Universität Wien
Prof. Dr. D. Burde
WS 2020/2021

Computational Commutative Algebra
— Exercises —

Exercise 1. Find an integer solution of the equation

$$x^2 + y^2 = z^2$$

with $x = 555555$ and $z - x < 100000$.

Exercise 2. Find a solution of the matrix equation

$$X^4 + Y^4 = Z^4$$

in $M_2(\mathbb{Z})$, where each of X, Y, Z has at most two zero entries.

Exercise 3. Determine all positive integers n such that $\mathbb{Z}/n\mathbb{Z}$ is a PID. Describe the ideals of the ring $\mathbb{Z}/n\mathbb{Z}$ for all integers n.

Exercise 4. Let ζ_3 be a primitive third root of unity. Show that the ring $\mathbb{Z}[\zeta_3]$ is a PID.

Exercise 5. Determine all units of the ring $\mathbb{Z}[\zeta_3]$ and show that $E(\mathbb{Z}[\zeta_3]) \cong C_6$.

Exercise 6. Let I and J be coprime ideals in a commutative ring R with unit. Show that the ideals I^m and J^n are coprime for all $m, n \in \mathbb{N}$. Is it true that $I^m J^n = I^m \cap J^n$?

Exercise 7 - extra. Let $\theta = \frac{1+\sqrt{-19}}{2}$. Show that the ring $\mathbb{Z}[\theta]$ is not a Euclidean domain.

Exercise 8. Compute the radical of all ideals in the ring $\mathbb{Z}/n\mathbb{Z}$ for every $n \geq 1$. In particular, compute the nilradical of $\mathbb{Z}/n\mathbb{Z}$.

Exercise 9. Let $R = \mathbb{Z}/6\mathbb{Z}$ and $S = \{\overline{1}, \overline{2}, \overline{4}\}$. Show that the ring of fractions $S^{-1}R$ is well-defined and is isomorphic to the ring $\mathbb{Z}/3\mathbb{Z}$.

Exercise 10. Show that every subring R (containing 1) of $\mathbb{Q}$ is Noetherian. Is every subring of $\mathbb{Z}[X]$ also Noetherian?

Exercise 11. Let A, B be rings (always commutative with 1) and $f\colon A \to B$ be a ring homomorphism. If J is an ideal of B, then the preimage $f^{-1}(J) = J^c$ is an ideal in A, called the *contraction* of J. If I is an ideal in A, the ideal I^e of B generated by $f(I)$ is called the *extension* of I. Show that the contraction of a prime ideal is always a prime ideal, while the extension of a prime ideal need not be a prime ideal.

Exercise 12. Let K be a field and I an ideal in $K[x, y, z]$ given by $I = (xy, x - yz)$. Show that

$$I = (x, z) \cap (y^2, x - yz)$$

is a primary decomposition of I.

Exercise 13. An affine algebraic set $X \subseteq \mathbb{A}^n$ is called *irreducible* if $X \neq \emptyset$ and X cannot expressed as $X = X_1 \cup X_2$ with X_1, X_2 affine algebraic sets different from X. Show that X is irreducible if and only if $I(X)$ is a prime ideal.

Exercise 14 - extra. Let K be an infinite field. Show that the irreducible algebraic sets in $\mathbb{A}^2$ are given by by $\mathbb{A}^2$ itself, any singleton $\{(a, b)\}$ for some $a, b \in K$, or by a set $V(f)$, where $f \in K[x, y]$ is an irreducible polynomial such that $V(f)$ is infinite.

Exercise 15. Let $V \subset \mathbb{A}^m$ and $W \in \mathbb{A}^m$ be two affine algebraic sets. Prove that their product set $V \times W \subset \mathbb{A}^{m+n}$ is an affine algebraic set, too.

Exercise 16. Let $J = (x^2y^3, xy^4) \subseteq K[x, y]$. Show that $\sqrt{J} = (xy)$ and determine the ideals $I(V(J))$ and $I(V(\sqrt{J}))$.

Exercise 17. Let X, Y be two affine algebraic sets in $\mathbb{A}^n$ over an algebraically closed field K. Show that we have

$$I(X \cup Y) = I(X) \cap I(Y),$$
$$I(X \cap Y) = \sqrt{I(X) + I(Y)}.$$

Show that $I(X \cap Y) = I(X_1) + I(X_2)$ does not hold in general.

Exercise 18. Consider the polynomial ring $\mathbb{Q}[x, y]$ together with the lexicographic order and $y \prec x$. Let $f = x^5 + y^5$ and $f_1 = x^3 + y^2$, $f_2 = y^2 + 1$. Use the multivariate division algorithm to find (unique) polynomials $q_1, q_2, r \in \mathbb{Q}[x, y]$ such that $f = q_1 f_1 + q_2 f_2 + r$.

Exercise 19. Show that a monomial ideal I in $K[x_1, \ldots, x_n]$ is prime if and only if it is generated by some of the variables in $\{x_1, \ldots, x_n\}$.

Exercise 20. Let K be a field of characteristic zero. Find all solutions over K of the polynomial equations

$$\begin{aligned} x^2y + 4y^2 - 17 &= 0 \\ 2xy - 3y^3 + 8 &= 0 \\ xy^2 - 5xy + 1 &= 0. \end{aligned}$$

Test your answer by computing a Groebner basis for the ideal generated by these polynomials in $K[x, y]$ with a computer algebra system.

Exercise 21 - extra. Let K be a field of characteristic zero. Using resultants of polynomials find all solutions over K of the polynomial equations

$$\begin{aligned} x^2 + y^2 + z^2 - 6 &= 0 \\ x^3 + y^3 + z^3 - 3xyz + 4 &= 0 \\ xy + xz + yz + 3 &= 0. \end{aligned}$$

Exercise 22. Which of the following subsets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ of $\mathbb{Q}[x, y, z]$ are a Groebner basis for the ideal generated by the polynomials, with the lexicographic order?

$$\begin{aligned} \mathcal{G}_1 &= \{x + y, y^2 - 1\} \text{ for } x \prec y, \\ \mathcal{G}_2 &= \{x^2 + y^2 - 1, xy - 1, x + x^3 - y\} \text{ for } x \prec y, \\ \mathcal{G}_3 &= \{xyz - 1, x - y, y^2z - 1\} \text{ for } x \prec y \prec z. \end{aligned}$$

Which sets are minimal or even reduced?

Exercise 23. Find polynomials $f, g, h \in \mathbb{Q}[x, y, z]$ such that the system of polynomial equations given by $f = g = h = 0$ has exactly the following 5 solutions

$$(0, 0, 0), (1, 1, 1), (-1, 1, -1), (1, -1, 2), (1, 1, -2).$$

Exercise 24. Use the integer solutions to the Pell equation $X^2 - 2Y^2 = 1$ with the units of the ring $\mathbb{Z}[\sqrt{2}]$ to show that there are infinitely many monic quadratic polynomials $f, g, h \in \mathbb{Z}[x]$ such that

$$\begin{pmatrix} x^2 - 2 & f(x) \\ g(x) & h(x) \end{pmatrix} \in SL_2(\mathbb{Z}[x]).$$

Write down the first three matrices in $SL_2(\mathbb{Z}[x])$ of your construction corresponding to the positive integer solutions $(X, Y) = (3, 2), (17, 12), (99, 70)$ of Pell's equation.

Exercise 25. Let R be an integral domain and I be a nonzero ideal. Let K be the quotient field of R. Show that $I \otimes_R K = K$.

Exercise 26. Let I and J be ideals in a ring R (always commutative with 1). Show that there is a unique R-module isomorphism

$$R/I \otimes_R R/J \cong R/(I + J),$$

where $\overline{x} \otimes \overline{y} \mapsto \overline{xy}$.

Exercise 27. Let $\varphi\colon M \to M$ be a surjective R-module homomorphism. Assume that M is a Noetherian R-module. Show that φ is an R-module isomorphism.

Exercise 28 - extra. Let R be a ring such that every localization R_P at a prime ideal P in R is Noetherian. Prove or disprove that R is Noetherian.

Exercise 29. Decide for each of the following rings R whether or not it is integrally closed and give a proof for it.

$$\mathbb{Z}[\sqrt{-5}],\ \mathbb{Z}[\sqrt{5}],\ \mathbb{Z}[\sqrt{2},\sqrt{3}],\ K[x,y]/(x^2-y^3) \cong K[t^2,t^3].$$

Exercise 30. Let K be a field. Determine the integral closure of the rings $K[t^2,t^3]$ and $K[t^3-t, t^2-t]$ in $K(t)$.

Exercise 31. A ring extension $A \subseteq B$ is called *finite* if B is finitely generated as an A-module. Find an example of an infinite integral ring extension.

Exercise 32. Show that the *Lying Over Theorem* and the *Going Up Theorem* don't hold for the ring extension $\mathbb{Z} \subset \mathbb{Q}$. Furthermore give an example for an integral ring extension $A \subset B = K[x,y]$, where the *Going Down Theorem* fails.

Exercise 33. Let d be a squarefree integer. Show that the ring $\mathbb{Z}[\sqrt{d}]$ has Krull dimension 1. However, such a ring is not a PID in general. Show that all rings $\mathbb{Z}[\sqrt{d}]$ for squarefree $d \leq -3$ are not UFD's.

Exercise 34. Which of the following rings is a DVR?

$$\mathbb{Z}_{(p)},\ \mathbb{Z}_p,\ \mathbb{C}[[x]],\ \mathbb{C}[[x,y]].$$

Here $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Exercise 35- extra. Let R be a Dedekind domain and $S \subset R$ be a multiplicatively closed subset of R. Show that $S^{-1}R$ is a Dedekind domain if and only if there is a nonzero prime ideal P in R with $P \cap S = \emptyset$.

Exercise 36. Find a ring R which is a Noetherian integral domain and has Krull dimension one, but which is not a Dedekind ring.

Exercise 37. Find a ring R which is a a Noetherian integral domain and is integrally closed, but which is not a Dedekind ring.

Exercise 38. Find a ring R which is an integrally closed domain of Krull dimension one, but which is not a Dedekind ring.

Due: January 31, 2021