Midterm exam

1. Consider a symmetric auction with N bidders and i.i.d. uniformly distributed values over $[0, 1]$.
   (a) Compute the equilibrium bidding strategy in the first-price, sealed-bid (FPSB) auction. What would be the equilibrium bid of a bidder with value x if $N = 2$? And if $N = 3$?
   (b) Compute the expected payment of a bidder with value x as well as the expected revenue to the seller in the FPSB auction with N bidders.
   (c) Consider instead the second-price, sealed-bid (SPSB) auction. If all bidders bid their values, who wins the object and at what price? What is the expected payment of a bidder with value x in a equilibrium? What is the expected revenue to the seller?
   (d) Suppose bidders are uncertain about the exact number of participants in the auction, but they all believe that the auction may have either 2 or 3 participants with the same probability. They do not expect the auction to have more than $N = 3$ participants.
      i. If bidders bid their values, what is the probability of winning the auction with value x if there are $n = 2$ participants? And if there are $n = 3$ participants? What is the total probability of winning the auction?
      ii. What is the expected payment to a bidder with value x in the SPSB auction now?
      iii. Compute the optimal bidding strategy in the corresponding FPSB auction.
2. Consider a first-price, sealed-bid auction for a single unit of a good with two different bidders. Suppose that bidder 1's value, X_1, and bidder 2's value, X_2, are respectively distributed according to the functions

$$F_1(x) = \frac{1}{4}(x-1)^2 \qquad \text{in the interval } [1,3]$$
$$F_2(x) = \exp\left\{\frac{2}{3}x - 2\right\} \qquad \text{in the interval } [0,3].$$

   (a) Compare the two distribution functions in a single diagram with values in $[0, 3]$. Does any of the two distributions dominate stochastically?
   (b) Write down the expected payoff to each of the bidders given that the opponent bids according to some strictly increasing and differentiable bidding strategy; that is, give expressions for $\Pi_i(x, b)$, the expected payoff to bidder $i = 1, 2$ when her value is x and her bid equals b, given that bidder $j \neq i$ follows a strictly increasing and differentiable strategy, $\beta_j(x_j)$.
   (c) Consider the bidding strategies

$$\beta_1(x) = x - 1 \qquad \beta_2(x) = \frac{2}{3}x.$$

      i. Show that these strategies constitute an equilibrium. *Hint:* For each player i, substitute $\beta_j^{-1}(b)$ and $F_j(x)$ in the expression you obtained for $\Pi_i(x, b)$ and check that $\beta_i(x)$ satisfies the f.o.c. for the maximization of $\Pi_i(x, b)$ with respect to b.
      ii. Compare both strategies in a diagram. How are they related to the distribution functions? Is the equilibrium efficient? Explain why or give a counterexample.
      iii. Show that the distribution of equilibrium prices is given by

$$L(p) = \frac{1}{4}p^2 \cdot \exp(p-2).$$

      Hint: Compute $L(p) = \text{Prob}(R \leq p)$, where R denotes revenue.
      iv. Compute expected revenue in equilibrium, $\mathbb{E}(R)$. *Hint:* You may want to use that

$$\int p^n \exp(p-2)dp = p^n \exp(p-2) - n \int p^{n-1} \exp(p-2)dp.$$