Universität Wien
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Homological Algebra
— Exercises —

Exercises SS 2022

Exercise 1. Find an example of a commutative ring R which is isomorphic to $R \times R$.

Exercise 2. Denote by $\mathbb{Z}/n$ the $\mathbb{Z}$-module of the integers modulo n. For two integers m and n, let $d = \gcd(m, n)$ be the greatest common divisor of m and n. Show that

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/d.$$

Exercise 3. Let $M_i, i \in I$ and N be R-modules. Show that in general

$$\mathrm{Hom_R}\left(\prod_{\mathrm{i\in I}} \mathrm{M_i}, \mathrm{N}\right) \cong \prod_{\mathrm{i\in I}} \mathrm{Hom_R}(\mathrm{M_i}, \mathrm{N}),$$

need not be true.

Exercise 4. Let $M = \prod_{i\geq 0} \mathbb{Z}$ be the $\mathbb{Z}$-module of integral sequences, and $e^n = (0, \dots, 0, 1, 0, \dots)$ be the sequence with all entries equal to zero, except for the n-th entry, which is 1. Let $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$. Show that $f(e^n) = 0$ for almost all $n \in \mathbb{N}$, and that if $f(e^n) = 0$ for all $n \geq 0$, then $f = 0$.

Exercise 5. Let $M = \prod_{i\in\mathbb{N}} \mathbb{Z}$. Show that the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ is free and is isomorphic to $N = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$.

Exercise 6. Use Exercise 5 to show that the $\mathbb{Z}$-module $M = \prod_{i\in\mathbb{N}} \mathbb{Z}$ is not free and conclude that the infinite product of free modules need not be free.

Exercise 7. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is flat.

Exercise 8. Use the definition of a projective module from the lecture to show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.

Exercise 9. Show that the $\mathbb{Z}$-modules $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}/2$ are both not flat.

Exercise 10. Let R be a PID and M be an R-module. Show that M is injective if and only if M is divisible. Conclude that the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Exercise 11. Let $R = K[x, y]$. For this integral domain R, find a divisible R-module M, which is not injective.

Exercise 12. Show that the direct product cannot be a coproduct in the category of groups.

Exercise 13. Show that the category of cyclic groups does not have all binary products, and hence is not an abelian category.

Exercise 14. Let p be a prime and $\mathbb{Z}_{(p)} = \{\frac{n}{m} \in \mathbb{Q} \mid p \nmid m\}$ the ring of p-local numbers. Let $\mathbb{Z}/p^\infty = \cup_{i\geq 0}\mathbb{Z}/p^i$ with the uniquely given embeddings $\mathbb{Z}/p_i \hookrightarrow \mathbb{Z}/p^{i+1}$. Show that there is a short exact sequence

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty \to 0$$

which does not split.

Exercise 15. Define the ring of p-adic integers by $\mathbb{Z}_p = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p^\infty)$. Show that there exists a unique ring monomorphism $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Z}_p$, and that $\mathbb{Z}_p$ is flat over $\mathbb{Z}_{(p)}$.

Exercise 16. Let M be a $\mathbb{Z}$-module, and $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ its dual $\mathbb{Z}$-module. Show that M need not be isomorphic with its bi-dual $(M^\vee)^\vee$. For which class of $\mathbb{Z}$-modules do we have $M \cong (M^\vee)^\vee$?

Exercise 17. Give an example of an Epi in the category of rings which is not a surjective map.

Exercise 18. Give an example of a Monic in the category of divisible abelian groups which is not an injective map.

Exercise 19. Show that the category of divisible abelian groups is additive, but not abelian.

Exercise 20. Show that the category of torsion-free abelian groups is additive, but not abelian.

Exercise 21. Show that the category of finitely generated abelian groups does not have non-trivial injective objects.

Exercise 22. Show that the category of R-modules has enough projective and injective objects.

Exercise 23. Verify that the functor $\mathrm{Hom}(-, \mathbb{Z})$ is not exact by applying it to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Exercise 24. Denote by $\mathbb{Z}_{(p)} = \{\frac{n}{q} \in \mathbb{Q} \mid p \nmid q\}$ the ring of p-local numbers, by $\mathbb{Z}_p$ the ring of p-adic integers, and by $\mathbb{Q}_p$ the field of p-adic numbers. Apply the Snake Lemma to show that

$$\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}.$$

Exercise 25. Suppose we are given a commutative diagram of R-module homomorphisms

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "i"] \arrow[d] & B_2 \arrow[r, "p"] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which all colums are exact, and $p \circ i = 0$. Use the Snake Lemma to show that if any two of the rows are exact, then so is the third.

Exercise 26. Show that

$$\mathrm{Tor}_k^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/gcd(n,m) & \text{for } k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 27. Let A be a torsion abelian group. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

Exercise 28. Let A be a finite abelian group. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}/m) \cong \frac{\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})}{m\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})}.$$

Exercise 29. Let A and B be a finite abelian groups. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, B) \cong A \otimes_{\mathbb{Z}} B.$$

Exercise 30. Let $R = \mathbb{Z}[t]/(t^n - 1)$. Show that

$$\mathrm{Tor}_k^R(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \mathbb{Z}/n & \text{for } k \text{ odd}, \\ 0 & \text{for } k > 0 \text{ even.} \end{cases}$$

Exercise 31. Show that

$$\mathrm{Ext}^k_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/gcd(n,m) & \text{for } k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 32. Let G be the trivial group, and A be a G-module. Compute the cohomology $H^n(G, A)$.

Exercise 33. Let L/K be a finite Galois extension. Show that

$$H^1(\mathrm{Gal}(L/K), L^\times) = 0.$$

Exercise 34. Show that the cohomology of the cyclic group C_n with trivial coefficients is given by

$$H^k(C_n, \mathbb{Z}) = \mathrm{Ext}^k_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ 0 & \text{for } k \text{ odd}, \\ \mathbb{Z}/n & \text{for } k > 0 \text{ even.} \end{cases}$$

Exercise 35. Show that the group $SL_2(\mathbb{Z})$ is generated by the elements

$$\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

of order 4 respectively order 6.

Exercise 36. The group $SL_2(\mathbb{Z})$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by Moebius transformations. Let $\mathcal{E}$ be the arc of the unit circle between i and $\omega = \frac{1+\sqrt{-3}}{2}$. Show that the stabilizers for this action at i, ω and $\mathcal{E}$ are given as follows:

$$\begin{aligned}
\mathrm{Stab}_{SL_2(\mathbb{Z})}(i) &= \langle \alpha \rangle = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \\
\mathrm{Stab}_{SL_2(\mathbb{Z})}(\omega) &= \langle \beta \rangle = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}, \\
\mathrm{Stab}_{SL_2(\mathbb{Z})}(\mathcal{E}) &= \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
\end{aligned}$$

Exercise 37. Show that the group $SL_2(\mathbb{Z})$ is isomorphic to a free product of C_4 and C_6 with the cyclic group C_2 amalgamated:

$$SL_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6.$$

Exercise 38. Show that the cohomology of $SL_2(\mathbb{Z})$ with trivial coefficients is given by

$$H^k(SL_2(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ 0 & \text{for } k \text{ odd}, \\ \mathbb{Z}/12 & \text{for } k > 0 \text{ even.} \end{cases}$$

Exercise 39. Let

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\} \subset SL_3(\mathbb{Z})$$

be the integral Heisenberg group. Determine the cohomology groups $H^n(H, \mathbb{Z})$ with trivial coefficients for all $n \geq 0$.

Due: Wednesday, June 29, 2022