sum_{j=1}^{i-1} x_{ji}E_{ji} + \sum_{j=2}^{n}\sum_{i=1}^{j-1} -x_{ij}E_{ij}\\
&= 0.
\end{aligned}$$

In the last two equations, the summands respectively the roles of i and j are changed. This means that the maps $\mathrm{ad}(E_{ii})$ with $1 \le i \le n$ are linearly dependent. Hence,

$$B := \mathrm{vct}\{\mathrm{ad}(E_{ii}) : \mathfrak{t}_n(K) \to \mathfrak{t}_n(K) \mid 1 \le i \le n\}.$$

has dimension $\dim(B) \le n-1$. It is clear that this is in fact an equality. Indeed,consider $C := \{\mathrm{ad}(E_{ii}) : \mathfrak{t}_n(K) \to \mathfrak{t}_n(K) \mid 1 \le i \le n-1\}$. For all $1 \le i \le n-1$, the only derivation of C which maps E_{in} to (a non-zero multiple of) E_{in} is $\mathrm{ad}(E_{ii})$. Therefore, all derivations of C are linearly independent. This completes the proof. □

Next proposition shows that $\mathrm{AID}(\mathfrak{t}_n(K)) = \mathrm{Inn}(\mathfrak{t}_n(K))$ for the Lie algebra $\mathfrak{t}_n(K)$ of all uppertriangular $(n \times n)$-matrices over the field K. The proof is more complicated than the case of the Lie algebra $\mathfrak{n}_n(K)$.

Proposition 3.5.6. *Let $\mathfrak{t}_n(K)$ be the Lie algebra of all uppertriangular $(n \times n)$-matrices over the field K. Then all almost-inner derivations are inner derivations.*

Proof. The statement is satisfied when $n = 1$, since $\mathfrak{t}_1(K)$ is abelian. The result for $\mathfrak{t}_2(K)$ is already shown in Example 3.5.4. Hence, it can be assumed without loss of generality that $n > 2$. Let $\varphi \in \mathrm{AID}(\mathfrak{t}_n(K))$ be an almost-inner derivation of $\mathfrak{t}_n(K)$. Then there exist parameters a_{ij}^{kl} with $1 \le i, j, k, l \le n$ such that

$$\begin{aligned}
\varphi(E_{11}) &= \sum_{t=2}^{n} a_{11}^{1t} E_{1t}; \\
\varphi(E_{jk}) &= \sum_{t=1}^{j} -a_{jk}^{tj} E_{tk} + \sum_{t=k}^{n} a_{jk}^{kt} E_{jt} \qquad \text{with} \quad 1 \le j < k \le n; \\
\varphi(E_{jj}) &= \sum_{t=1}^{j-1} -a_{jj}^{tj} E_{tj} + \sum_{t=j+1}^{n} a_{jj}^{jt} E_{jt} \qquad \text{with} \quad 2 \le j \le n-1; \\
\varphi(E_{nn}) &= \sum_{t=1}^{n-1} -a_{nn}^{tn} E_{tn}.
\end{aligned}$$

In the notation of the parameters, the second index is again written as a superscript. By Lemma 2.3.6, it suffices to prove that all basis vectors are fixed. This is done in different steps.

- Choose E_{1i} arbitrarily with $1 < i < n$. The visible parameters belonging to E_{1i} are a_{11}^{1i} and a_{ij}^{1i} (with $1 < i \le j \le n$).

  For an arbitrary $i < j \le n$, the equations

  $$\begin{aligned}
  0 &= \varphi([E_{11}, E_{ij}]) \\
  &= [\varphi(E_{11}), E_{ij}] + [E_{11}, \varphi(E_{ij})] \\
  &= [\sum_{t=2}^{n} a_{11}^{1t} E_{1t}, E_{ij}] + [E_{11}, \sum_{t=1}^{i} -a_{ij}^{ti} E_{tj} + \sum_{t=j}^{n} a_{ij}^{jt} E_{it}] \\
  &= a_{11}^{1i} E_{1j} - a_{ij}^{1i} E_{1j}
  \end{aligned}$$

  are satisfied. Therefore, $a_{11}^{1i} = a_{ij}^{1i}$ holds, where $1 < i < j \le n$.

  Further,

  $$\begin{aligned}
  0 &= \varphi([E_{11}, E_{ii}]) \\
  &= [\varphi(E_{11}), E_{ii}] + [E_{11}, \varphi(E_{ii})] \\
  &= [\sum_{t=2}^{n} a_{11}^{1t} E_{1t}, E_{ii}] + [E_{11}, \sum_{t=1}^{i-1} -a_{ii}^{ti} E_{ti} + \sum_{t=i+1}^{n} a_{ii}^{it} E_{it}] \\
  &= a_{11}^{1i} E_{1i} - a_{ii}^{1i} E_{1i}
  \end{aligned}$$

  implies that $a_{11}^{1i} = a_{ii}^{1i}$. Those two observations show that all values a_{ij}^{1i} with $1 < i \le j \le n$ are the same and equal to a_{11}^{1i}. Those are exactly the visible parameters belonging to E_{1i}. Since $1 < i < n$ was arbitrary, this means that E_{1i} is fixed for all values $1 < i < n$.

- The visible parameters belonging to E_{1n} are a_{11}^{1n} and a_{nn}^{1n}. By definition of a derivation,

  $$\begin{aligned}
  0 &= \varphi([E_{11}, E_{nn}]) \\
  &= [\varphi(E_{11}), E_{nn}] + [E_{11}, \varphi(E_{nn})] \\
  &= [\sum_{t=2}^{n} a_{11}^{1t} E_{1t}, E_{nn}] + [E_{11}, \sum_{t=1}^{n-1} -a_{nn}^{tn} E_{tn}] \\
  &= a_{11}^{1n} E_{1n} - a_{nn}^{1n} E_{1n}
  \end{aligned}$$

holds. Therefore, $a_{11}^{1n} = a_{nn}^{1n}$ is satisfied, which implies that E_{1n} is fixed.

- Choose E_{jn} arbitrarily with $1 < j < n$. The visible parameters belonging to E_{jn} are a_{ij}^{jn} and a_{nn}^{jn} (with $1 \le i \le j < n$). It will be shown that all these visible parameters are equal. Let $1 \le i < j$ be arbitrary. By definition,

$$\begin{aligned} 0 &= \varphi([E_{ij}, E_{nn}]) \\ &= [\varphi(E_{ij}), E_{nn}] + [E_{ij}, \varphi(E_{nn})] \\ &= [\sum_{t=1}^{i} -a_{ij}^{ti} E_{tj} + \sum_{t=j}^{n} a_{ij}^{jt} E_{it}, E_{nn}] + [E_{ij}, \sum_{t=1}^{n-1} -a_{nn}^{tn} E_{tn}] \\ &= a_{ij}^{jn} E_{in} - a_{nn}^{jn} E_{in} \end{aligned}$$

is fulfilled. This implies that $a_{ij}^{jn} = a_{nn}^{jn}$ holds, for all $1 \le i < j < n$.

Moreover, by the same reasoning,

$$\begin{aligned} 0 &= \varphi([E_{jj}, E_{nn}]) \\ &= [\varphi(E_{jj}), E_{nn}] + [E_{jj}, \varphi(E_{nn})] \\ &= [\sum_{t=1}^{j-1} -a_{jj}^{tj} E_{tj} + \sum_{t=j+1}^{n} a_{jj}^{jt} E_{jt}, E_{nn}] + [E_{jj}, \sum_{t=1}^{n-1} -a_{nn}^{tn} E_{tn}] \\ &= a_{jj}^{jn} E_{jn} - a_{nn}^{jn} E_{jn} \end{aligned}$$

is satisfied. Therefore, $a_{jj}^{jn} = a_{nn}^{jn}$ holds, where $2 \le j < n$. Together, above calculations show that all values a_{ij}^{jn} with $1 \le i \le j < n$ are the same and equal to a_{nn}^{jn}. Those are exactly the visible parameters belonging to E_{jn}. Therefore, E_{jn} is fixed for all values $1 < j < n$.

- When $n = 3$, the only basis vectors E_{ij} with $1 \le i < j \le n$ were treated in the previous cases. Suppose in this case that $n \ge 4$. Choose E_{jk} arbitrarily with $1 < j < k < n$. The visible parameters belonging to E_{jk} are

$$a_{ij}^{jk} \quad \text{and} \quad a_{kl}^{jk} \qquad \text{with} \quad 1 \le i \le j < k \le l \le n.$$

Let $1 \le i < j$ and $k < l \le n$ be arbitrary values. It follows from the definition of a derivation that

$$\begin{aligned} 0 &= \varphi([E_{ij}, E_{kl}]) \\ &= [\varphi(E_{ij}), E_{kl}] + [E_{ij}, \varphi(E_{kl})] \\ &= [\sum_{t=1}^{i} -a_{ij}^{ti} E_{tj} + \sum_{t=j}^{n} a_{ij}^{jt} E_{it}, E_{kl}] + [E_{ij}, \sum_{t=1}^{k} -a_{kl}^{tk} E_{tl} + \sum_{t=l}^{n} a_{kl}^{lt} E_{kt}] \\ &= a_{ij}^{jk} E_{il} - a_{kl}^{jk} E_{il}. \end{aligned}$$

This means that

$$a_{ij}^{jk} = a_{kl}^{jk} \qquad \text{where} \quad 1 \le i < j < k < l \le n. \tag{3.7}$$

The definition of the Lie brackets implies that

$$\varphi([E_{ij}, E_{jj}]) = \varphi(E_{ij}) = \sum_{k=1}^{i} -a_{ij}^{ki} E_{kj} + \sum_{k=j}^{n} a_{ij}^{jk} E_{ik}$$

holds. Further, also

$$\begin{aligned}
\varphi([E_{ij}, E_{jj}]) &= [\varphi(E_{ij}), E_{jj}] + [E_{ij}, \varphi(E_{jj})] \\
&= [\sum_{k=1}^{i} -a_{ij}^{ki} E_{kj} + \sum_{k=j}^{n} a_{ij}^{jk} E_{ik}, E_{jj}] + [E_{ij}, \sum_{k=1}^{j-1} -a_{jj}^{kj} E_{kj} + \sum_{k=j+1}^{n} a_{jj}^{jk} E_{jk}] \\
&= \sum_{k=1}^{i} -a_{ij}^{ki} E_{kj} + a_{ij}^{jj} E_{ij} + \sum_{k=j+1}^{n} a_{jj}^{jk} E_{ik}
\end{aligned}$$

is fulfilled. Combining these equations gives

$$a_{ij}^{jk} = a_{jj}^{jk} \qquad \text{where} \qquad 1 \le i < j < k \le n. \tag{3.8}$$

By definition of the Lie brackets,

$$\varphi([E_{kk}, E_{kl}]) = \varphi(E_{kl}) = \sum_{j=1}^{k} -a_{kl}^{jk} E_{jl} + \sum_{j=l}^{n} a_{kl}^{lj} E_{kj}$$

holds. Since φ is a derivation,

$$\begin{aligned}
\varphi([E_{kk}, E_{kl}]) &= [\varphi(E_{kk}), E_{kl}] + [E_{kk}, \varphi(E_{kl})] \\
&= [\sum_{j=1}^{k-1} -a_{kk}^{jk} E_{jk} + \sum_{j=k+1}^{n} a_{kk}^{kj} E_{kj}, E_{kl}] + [E_{kk}, \sum_{j=1}^{k} -a_{kl}^{jk} E_{jl} + \sum_{j=l}^{n} a_{kl}^{lj} E_{kj}] \\
&= \sum_{j=1}^{k-1} -a_{kk}^{jk} E_{jl} - a_{kl}^{kk} E_{kl} + \sum_{j=l}^{n} a_{kl}^{lj} E_{kj}
\end{aligned}$$

is satisfied. It follows from these equations that

$$a_{kl}^{jk} = a_{kk}^{jk} \qquad \text{where} \qquad 1 \le j < k < l \le n. \tag{3.9}$$

Combining the equations (3.7), (3.8) and (3.9), it is clear that $a_{ij}^{jk} = a_{kl}^{jk}$ holds, where $1 \le i \le j < k \le l \le n$. This means that all visible parameters belonging to E_{jk} are the same. Since $1 < j < k < n$ were arbitrary, E_{jk} is fixed for all $1 < j < k < n$.

- For the basis vectors E_{jj} with $1 \le j \le n$, there is another approach needed. The visible parameters belonging to E_{jj} are a_{ij}^{jj} for all $1 \le i < j$ when $j > 1$ and a_{jk}^{jj} for all $j < k \le n$ when $j < n$. The basis vectors E_{ij} are fixed, where $1 \le i < j \le n$. Hence, there exist $\binom{n}{2}$ values $a_{ij} \in K$ such that $a_{ij} = a_{kl}^{ij}$ for all $1 \le i < j \le n$ and all $1 \le k, l \le n$. Using this, it's possible to reformulate the beginning of the proof.

Hence, for all $1 \leq j \leq k \leq n$, the image $\varphi(E_{jk})$ is given by

$$\begin{aligned}
\varphi(E_{11}) &= \sum_{t=2}^{n} a_{1t}E_{1t}; \\
\varphi(E_{jk}) &= (a_{jk}^{kk} - a_{jk}^{jj})E_{jk} + \sum_{t=1}^{j-1} -a_{tj}E_{tk} + \sum_{t=k+1}^{n} a_{kt}E_{jt} \qquad \text{with} \quad 1 \leq j < k \leq n. \\
\varphi(E_{jj}) &= \sum_{t=1}^{j-1} -a_{tj}E_{tj} + \sum_{t=j+1}^{n} a_{jt}E_{jt} \qquad \text{with} \quad 2 \leq j \leq n-1; \\
\varphi(E_{nn}) &= \sum_{t=1}^{n-1} -a_{tn}E_{tn},
\end{aligned}$$

where $\varphi \in \mathrm{AID}(\mathfrak{t}_n(K))$ is an almost-inner derivation. Define now for all $1 \leq j < k \leq n$ the value b_{jk} as

$$b_{jk} := a_{jk}^{kk} - a_{jk}^{jj};$$

there are thus $\binom{n}{2}$ such values. Choose $1 \leq i < j < k \leq n$ arbitrarily. By definition,

$$\varphi([E_{ij}, E_{jk}]) = \varphi(E_{ik}) = \sum_{t=1}^{i} -a_{ik}^{ti}E_{tk} + \sum_{t=k}^{n} a_{ik}^{kt}E_{it}$$

holds. Since φ is a derivation,

$$\begin{aligned}
\varphi([E_{ij}, E_{jk}]) &= [\varphi(E_{ij}), E_{jk}] + [E_{ij}, \varphi(E_{jk})] \\
&= [\sum_{t=1}^{i} -a_{ij}^{ti}E_{tj} + \sum_{t=j}^{n} a_{ij}^{jt}E_{it}, E_{jk}] + [E_{ij}, \sum_{t=1}^{j} -a_{jk}^{tj}E_{tk} + \sum_{t=k}^{n} a_{jk}^{kt}E_{jt}] \\
&= \sum_{t=1}^{i} -a_{ij}^{ti}E_{tk} + a_{ij}^{jj}E_{ik} - a_{jk}^{jj}E_{ik} + \sum_{t=k}^{n} a_{jk}^{kt}E_{it}
\end{aligned}$$

is satisfied. In particular, the coefficients of E_{ik} have to be equal in both equations. Hence, it follows that

$$-a_{ik}^{ii} + a_{ik}^{kk} = -a_{ij}^{ii} + a_{ij}^{jj} - a_{jk}^{jj} + a_{jk}^{kk} \qquad \text{where} \quad 1 \leq i < j < k \leq n. \tag{3.10}$$

Let $1 < i \leq n$ and consider $a_{1i}^{ii} - a_{1i}^{11}$ where $1 < i \leq n$. It follows from equation (3.10) that

$$a_{jk}^{kk} - a_{jk}^{jj} = (a_{1k}^{kk} - a_{1k}^{11}) - (a_{1j}^{jj} - a_{1j}^{11}) \qquad \text{where} \quad 1 < j < k \leq n.$$

Hence, the values for b_{jk}, with $1 < j < k \leq n$) are linear combinations of the $n-1$ values b_{1i} where $1 < i \leq n$.

From the above observations, it is clear that an upper bound for the dimension of $\mathrm{AID}(\mathfrak{t}_n(K))$ is given by

$$\dim(\mathrm{AID}(\mathfrak{t}_n(K))) \leq \binom{n}{2} + (n-1) = \frac{n^2 - n + 2(n-1)}{2} = \frac{n^2 + n - 2}{2}.$$

The claim now follows from Proposition 3.5.5. $\square$

In total, five different classes were studied: the low-dimensional complex Lie algebras, the metabelian filiform Lie algebras, Lie algebras defined by graphs, free nilpotent Lie algebras with nilindex two or three and (strictly) uppertriangular matrices. For most of the previous classes, the only almost-inner derivations are the inner ones. Only for the metabelian filiform Lie algebras which are not standard graded, there is a different result. Even in that case, the dimension of $\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is small (namely equal to one). This may give the impression that an almost-inner derivation which is not inner is a very rare phenomenom. Nevertheless, for general Lie algebras, this is not the case. Illustrations of this fact are provided in Examples 2.3.2, 3.2.7 and 3.3.4, were the first and last are more general two-step nilpotent Lie algebras. The second is an instance of a three-step solvable filiform. However, it is more difficult to classify such Lie algebras and prove a general result, since there is, contrary to the treated classes, no general way to write the basis and Lie brackets.

Conclusion

The aim of this thesis is the study of almost-inner derivations of Lie algebras. Those derivations arise in a geometric context by the construction of isospectral and non-isometric manifolds, but are here treated algebraically.

The first chapter contains an introduction to Lie algebras. The geometrical definition is given with the aid of Lie goups. Then, some basic concepts about Lie algebras are introduced.

The second chapter is devoted to almost-inner derivations. First, the set of all derivations $\mathrm{Der}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is introduced, as well as the subsets

$$\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g}).$$

Then, some basic notions concerning spectral geometry are explained to understand the geometric importance of the almost-inner derivations. Further, the notions of the parameters and a fixed vector are defined. There is a procedure to calculate $\mathrm{AID}(\mathfrak{g})$, which consists of different steps. There has to be an easy basis for $\mathfrak{g}$ and an efficient way to construct the non-vanishing Lie brackets. Further, the matrix representation containing the parameters with respect to this basis can be worked out. The conditions due to the definition of a derivation and an almost-inner derivation reveal the underlying relations on the parameters.

Some properties concerning almost-inner derivations were elaborated, in particular the two Lemmas 2.3.7 and 2.3.8. They ease the check on the conditions due to the definition.

In the third chapter, this procedure is used for some types of Lie algebras. Those classes are the low-dimensional Lie algebras, the filiform Lie algebras, the two-step nilpotent Lie algebras defined by graphs, the free nilpotent Lie algebras and the (strictly) uppertriangular matrices. There is a focus on nilpotent Lie algebras, because those are, for this notion, geometrically of most importance.

For certain classes, the only almost-inner derivations are inner. This is the case for the low-dimensional complex Lie algebras, the standard graded filiform Lie algebras, the two-step nilpotent Lie algebras defined by graphs and the (strictly) uppertriangular matrices. The same result holds for free nilpotent Lie algebras with nilindex two and three. It is not sure if this is also true for higher nilindices. The difficulty in that case is the description of the Lie brackets.

For metabelian filiform Lie algebras (which are not standard graded), the dimension of $\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is equal to one. However, this is not true for more general filiform Lie algebras.

The appendix contains some computer programs implemented in Matlab. The first part checks whether or not given structure constants define a Lie algebra. The second section is devoted to some algorithms concerning the computation of a basis for $\mathrm{Der}(\mathfrak{g})$.

Appendix A

Algorithms

In the appendix, some computer algorithms implemented in Matlab are given. Those algorithms ease the computations. The first part is devoted to the check whether or not the Jacobi identity is satisfied for a given set of structure constants. The second section concerns algorithms which allow to compute a basis and the dimension of $\mathrm{Der}(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. Note that the algorithms are programmed to work only for $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. However, in some cases, it is not very difficult to modify it to be valid for more general fields. For example, the first two algorithms can be adjusted for the field $\mathbb{F}_p$ with p elements, where p is a prime.

A.1 Check of the Jacobi identity

An n-dimensional Lie algebra $\mathfrak{g}$ is represented by a $(n \times n \times n)$-matrix C where the entry $C(i, j, k)$ stands for the structure constant c_{ij}^k. For the convenience, only the non-zero entries $C(i, j, k)$ are implemented when $i < j$. The first algorithm, `complete(C)` fills up the matrix `C` so that the skew-symmetry is respected.

Code A.1: complete.m

```
function [C] = complete(C)
%   Input
%   'C' is an (n x n x n)-matrix with all non-zero
%   structure constants (for which only C(i,j,k)
%   is implemented when i<j).
%
%   Output
%   'C' is an (n x n x n)-matrix with all non-zero structure
%   constants.

    n = size(C,1);
    for i=1:n
        for j=i:n
            for k=1:n
                C(j,i,k)= -C(i,j,k);
            end
```

```
        end
    end
end
```

The algorithm `checkJacobi(C)` verifies whether or not the Jacobi identity is satisfied for the given structure constants, represented by the $(n \times n \times n)$-matric `C`. Therefore, the equations (1.1) are checked.

Code A.2: checkJacobi.m

```
function [B] = checkJacobi(C)
%   Input
%   'C' is a (n x n x n)-matrix with all non-zero structure
%   constants (for which only C(i,j,k) is implemented when i<j).
%
%   Output
%   'B' is true or false, depending whether or not the structure
%   constants define a Lie algebra.

    C = complete(C);
    n = size(C,1);
    D = zeros(1,n);
    B = true;
    for i=1:n
        for j=1:n
            for k=1:n
                for l=1:n
                    for m=1:n
                        D(m) = C(j,k,m)*C(i,m,l)+C(k,i,m)*C(j,m,l)+C(i,j,m)*C(k,m,l);
                    end
                    if ~isequal(sum(D),0)
                        B = false;
                    end
                end
            end
        end
    end
    if ~B
        error('The given structure constants do not define a Lie algebra!')
    end
end
```

A.2 Computation of a basis for Der($\mathfrak{g}$)

Let $\mathfrak{g}$ be an n-dimensional Lie algebra. With the next two algorithms, it is possible to compute the dimension of Der($\mathfrak{g}$). The algorithm `derivations(C)` implements the equations (2.3) for an arbitrary derivation of $\mathfrak{g}$, represented by the $(n \times n \times n)$-matrix `C`. The output is an $(n^2 \times n^2)$-system where all relations on the matrix entries are listed. Column $(i-1)n+j$ in the system represents the information for d_{ij}. This algorithm is based on [1].

Code A.3: derivations.m

```
function [D] = derivations(C)
%   Input
%   'C' is an (n x n x n)-matrix with all non-zero structure
%   constants (for which only C(i,j,k) is implemented when i<j).
%
%   Output
%   'D' is an (n^2 x n^2)-system, for which every row represents
%   a relation on the matrix entries of an arbitrary derivation.
%   The ((i-1)n+j)-th column represents the matrix entry d_{ij}.

    checkJacobi(C);
    C = complete(C);
    n = size(C,1);
    D = zeros(n^2);
    L = zeros(1,n^2);
    r1 = zeros(1,n^2);
    r2 = zeros(1,n^2);
    t = 1;
    for i = 1:n-1
       for j = i+1:n
           for k = 1:n
               for l = 1:n
                   L((l-1)*n+k) = C(i,j,l);
                   r1((i-1)*n+l) = C(l,j,k);
                   r2((j-1)*n+l) = C(i,l,k);
               end
               for m = 1:n^2
                   D(t,m) = L(m) - r1(m) - r2(m);
               end
               L = zeros(1,n^2);
               r1 = zeros(1,n^2);
               r2 = zeros(1,n^2);
               t = t+1;
           end
       end
    end
end
```

Let $\mathfrak{g}$ be n-dimensional Lie algebra represented by the $(n \times n \times n)$-matrix `C`. The algorithm `dimensionDerivations(A)` computes the dimension of $\mathrm{Der}(\mathfrak{g})$. As input, the output of `derivations(C)` can be used.

Code A.4: dimensionDerivations.m

```
function [d] = dimensionDerivations(A)
%   Input
%   'A' is a matrix with n^2 columns, for which every row contains
%   a relation on the matrix entries.
%   The ((i-1)n+j)-th column represents the matrix entry d_{ij}.
%
%   Output
%   'd' is the dimension of the space of all derivations.

    [R,jb] = rref(derivations(A));
    d = size(R,2) - size(jb,2);
end
```

For low-dimensional Lie algebras, the algorithm `makeBasisDerivations(C)` is very useful. The output is an $(n \times n)$-matrix, which is a matrix representation of an arbitrary derivation for the Lie algebra $\mathfrak{g}$, represented by the $(n \times n \times n)$-matrix `C`. Moreover, also the dimension of $\mathrm{Der}(\mathfrak{g})$ is printed. When n is large, it takes too long to compute the right answer.

Code A.5: makeBasisDerivations.m

```
function [D,m] = makeBasisDerivations(C)
%   Input
%   'C' is an (n x n x n)-matrix with all non-zero structure
%   constants (for which only C(i,j,k) is implemented when i<j).
%
%   Output
%   'D' is an (n x n)-matrix, which gives a matrix representation
%   for an arbitrary derivation.
%   'm' is the dimension of the space of all derivations.

    A = derivations(C);
    [D,m] = makeBasis(A);
end
```

In this algorithm, `makeBasis(A)` is used. The input is a matrix with n^2 columns for which every row represents a relation between the matrix entries. This algorithm visualises how an arbitrary $(n \times n)$-matrix which satisfy certain conditions looks like. Here, the values for d_{ij} can be chosen arbitrarily. The conditions are listed in a matrix with n^2 columns and implemented as input. The $((i-1)n+j)$-th column corresponds with the matrix entry d_{ij}.

Code A.6: makeBasis.m

```
function [D,m] = makeBasis(A)
%   Input
%   'A' is a matrix with n^2 columns, for which every row contains
```

```
%   a relation on the matrix entries.
%   The ((i-1)n+j)-th column represents the matrix entry d_{ij}.
%
%   Output
%   'D' gives a matrix representation for an arbitrary matrix which
%   satisfies all conditions listed in 'A'.
%   'm' is the dimension of the space of matrices which satisfy all
%   conditions listed in 'A'.

    [R,jb] = rref(A);
    m = size(R,2) - size(jb,2);
    d = sqrt(size(A,2));
    D = sym('d',[d,d]);
    V = sym('v',[1,d^2]);
    T = 0;
    k = 1;
    for i = 1:size(R,1)
        for j=k:d^2
            g = R(size(R,1)-i+1,d^2-j+1);
            if g ~= 0 && T==0;
                for p=1:d^2
                    V(p)= R(size(R,1)-i+1,p)*D(ceil(p/d),isequal(0,mod(p,d))*d+mod(p,d));
                end
                V(d^2-j+1)=0;
                D(ceil((d^2-j+1)/d),isequal(0,mod(d^2-j+1,d))*d+mod(d^2-j+1,d)) = -1/g*sum(V);
                T = 1; k = j;
            end
        end
        T=0;
    end
end
```

Note that this is only useful for low-dimensional Lie algebras, since this algorithm requires a long computational time.

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